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A NOTE ON CONTINUITY OF PSEUDODIFFERENTIAL OPERATORS IN HARDY SPACES

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The following sharp result concerning L_p-bounds for pseudo differential operators was proven by C. Fefferman in [3]: If T is a pseudo differential operator of class $L_{\varrho,\delta}^{-m}$ where $0 \leqq \delta < \varrho \leqq 1$ and $m \geqq (1-\varrho)|n/2 - n/p|$ then

$$T: L_p \to L_p, \quad 1 < p < \infty . \tag{0.1}$$

Moreover, if $m \geqq (1-\varrho)n/2$ then

$$T: H_1 \to L_1 . \tag{0.2}$$

Here L_p is the Lebesgue space in R^n and H_1 is the Hardy space in the sence of Fefferman and Stein (cf. [5]).

There arises the question if it is possible to extend this result to the case $0 < p < 1$ and further whether it is possible for $p = 1$ to have the same space of both sides. There is indeed a natural candidate for such a generalization, using the local or non-homogeneous Hardy spaces h_p (cf. [6], [10] or [12] p. 124]).

We have not been able to prove this but only the following weaker result.

THEOREM. *Let* $m \in \mathsf{R}$, $0 \leqq \delta < \varrho \leqq 1$ *and* $T \in L_{\varrho,\delta}^m$. *Then for all* $0 < p, q, r < \infty$, $s \in \mathsf{R}$ *and* $s_1 < s + m - (1-\varrho)n|1/p - 1/2|$ *it holds*

$$T: F_{pq}^s \to F_{pr}^{s_1} . \tag{1}$$

For the definition and properties of Triebel spaces F_{pq}^s we refer to [9] or [12]. Note that $h_p = F_{p2}^0$ for $0 < p < \infty$.

REMARK 1. For $s_1 > s + m - (1-\varrho)n|1/p - 1/2|$ the claim is clearly false.

REMARK 2. We recall that a symbol $r(x,\xi)$ is said to be in class $S_{\varrho,\delta}^m$, $0 \leqq \varrho, \delta \leqq 1$, if

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$$|D_\xi^\alpha D_x^\beta r(x,\xi)| \leqq C_{\alpha\beta}(1+|\xi|)^{m+\delta|\beta|-\varrho|\alpha|}$$

holds for any multi-indexes α and β and for each pair $(x,\xi) \in R^n \times \mathrm{R}^n$. If $r(x,\xi) \in S^m_{\varrho,\delta}$ we say that the corresponding pseudodifferential operator $r(x,D)$ belongs to the class $L^m_{\varrho,\delta}$.

PROOF OF THEOREM. If suffices to show that

$$T\colon F^s_{pq} \to F^{s_1}_{pq} \qquad \text{for } s_1 < s+m-(1-\varrho)(n/p+\text{const.})$$

In fact, if we combine this with Hörmander's L_2 estimate (cf. [7])

$$T\colon F^s_{22} \to F^s_{22}, \qquad T \in L^0_{\varrho,\delta}$$

the desired result follows by non-trivial interpolation (cf. [4] or [12 p. 73]). Thus for (1) it is sufficient to prove the following lemma.

LEMMA. *Let* $T \in L^{-m}_{\varrho,\delta}$, $0 \leqq \delta < \varrho \leqq 1$ *and* $m \geqq (1-\varrho)(n/\min(p,q)+n+1)$. *Then for all* $0<p,q<\infty$ *and* $s \in \mathrm{R}$

$$T\colon F^s_{pq} \to F^s_{pq}\,.$$

PROOF. For simplicity we suppose that $s=0$. We write $r(x,D)$ for T. Let (φ_k) be the sequence of test functions as in the standard definition of F^s_{pq} (cf. [8]). What we should do is to estimate the norm

$$\|(\varphi_j(D)r(x,D)f(x))_{j=0}^\infty\|_{L_p(l_q)}$$

by the norm $\|(\varphi_j(D)f)_{j=0}^\infty\|_{L_p(l_q)}$. Thus the main task will be to commute the operators $\varphi_j(D)$ and $r(x,D)$. We shall do this in the well-known manner by first invoking the Leibniz rule (cf. [11 p. 46]).

$$\varphi_j(D)r(x,D) \sim \sum_{\beta \geqq 0} \frac{1}{\beta!} r_{(\beta)}(x,D)\varphi_j^{(\beta)}(D)\,.$$

Here we have used the notation $p_{(\beta)}^{(\alpha)}(x,\xi) = D_x^\beta(iD_\xi)^\alpha p(x,\xi)$.

Next we choose another sequence of test functions $(\psi_j)_{j=0}^\infty$ with $\psi_j(D)\varphi_j(D) = \varphi_j(D)$ valid for all j. Moreover we suppose that $\psi_j(\xi)$ is supported in a set where $|\xi| \sim 2^j$. To estimate $r_{(\beta)}(x,D)\varphi^{(\beta)}(D)f(x)$ we write it in the integral form

$$r_{(\beta)}(x,D)\varphi_j^{(\beta)}(D)f(x) = \int K_\beta^j(x,y)f_j(y)\,dy \tag{2}$$

where $f_j = \psi_j(D)f$ and

$$K_\beta^j(x,y) = \int e^{i(x-y)\xi} r_{(\beta)}(x,\xi)\varphi_j^{(\beta)}(\xi)\,d\xi\,.$$

For the kernel $K^j_\beta(x, y)$ we can get the following estimate

$$\text{(3)} \qquad |K^j_\beta(x,y)| \leqq C_\lambda \frac{2^{jn}}{(1+2^j|x-y|)^\lambda}, \quad \text{for } \lambda \leqq [m]/(1-\varrho)\,.$$

Namely, by partial integration one obtains for each α, $|\alpha| \leqq [m]/(1-\varrho)$

$$|(x-y)^\alpha K^j_\beta(x,y)| \leqq C_{\alpha\beta} \sum_{\gamma \leqq \alpha} 2^{jn}(1+2^j)^{-m+(1-\varrho)|\gamma|} 2^{-j|\alpha|} \leqq C_{\alpha\beta} 2^{jn} 2^{-j|\alpha|}\,.$$

By using $\lambda > (n/\min(p,q)) + n$ it follows from (2) and (3) that

$$|r_{(\beta)}(x, D)\varphi_j^{(\beta)}(D)f(x)| \leqq Cf^*_j(\mu, x)$$

with $\mu > n/\min(p,q)$. Here $f^*_j(\mu, x)$ is the Fefferman-Stein maximal function defined by

$$f^*_j(\mu, x) = \sup_{y \in \mathbf{R}^n} \frac{|\varphi_k(D)f(y)|}{(1+2^k|x-y|)^n}\,.$$

Hence, if we write

$$\varphi_j(D)r(x,D)f(x) := \sum_{|\beta|<N} \frac{1}{\beta!} r_{(\beta)}(x,D)\varphi_j^{(\beta)}(D)f(x) + R_j^N f(x) := g_j^0(x) + g_j^1(x)$$

we obtain from the Fefferman-Stein-Peetre inequality (cf. [5], [9] or [12, p. 47]) that

$$\|(g_j^0(x))_{j=0}^\infty\|_{L_p(l_q)} \leqq C_N \|f\|_{F^0_{pq}}\,.$$

It remains to give a similar estimate for the remainder $R_j^N f(x)$. In order to do that we write $R_j^N f$ in the form

$$R_j^N f(x) = \int e^{ix(y+\xi)} \hat{f}(\xi) p_j^N(\eta, \xi)\, d\xi\, dy$$

where

$$p_j^N(\eta, \xi) = \hat{r}(\eta, \xi)\left(\varphi_j(\eta+\xi) - \sum_{|\beta|<N} \frac{1}{\beta!} \varphi_j^{(\beta)}(\xi)\eta^\beta\right)$$

and $\hat{r}(\eta, \xi)$ is the Fourier transform of $r(x, \xi)$ with respect to x. By using Lagrange's remainder term in Taylor's formula and by taking N large enough one can prove that

$$\|(g_j^1)_{j=0}^\infty\|_{L_p(l_q)} \leqq C\|f\|_{F^0_{pq}}\,.$$

For the details cf. [8].

REMARKS. The use of interpolation yields the corresponding result for Besov spaces.

Finally, we ask whether Fefferman's theorem remains true if $0 \leqq \delta < \varrho \leqq 1$ is replaced by $0 \leqq \delta = \varrho < 1$ or more generally whether the following result holds: Supposing $0 < p < \infty$, $0 \leqq \varrho < 1$ and $T \in L_{\varrho,\varrho}^{(1-\varrho)n|1/p-1/2|}$ we have

$$T: h_p \to h_p .$$

For $p = 2$ this is true according to theorem of Calderón and Vaillancourt (cf. [1]). For $1 < p < \infty$ and $\varrho = 0$ it is proved by Coifman and Meyer in [2 p. 140].

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