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# CAPACITARY INTEGRALS IN DIRICHLET SPACES

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## Introduction.

In the present paper we deal with capacitary integrals in Dirichlet spaces. This stems from a wish to generalize the following problem and its consequences, and also to put matters in their proper context.

Let $E$ be an open, say, set in $\mathsf{R}^d$, $d \geqq 3$, and define $H(E)$ as the closure of all $C_0^\infty(\mathsf{R}^d)$ functions harmonic in $E$, with respect to the norm $(\int |\nabla f|^2 \, dx)^{\frac{1}{2}}$. Characterize the points $x$ such that the map $f \to f(x), f \in H(E)$, is bounded in this norm. In [17] the author proved that such points are precisely those for which the harmonic measure $\delta_x^{\complement E}$ for $E$ is of finite energy, and also that this was equivalent to a certain Wiener criterion. At about the same time, Fuglede [13] investigated the spaces $H(E)$ from a fine-topological point of view.

Another interesting fact was that the Wiener criterion could be expressed by means of certain capacitary integrals. With this tool, Adams [2] characterized the obstacles for which the obstacle problem in the Sobolev space $W_0^{1,2}(\Omega)$ has a solution.

These results form the starting-point of this paper.

Here we will concentrate on the capacitary approach. In [19] we treat the fine-potential theoretic aspects of the problem described. We also give a general theory for fine superharmonicity in Dirichlet spaces.

The appropriate spaces in which to work are the so-called Dirichlet spaces, originally introduced by Beurling and Deny. By results of Fukushima [14], this is, under certain regularity conditions, equivalent to working with symmetric Hunt processes, so the probabilistic theory is included, although the approach of this paper is analytic and not probabilistic. In fact, a good deal of work had to be done in order to "translate" the probabilistic theory to the usual set-up of potential theory.

Although we build up the Dirichlet space $W$ from a sub-Markov transition function $p(t, x, E)$ and not from a Dirichlet form as in [14], this is merely a difference of exposition. We have however added some extra assumptions, viz.

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(a) the Green operator is proper, (section 1.2),
(b) the measures $(t, x, .)$ are absolutely continuous for $t > 0$, (section 3.1),
(c) the excessive functions are lower semi-continuous, (section 3.3).

The following are consequences of (a)–(c):

—The fine topology is generated by the excessive functions, which puts us in a situation where we can use the axiomatics of Brelot [4]. (This follows from (a).)

—From (b) follows that potentials of measures can be defined everywhere, and not only quasi everywhere. We also get a simpler situation concerning exceptional sets.

—Condition (c) implies that the balayage operation $f \to \hat{R}_f$ is an upper capacity (with values in the lattice of excessive functions) for which the theory in [11] is applicable. We will use this fact on several occassions below and in [19].

A general theory with emphasis on fine topology and quasi topology in Dirichlet spaces is sketched: In section 1 we construct the Dirichlet space, and in section 3 we consider excessive functions and related concepts. In Theorem 3.10 we deduce the Choquet property, essential to fine-topological considerations.

In section 2 we prove a strong-type capacitary inequality (cf. [1]) for functions in the Dirichlet space $W$, and in section 4 we prove a converse result: if $u$ is excessive and a certain $L^2$-capacitary integral of $u$ is finite, then $u \in W$. We also consider a parallel of this result for non-linear potentials.

The fifth section is devoted to a study of the obstacle problem, which in its turn means a closer look at the just mentioned functional $\hat{R}_f$. We use the Choquet capacitability theorem and the earlier results of sections 2 and 4 to prove that for a large class of functions $f$, including the Borel functions, we get a solution to the obstacle problem with obstacle $f$, if and only if the $L^2$-capacitary integral of $f$ is finite.

Disregarding considerations of the measurability of the obstacle, this is precisely the Adams result alluded to above.

Apart from the articles [2] and [17], referred to already, we would also like to mention the work of Hansson [15], which has a lot in common with the material in our sections 2 and 4.

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## 0. Preliminaries.

Throughout this paper, $X$ will denote a locally compact Hausdorff space with a countable base for its topology $\mathscr{T}$. We write $C=C(X)$ for the continuous and real-valued functions on $X$, $C_0=C_0(X)$ denotes all compactly supported functions in $C$, and $C_\infty=C_\infty(X)$ is the closure of $C_0$ with respect to the supremum norm. That is, $C_\infty$ consists of all continuous functions which vanish at infinity. The Borel sets are denoted by $\mathscr{B}=\mathscr{B}(X)=\sigma(\mathscr{T})$, that is, $\mathscr{B}$ is the sigma-closure of $\mathscr{T}$. We write $\mathscr{M}=\mathscr{M}(X)$ for the class of (real-valued) Radon measures on $(X,\mathscr{B})$. By the Riesz representation theorem, valid in $(X,\mathscr{B})$, $\{\mu \in \mathscr{M}: \int_X d|\mu| < \infty\}$ is the dual of $C_\infty$. The $\sigma$-algebra $\mathscr{B}^*$ of universally measurable sets is defined as the intersection $\bigcap_\mu \mathscr{B}^\mu$, where $\mathscr{B}^\mu$ denotes the completion of $\mathscr{B}$ with respect to the measure $\mu$, and where $\mu$ ranges over all probability measures on $(X,\mathscr{B})$.

Any function $f: X \to [-\infty, +\infty]$ taken under consideration, is tacitly assumed to be $\mathscr{B}^*$-measurable: $f \in \mathscr{B}^*$. (As a general rule, we will write $f \in \mathscr{F}$ to indicate that $f$ is measurable with respect to the $\sigma$-algebra $\mathscr{F}$.) Whenever convenient, we may consider any $\mu \in \mathscr{M}$ as a measure on $(X,\mathscr{B}^*)$.

Given a collection $\mathscr{A}$ of measures of functions, we denote by $\mathscr{A}^+$ (or $\mathscr{A}_+$) its positive members. In the case when $\mathscr{A} \subset \mathscr{M}$, we denote by $\mathscr{A}(M)$ the class of all $\mu \in \mathscr{A}$ carried by the (universally measurable) set $M$. We will use the notation $1_M$ for the indicator (characteristic function) of $M$. The restriction of a measure $\mu$, or a function $f$, to $M$ will be denoted by $\mu \mid M$ or $f \mid M$, respectively. As usual, $u^+ = \max(u,0) = u \vee 0$, and $u^- = -\min(u,0) = -(u \wedge 0)$, denotes the positive and negative parts of a function $u$. We will write $E_1 - E_2$ for the vector difference of two sets: $E_1 - E_2 = \{x_1 - x_2 : x_i \in E_i\}$, whereas $E_1 \setminus E_2$ is the set-theoretic difference.

General references are Fukushima [19], Blumenthal and Getoor [3], Brelot [4], and Fuglede [11]. On quasi topology, see also Fuglede [10, 12].

## 1. Construction of the Dirichlet space.

The construction to be outlined below is essentially equivalent to that in [14]. For practical reasons we work with a sub-Markov transition function instead of a Dirichlet form.

To clarify what is actually assumed, we mark the assumptions (A1), (A2), etc.

1.1. A *sub-Markov transition function* on $(X,\mathscr{B})$ is a function

$$p = p(t,x,E), \qquad t>0,\ x \in X,\ E \in \mathscr{B}$$

satisfying (A1)–(A5) below.

(A1) $$0 \leqq p \leqq 1 ,$$

(A2) $$p(t,x,.) \in \mathscr{M}, \quad t > 0, \quad x \in X ,$$

(A3) $$p(t,.,E) \in \mathscr{B}, \quad t > 0, \quad E \in \mathscr{B} .$$

Defining

$$p_t f(x) = \int_X p(t,x,dy) f(y), \quad f \in \mathscr{B}_+^* ,$$

where $p(t,x,dy)$ is the Radon measure given by (A2), we also assume that $(p_t)_{t>0}$ is a *semigroup*:

(A4) $$p_t p_s = p_{t+s}, \quad t,s>0 ,$$

and that

(A5) $$\lim_{t\downarrow 0} p(t,x,\cdot) = \delta_x, \quad x \in X .$$

Writing $p_0 = 1$, the identity (A4) is thus valid for all $t,s \geqq 0$.

The convergence in (A5) means of course that $p_t\varphi(x) \to \varphi(x)$ for any $\varphi \in C_\infty(X)$.

1.2. We define the *Green operator* by

$$Gf = \int_0^\infty p_t f\, dt, \quad f \in \mathscr{B}_+^* ,$$

and we assume that $G$ is *proper*:

(A6) $$K \text{ compact} \Rightarrow G1_K(x) < \infty, \quad \forall x \in X .$$

(A6) says that $E \to G1_E(x)$ is a Radon measure for all $x$.

This is the classical condition for transience. In [14], a less restrictive notion of transience is employed, however.

We note that if $p_t$ satisfies (A1)–(A5), then $e^{-\lambda t}p_t$, $\lambda>0$, satisfies (A6). Thus we have already taken care of the possible passage from $p_t$ to $e^{-\lambda t}p_t$.

Next we assume that a measure $m \in \mathscr{M}^+$ with $\operatorname{supp} m = X$ is given, such that $(p_t)_{t>0}$ is symmetric with respect to $m$; that is

(A7) $$\int_X f\cdot p_t g\, dm = \int_X p_t f\cdot g\, dm, \quad f,g \in \mathscr{B}_+^* .$$

1.3. We shall now introduce the Hilbert space $W$ in which things will take place. We note that by [14, Lemma 1.4.3], $(p_t)_{t>0}$ is a strongly continuous semi-group on $L^2(X,m)$, so the theory in Fukushima [14] is applicable. For $u = Gf$, $v = Gg$, $f,g \geqq 0$, we define

(1.1) $$(u \mid v) = \int_X Gf \cdot g \, dm = \int_X Gg \cdot f \, dm .$$

It follows from the symmetry and the semi-group property that

$$\int_X G(f_1 - f_2) \cdot (f_1 - f_2) \, dm \geqq 0, \; f_1, f_2 \in \mathscr{B}_+^* ,$$

whenever this makes sense. Thus we may extend (1.1) to (extended) real-valued functions $Gf$ to get an inner product. The real Hilbert space obtained by taking the completion of all $u = Gf = G(f_1 - f_2)$, $f_1, f_2 \geqq 0$, such that $\int_X Gf_i f_i \, dm < \infty$ is denoted by $W$. This corresponds to the construction in [14, section 1.5], and in the terminology of that book, $(p_t)_{t>0}$ is *transient* and $W$ an *extended Dirichlet space* (see the remark following (A6)).

A further assumption to be made is that $W$ be *regular*:

(A8) $C_\infty \cap W$ *is dense in* $C_\infty$ *and in* $W$.

1.4. By the results of Fukushima [14, Chapter 6], there is a Hunt process (essentially a strong Markov process with right-continuous trajectories, see Blumenthal and Getoor [3, p. 45]) on $(X_\partial, \mathscr{B}_\partial)$ such that

$$p(t, x, E) = P_x(Z_t \in E) ,$$

where $Z_t$ denote the process at time $t$, and where $P_x$ is the corresponding probability law for a "particle" starting at $x$. (Here $\partial$ is the point at infinity (added as an isolated point in case $X$ is compact), $X_\partial = X \cup \partial$, the compactification of $X$, and $\mathscr{B}_\partial$ the corresponding $\sigma$-algebra.)

This means that the theory in Blumenthal and Getoor [3] is at our disposal.

1.5. Examples. Let $X = \mathbf{R}^d$, $d \geqq 3$; the case when the semigroup is given by the normal density will be referred to as the *Newtonian case*. That is,

$$p(t, x, E) = (2\pi t)^{-d/2} \int_E \exp\left(-\frac{|x-y|^2}{2t}\right) dy ,$$

and $Gf$ is the classical Newton potential:

$$Gf(x) = \int_{\mathbf{R}^d} |x-y|^{2-d} f(y) \, dy .$$

(Let us make the agreement that all "explicit" formulae are to be read modulo constants of no importance.)

The scalar product is defined by

$$(u \mid v) = \int_{\mathbf{R}^d} \nabla u \cdot \nabla v \, dx \, .$$

We obtain $W$ by taking the closure of $C_0^\infty(\mathbf{R}^d)$ with respect to the corresponding norm. (The elements of $W$ are known in the literature as BL (Beppo Levi) or BLD (D as in Deny) functions.)

If we replace $p_t$ in this example with $e^{-t}p_t$, we obtain the Sobolev space $W^{1,2}(\mathbf{R}^d)$, with inner product

$$(u \mid v) = \int_{\mathbf{R}^d} (\nabla u \cdot \nabla v + uv) \, dx \, ,$$

and as above, we obtain the Dirichlet space when taking the closure of $C_0^\infty(\mathbf{R}^d)$ in this norm. (In this case, the assumption $d \geqq 3$ is unnecessary.) After a Fourier transformation, we get

$$(u \mid v) = \int_{\mathbf{R}^d} \hat{u}\bar{\hat{v}}(1 + |\xi|^2) \, d\xi \, ,$$

which we recognize as a case of Bessel potentials. In fact, if we replace $(1 + |\xi|^2)$ by $(1 + |\xi|^2)^\alpha$, $0 < \alpha < 1$, this is also the scalar product of a Dirichlet space. This is not the case when $\alpha > 1$, however.

A fourth example is obtained from the $\Gamma$-semi-group on $[0, \infty)$. If

$$\lambda_t(dx) = \frac{x^{t-1}}{\Gamma(t)} e^{-x} \, dx, \qquad x > 0, \ t > 0 \, ,$$

we can symmetrize $\lambda_t$ to get a convolution semi-group of probability measures $\mu_t$ on $X = \mathbf{R}$:

$$p_t f(x) = \int_{-\infty}^{\infty} f(x+y) \, d\mu_t(x) \, ,$$

where

$$\hat{\mu}_t(\xi) = |\hat{\lambda}_t(\xi)|^2 = \frac{1}{(1+\xi^2)^t} = \exp\{-t \log(1+\xi^2)\} \, ,$$

and where $\hat{\mu}$ denotes the Fourier transform of a measure $\mu$. The Dirichlet space $W$ is characterized by the inner product

$$(u \mid v) = \int_{-\infty}^{\infty} \hat{u}(\xi)\overline{\hat{v}(\xi)}\psi(\xi) \, d\xi$$

and

$$W = \{u \colon \hat{u} \in L^2(\psi(\xi) d\xi)\}, \qquad \psi(\xi) = 1 + \log(1+\xi^2) \, .$$

(If $\psi$ is replaced by $\log(1+\xi^2)$, we do not obtain a transient case, see [14, Example 1.5.2, pp. 38–40].)

We note that this is a limiting case of Bessel potentials.

—In all these examples, $m$ is the Lebesgue measure.

1.6. The *capacity* of a compact set $K$ in $X$ is defined as

$$\operatorname{cap} K = \inf\{\|u\|^2 : u \in C_0(X) \cap W,\ u \geqq 1_K\}, \tag{1.2}$$

and cap is then extended as an outer capacity in the usual manner. The set function obtained is a *Choquet-capacity*, see [14, Theorem 3.1.1]. We will use the term "quasi everywhere" (q.e.) meaning outside a set of zero capacity; we write "for q.e. $x$", to be read "for quasi every $x$." More or less by definition, in view of (A8), functions in $W$ are *quasi continuous*: Given $\varepsilon > 0$ and $u \in W$ we can redefine $u$ so that $u \mid X \setminus e$ is continuous in the relative topology of $X \setminus e$, where $\operatorname{cap} e < \varepsilon$ and $e$ may be chosen as an open set.

Strictly speaking, it is of course a question of identifying abstract elements with functions defined q.e. on $X$, but from (A8) we know that continuous functions are dense in $W$.

We are always referring to quasi continuous representatives when speaking of functions in $W$.

1.7. The following expression of the scalar product will be useful. Let $u, v \in W$ and define, for $t > 0$,

$$E_t(u,v) = \frac{1}{t}\int_X (1-p_t)u \cdot v\, dm, \tag{1.3}$$

and

$$E_t(u) = E_t(u,u). \tag{1.4}$$

Then (see [14, section 1.3], and also [14, Lemma 1.5.4], where the proof is carried out for the corresponding approximation with the resolvent)

$$\lim_{t \downarrow 0} E_t(u) = \|u\|^2 \tag{1.5}$$

and the limit is increasing. The symmetry of $(p_t)$ with respect to $m$ yields easily the following representation of $E_t$.

$$E_t(u) = \frac{1}{2t}\int_{X\times X} (u(x)-u(y))^2 \sigma_t(dx,dy) + \frac{1}{t}\int_X u(x)^2(1-p(t,x,X))\, m(dx), \tag{1.6}$$

where $\sigma_t(dx, dy) = p(t, x, dy)m(dx)$ is a symmetric Radon measure on $X \times X$. From (1.6) it is clear that every *normal contraction operates continuously on W.* (The map $u \to Tu$ is a normal contraction if for every $x, y$ in $X$, $|Tu(x)| \leqq |u(x)|$ and $|Tu(x) - Tu(y)| \leqq |u(x) - u(y)|$.)

## 2. Capacitary integrals.

2.1. For a continuous function $u$ in $W$, the estimate

$$t^2 \operatorname{cap}(|u| > t) \leqq \|u\|^2, \quad t \geqq 0,$$

is trivial. The following result shows that a *strong-type estimate* holds.

THEOREM. *If $u \in W$, then*

$$\int_0^\infty \operatorname{cap}(|u| > t)\, dt^2 \leqq \text{const.} \|u\|^2. \tag{2.1}$$

(Here $dt^2 = d(t^2) = 2tdt$; the integral in (2.1) is called a capacitary integral.)

PROOF. We start by proving that

$$\sum_{n=-\infty}^{\infty} 2^{2n} \operatorname{cap} A_n \leqq \text{const.} \|u\|^2, \tag{2.2}$$

where

$$A_n = \{x : 2^n < |u(x)| \leqq 2^{n+1}\}, \quad n \in \mathbf{Z}. \tag{2.3}$$

We may assume that $u \geqq 0$, and to avoid the term q.e., we also assume $u \in C_0 \cap W$.

Let $H(s)$, $s \in \mathbf{R}$, equal 1 if $2 \leqq s \leqq 4$, 0 if $s \leqq 1$ or $s \geqq 8$, and let $H$ be linear elsewhere so that $H$ is Lipschitz with $\|H'\|_\infty \leqq 1$. Then the map $v \to H(v)$ is a normal contraction, so defining

$$u_n = H(2^{-n+1}u),$$

$u_n \in W$. Further, $u_n = 1$ on $A_n$ and $\operatorname{supp} u_n \subset K_n \equiv \{2^{n-1} \leqq u \leqq 2^{n+2}\}$. In particular,

$$\operatorname{cap} A_n \leqq \|u_n\|^2. \tag{2.4}$$

We shall now use (1.5) and (1.6) to obtain (2.2). Since $(u_n(x) - u_n(y))^2 \neq 0$ only if $(x, y) \in (K_n \times X) \cup (X \times K_n)$, the symmetry of $\sigma_t$ gives

$$\frac{1}{2t} \int_{X \times X} (u_n(x) - u_n(y))^2 \sigma_t(dx, dy)$$

$$= \frac{1}{t} \int_{K_n \times X} (u_n(x) - u_n(y))^2 \sigma_t(dx, dy) \tag{2.5}$$

$$\leqq (2^{-n+1})^2 \frac{1}{t} \int_{K_n \times X} (u(x) - u(y))^2 \sigma_t(dx, dy) ,$$

because $u_n$ is a normal contraction of $2^{-n+1}u$. For the same reason,

$$\frac{1}{t} \int_X u_n(x)^2 m_t(dx) \leqq (2^{-n+1})^2 \frac{1}{t} \int_{K_n} u(x)^2 m_t(dx) , \tag{2.6}$$

where we have written $m_t(dx) = (1 - p(t, x, X))m(dx)$.

From (2.5) and (2.6) it follows that

$$E_t(u_n) \leqq 8 \cdot 2^{-2n} \left\{ \frac{1}{2t} \int_{K_n \times X} (u(x) - u(y))^2 \sigma_t(dx, dy) + \frac{1}{t} \int_{K_n} u(x)^2 m_t(dx) \right\} . \tag{2.7}$$

Multiplying (2.7) by $2^{2n}$ and summing over $n \in \mathbf{Z}$, gives

$$\sum_n 2^{2n} E_t(u_n) \leqq 24 \cdot E_t(u) , \tag{2.8}$$

where we have used that $\sum 1_{K_n} \leqq 3$. The right-hand side of (2.8) is no greater than $24 \cdot \|u\|^2$, so, letting $t \downarrow 0$ in the left-hand side of (2.8) we obtain

$$\sum_n 2^{2n} \|u_n\|^2 \leqq \text{const.} \|u\|^2 , \tag{2.9}$$

by monotone convergence. Now (2.2) follows from (2.4) and (2.9).—Let $a_n = \operatorname{cap} A_n$ and $b_n = \operatorname{cap} (u > 2^n)$. By subadditivity,

$$b_n \leqq \sum_{k \geqq n} a_k , \tag{2.10}$$

so that

$$\sum_n 2^{2n} b_n \leqq \sum_n 2^{2n} \sum_{k \geqq n} a_k = \sum_k a_k \sum_{n \leqq k} 2^{2n} = \sum_k a_k 2^{2k} \sum_{j \leqq 0} 2^{2j} = \frac{4}{3} \sum_k a_k 2^{2k} . \tag{2.11}$$

Since the left-hand sides of (2.1) and (2.11) are comparable to each other, (2.2) and (2.11) proves the theorem.

2.2. REMARKS.

1. Strong type capacitary estimates were originally introduced by Maz'ja in

the Sobolev spaces $W^{1,p}(\mathbf{R}^n)$ (see e.g. [20, Satz 4.1, pp. 92–94]) in connection with the Schrödinger equation. If $L=-(\Delta+q)$, where $q$ stands for multiplication by the distribution $q$, one wants to know when $L$ maps $W^{1,2}(\mathbf{R}^n)$ into its dual $W^{-1,2}(\mathbf{R}^n)$.

If $q=\mu\in\mathcal{M}^+(\mathbf{R}^n)$, $\mu\neq 0$, an easy application of the theorem shows that this happens if and only if $\mu(E)\leqq A\cdot\operatorname{cap}(E)$ for all $E\in\mathcal{B}$.

2. If $\mu\in\mathcal{M}^+$ and $\mu(E)\leqq\operatorname{cap}(E)$ for all $E\in\mathcal{B}$, then

$$\int_X f^2\,d\mu = \int_0^\infty \mu(|f|>t)\,dt^2 \leqq \int_0^\infty \operatorname{cap}(|f|>t)\,dt^2 .$$

Let us write $f\in L^2(X,\operatorname{cap})$ if this last integral is finite. (To the author's knowledge, this suggestive notation, which coincides with the usual one in case cap is additive, i.e. a measure, is due to D. R. Adams [2]. Integrals of the type $\int_0^\infty \operatorname{cap}(|f|>t)\,dt$ occur already in Choquet's fundamental treatise [6, chapter 48], though.) It follows that

$$L^2(X,\operatorname{cap}) \subset \bigcap_{0\leqq\mu\leqq\operatorname{cap}} L^2(X,\mu)$$

where the right-hand side may be normed by $\sup_{\mu\leqq\operatorname{cap}}\{\int_X f^2\,d\mu\}^{\frac{1}{2}}$.

3. Clearly $L^2(X,\operatorname{cap})$ is not contained in $W$. For instance, any function $f$ in $C_0(X)$ is a member of the former space, but in general not of the latter. For a "converse result", see section 4.

4. In [18] we use capacitary integrals (with respect to condenser capacities) to solve a problem on removable singularities for functions in the Sobolev space $W^{1,p}$, $1<p<\infty$.

See also the introduction and 4.4 below for further applications.

## 3. Excessive functions and related topics.

3.1. Our next objective is to introduce excessive functions. First however, we add the following to our list of assumptions

$$\text{(A9)}\qquad p(t,x,E) = \int_E p(t,x,y)\,dm(y), \qquad t>0,\ x\in X,\ E\in\mathcal{B}\,.$$

In other words, $p(t,x,dy)$ are absolutely continuous with respect to $m$, with densities $p(t,x,y)$ for all $t>0$. The symmetry condition then takes the form

$$\text{(A7)}'\qquad p(t,x,y) = p(t,y,x), \qquad m\times m-\text{a.e.}\,,$$

and the semi-group property means that

(A4)′
$$p(s+t,x,y) = \int_X p(s,x,z)p(t,z,y)\,dm(z)\,.$$

The condition (A9) has an important consequence that concerns *exceptional sets.* (For definitions, see [3, Chapter II].) It turns out that the notions of polar sets, semi-polar sets and sets of capacity zero all coincide, and moreover, (A9) is also necessary for this situation to occur; see [14, Theorem 4.3.4]. Thus we may freely use the preferable adjective "polar" in place of the clumsier "of capacity zero."

(In [14] one uses the capacity obtained from the semi-group $(e^{-t}p_t)_{t>0}$. This capacity has, however, the same null-sets as the capacity used here (see [14, Theorem 3.1.5, p. 68]), although these capacities are not comparable in general.)

3.2. Definition. A function $f\colon X \to [0, +\infty]$ is *excessive* (with respect to the semi-group $(p_t)$) if

(3.1)
$$\text{(a)}\quad p_t f \leqq f, \quad t \geqq 0, \quad \text{and}$$
$$\text{(b)}\quad \lim_{t\downarrow 0} p_t f(x) = f(x), \quad x \in X\,.$$

The class of excessive functions is denoted by $\mathscr{S}$. We note that $1 \in \mathscr{S}$ and $+\infty$ also. In view of (A9), the kernel of the Green operator is given by *Green's function*

$$G(x,y) = \int_0^\infty p(t,x,y)\,dt\,,$$

so for $\mu \in \mathscr{M}^+$ we may define the *potential* of $\mu$ by

$$G\mu(x) = \int_X G(x,y)\,d\mu(y)\,.$$

Then $G\mu$ *is excessive* because

$$p_t G\mu(x) = \int_t^\infty \int_X p(s,x,y)\,d\mu(y)\,ds \uparrow G\mu(x), \quad t\downarrow 0\,.$$

When $\mu$ is absolutely continuous the definition of $G\mu$ agrees with that given in section 1.1 (with $d\mu = f\,dm$).

Without condition (A9), $G\mu$ will satisfy (3.1) q.e. This is rather much as in [14]. It is convenient to have potentials $G\mu$ (and not only those of the form $Gf$, $f \in \mathscr{B}_+^*$) defined everywhere.

3.3. The last assumption to be made is the following:

(A10) *excessive functions are* l.s.c. ,

(lower semi-continuous). Then $G1_E(x)=0$ for all $x \in X$ ($E \in \mathscr{B}^*$), if and only if $mE=0$, since by [14, Lemma 4.2.1], $mE=0$ iff $G1_E(x)=0$ for $m$-a.e. $x$. By (A10), the set $\{G1_E>0\}$ is open, hence of positive $m$-measure if it is non-empty. (Recall that $\operatorname{supp}(m)=X$.) Thus $m$ is a *representing measure* for $(p_t)_{t>0}$. See Blumenthal-Getoor [3, p. 196 ff.].

In the sequel, we assume (A1)–(A10) to hold unless otherwise is explicitly stated.

3.4. Since $m$ is representing, the theory developed by Doob and Meyer, see [3, p. 196 ff.], holds. This implies that $\mathscr{S}$ is a lattice with particularly nice properties. For details on this, we also refer to Chapter II.2-3 of the same book.

For a given function $f$, let $\hat{f}$ denote its l.s.c. envelope:

$$\hat{f}(x) = \liminf_{y \to x} f(y) . \tag{3.2}$$

For any family $(u_i)$ of excessive functions, we define

$$\bigwedge_i u_i = \left(\inf_i u_i\right)^{\wedge} . \tag{3.3}$$

Then $\bigwedge_i u_i \in \mathscr{S}$, and there is a sequence $(u_{i_n})_{n\geq 1} \subset (u_i)$ with the same pointwise infimum q.e. (Clearly $u=\inf_i u_i$ is super mean-valued, i.e. (3.1.a) is fulfilled. Then $\hat{u}$ is finely continuous (section 3.8) and fulfils (3.1a), which is known to imply that $\hat{u}$ is in $\mathscr{S}$. In fact, $\hat{u}=\lim_{t\downarrow 0} p_t u$.—The regularization (3.2) may just as well be performed with respect to the fine topology, see [19].) Also, the function defined in (3.3) agrees with the pointwise infimum q.e. When the index set is finite, there is no need for regularization; that is, in this case $\bigwedge_i u_i = \inf_i u_i$. In particular, $u \wedge v = \inf(u,v) \in \mathscr{S}$ if $u, v \in \mathscr{S}$.

As to the pointwise supremum of a family of excessive functions, we must demand that the family be pointwise *upper directed*. If this is the case, then

$$\bigvee_i u_i = \sup_i u_i ,$$

is an excessive function.

3.5. For measures $\mu \in \mathscr{M}$ with $\int_X G|\mu| \, d|\mu| < +\infty$, we have, in analogy with section 1.4,

$$I(\mu) \equiv \int G\mu \, d\mu = \|G\mu\|^2 \geqq 0 . \tag{3.4}$$

More generally, for such measures,

$$\int_X \varphi \, d\mu = (G\mu \mid \varphi), \qquad \varphi \in W, \tag{3.5}$$

and the *energy* $I(\mu)$ is finite if and only if the measure $\mu$ belongs to the dual $W'$ of $W$. We write $\mathscr{E}$ for the *positive* measures of finite energy. That is,

$$\mathscr{E} = \{\mu \in \mathscr{M}^+ : I(\mu) < +\infty\} .$$

3.6. We know from section 3.2 that $G\mu \in \mathscr{S}$ for $\mu \in \mathscr{M}^+$. Thus, by section 3.5, we have $\{G\mu : \mu \in \mathscr{E}\} \subset \mathscr{S} \cap W$. But if $u \in \mathscr{S} \cap W$, then (1.3) and (1.5) show that the map $\varphi \to (u \mid \varphi)$ is positive. Using the regularity condition (A8), one finds that this map is given by a positive Radon measure: $(u \mid \varphi) = \int_X \varphi \, d\mu$, with $\mu \in \mathscr{M}^+ \cap W'$, and it follows that $u = G\mu$. Hence we obtain the following equivalent characterizations of $\mathscr{S} \cap W$:

$$\mathscr{S} \cap W = \{G\mu : \mu \in \mathscr{E}\} = \{u \in W : (u \mid \varphi) \geqq 0, \ \varphi \in W^+\} . \tag{3.6}$$

For this, cf. [14, Theorem 3.2.1].

Using (1.3)–(1.5), one sees that among excessive functions, the norm of $W$ is *order preserving*:

$$[u \leqq v, \ u, v \in \mathscr{S}] \ \Rightarrow \ [\|u\| \leqq \|v\|] . \tag{3.7}$$

In particular, $\mathscr{S} \cap W$ is a *hereditary subcone* of $\mathscr{S}$: If $v$ in (3.7) belongs to $W$, then so does $u$.

3.7. At this stage, an alternative way of defining $W$ is available. By Yosida [29, Chapter IX.11] there is a semigroup $(q_t)_{t>0}$ such that $H = \int_0^\infty q_t \, dt$ satisfies $G = H^2$, that is $G\mu = H(H\mu)$. (From the formulae given in [23], it is obvious that $(q_t)$ inherits the relevant properties of $(p_t)$, such as symmetry, $0 \leqq q(t, x, E) \leqq 1$, etc.) A simple calculation shows that $\int_X G\mu \, d\mu = \int_X (H\mu)^2 \, dm$ and $W$ could just as well be defined as the class of all $u = Hf$ such that $f \in L^2(X, m)$, as one sees from a limiting argument. $H$ can also be calculated directly from $p_t$:

$$H = \frac{1}{\sqrt{\pi}} \int_0^\infty p_t \frac{dt}{\sqrt{t}} .$$

3.8. Because of (A6), Theorem II.4.6. of [3] permits us to define the *fine topology* as the coarsest topology making all excessive functions continuous. Furthermore, condition (A10) ensures that the theory of Brelot [4, Chapters

I–III] can be applied. (The fine topology is the weak topology generated by a positive cone of l.s.c. functions.) — As a consequence, *all functions in W are finely continuous* q.e. (given $u \in W$, there is a set $e$ with $\operatorname{cap} e = 0$ such that $u \mid X \setminus e$ is continuous in the relative fine topology of $X \setminus e$, [14, Theorem 4.3.2]). Using the probabilistic approach hinted at above, we may introduce *thinness*, see [3, p. 79]. Since we do not need this concept itself at the moment, rather its consequences, we content ourselves with defining it through the classical characterization of fine neighbourhoods. $V$ is a fine neighbourhood of $x \in V$ if and only if $\complement V$, the complement of $V$, is thin at $x$. See however Brelot [4].

If $A$ is any set, we define its *base* as the set $b(A) = \{x \in X : A \text{ is non-thin at } x\}$; the *fine interior* of $A$ is denoted by $A'$ and the *fine closure* of $A$ is the set $\tilde{A} = A \cup b(A)$. We note the *Kellogg property*: $A \setminus b(A)$ is polar, see [3, Corollary V.1.14].

3.9. The relation between the fine topology and the original topology of $X$ is perhaps best described by "quasi topological" notions. Let us say that a set $E$ is *quasi open* if

$$\inf_{\omega} \operatorname{cap}(E \triangle \omega) = 0, \quad \omega \text{ open}, \tag{3.8}$$

$\triangle$ denoting symmetric difference. We can also define what it means for a set to be *quasi compact*, *quasi closed*, *quasi Borel* or *quasi analytic* (analytic set = Suslin set, see [6]) in exactly the same way. We remark here that because cap is outer, we may replace (3.8) by

$$\inf_{\omega} \operatorname{cap}(\omega \setminus E) = 0, \qquad \omega \text{ open}, \ \omega \supset E. \tag{3.9}$$

As for quasi continuity, a function $u$ is quasi continuous iff $u^{-1}(V)$ is quasi open for all open $V \subset [-\infty, +\infty]$. That $u$ is quasi l.s.c. means that $\{u > \lambda\}$ is quasi open for all $\lambda$, and similarly for quasi upper semi-continuous functions.

The *quasi Lindelöf property* discovered by Doob [8] holds because $m$ is representing. Thus, for any family $(V_i, i \in I)$ of finely open sets, one can extract a sequence $(i_n)_{n \geq 1} \subset I$ such that

$$\bigcup_{i \in I} V_i \setminus \bigcup_{n \geq 1} V_{i_n} \text{ is polar}. \tag{3.10}$$

See [3, Chapter V.1.17, p. 203].

3.10. We shall use the Kellogg and quasi Lindelöf properties to prove the following indeed useful result.

THEOREM. *The capacity introduced by* (1.2) *has the Choquet property*, i.e. *given any number* $\varepsilon>0$ *and any set* $E\subset X$, *there is an open set* $\omega$ *such that* $X\setminus b(E)\subset\omega$ *and*

$$\operatorname{cap}(E\cap\omega) < \varepsilon . \tag{3.11}$$

For Choquet's original proof in the Newtonian case, see [5, Theorem 1].

PROOF. For simplicity we write $e(E)=X\setminus b(E)$. By the proof of [4, Theorem I.3], we have

$$e(E)\setminus E \subset \bigcup\{V\in\mathscr{V}\} \subset \complement E , \tag{3.12}$$

where each $V\in\mathscr{V}$ is of the form

$$V = \{u<\lambda\}\cap\Omega ,$$

for some open set $\Omega$, some $u\in\mathscr{S}$, and some number $\lambda$ with $\lambda=\inf u(y)$, $y\in E\cap\Omega$. Using (3.7) and the balayage operation, defined in section 3.12 below, it is not hard to see that the $u$'s may be chosen from $W$, and this implies that all $V\in\mathscr{V}$ can be chosen as quasi open. By the quasi Lindelöf property (3.10) we can find sets $V_1, V_2,\ldots\in\mathscr{V}$ such that $\bigcup\{V\in\mathscr{V}\}\setminus\bigcup_1^\infty V_i$ is polar. This together with (3.12) and the fact that $e(E)\cap E$ is polar, gives us an open set $\omega_0$ with $\operatorname{cap}\omega_0<\varepsilon/2$ such that $e(E)\subset\bigcup_1^\infty V_i\cup\omega_0$, where $\varepsilon>0$ is a given small number.

Now choose open sets $\omega_1,\omega_2,\ldots$ with $V_i\subset\omega_i$ such that

$$\operatorname{cap}(\omega_i\setminus V_i) < \varepsilon\cdot 2^{-(i+1)}, \qquad i\geqq 1 .$$

Then

$$e(E) \subset \bigcup_0^\infty \omega_i .$$

For $i\geqq 1$ we get

$$E\cap\omega_i = (E\cap V_i)\cup(E\cap(\omega_i\setminus V_i)) = E\cap(\omega_i\setminus V_i) \subset \omega_i\setminus V_i ,$$

since $V\subset\complement E$ for all $V\in\mathscr{V}$. Thus

$$\operatorname{cap}\left(E\cap\bigcup_{i=0}^\infty \omega_i\right) \leqq \operatorname{cap}\omega_0+\sum_{i=1}^\infty \operatorname{cap}(\omega_i\setminus V_i) < \varepsilon ,$$

so, with $\omega=\bigcup_0^\infty\omega_i$, we have the desired set.

3.11. REMARK. It is well known that the Choquet property implies that the finely open sets are quasi open, and similarly for finely closed sets. Thus *the concepts "quasi continuity" and "fine continuity* q.e." *are equivalent*, and similar statements may of course be formulated on semi-continuity. We refer the reader to [4, Chapter IV]. — We mention also that if we use a semi-group of the form $(e^{-t}p_t)_{t>0}$, then Choquet's original proof carries over more or less directly.

3.12. Let $f\colon X \to [0, +\infty]$ and define its *reduced function* $R_f$ by

$$(3.13) \qquad R_f(x) = \inf\{v(x)\colon v \in \mathscr{S},\ v \geqq f\}\ .$$

Write also

$$(3.14) \qquad R_f^A = R_{f\cdot 1_A}, \quad A \subset X\ .$$

We have (see section 3.4)

$$(3.15) \qquad \hat{R}_f = \bigwedge\ \{v \in \mathscr{S}\colon v \geqq f\}\ ,$$

and by Doob's Theorem, [3, p. 203],

$$(3.16) \qquad \hat{R}_f = R_f \quad \text{q.e.}$$

(In fact, $\hat{R}_f = \text{fine}\liminf_{y \to .} R_f(y)$; see [19].) Consider now the case of a function $u \in \mathscr{S}$. Then

$$(3.17) \qquad \hat{R}_u^A = \bigwedge\ \{v \in \mathscr{S}\colon v = u \text{ on } A\}\ .$$

Clearly $\hat{R}_u^A \leqq u$. Furthermore, $\hat{R}_u^A = u$ q.e. on $A$. If $u = G\mu$, $\mu \in \mathscr{E}$, also $\hat{R}_u^A \in \mathscr{S} \cap W$ by (3.6) and (3.7). Hence, for some measure $\mu^A \in \mathscr{E}$,

$$(3.18) \qquad \hat{R}_{G\mu}^A = G\mu^A\ .$$

$\hat{R}_u^A$ is the outer) *balayage* of $u$ onto $A$, and $\mu^A$ the (outer) balayage of $\mu$ onto $A$.
One can prove that $\mu^A \in \mathscr{E}(\tilde{A})$ and that

$$(3.19) \qquad \mu^A = \mu^{\tilde{A}} = \mu^{b(A)}, \quad \mu \in \mathscr{E}\ .$$

This follows e.g. from [3, Theorem I.11.4], and the Kellogg property.

## 4. Converse results on capacitary integrals.

4.1. The following theorem is the result referred to in section 2.2.4. As is easily seen, it is best possible.

THEOREM. *Let $u \in \mathscr{S}$ and assume that*

$$(4.1) \qquad \int_0^\infty \operatorname{cap}(u > t)\, dt^2 < \infty\ .$$

*Then* $u \in W$ *and*

$$\|u\|^2 \leqq \text{const.} \int_0^\infty \operatorname{cap}(u>t)\,dt^2 . \tag{4.2}$$

*If* $u = G\mu$, $\mu \in \mathcal{M}^+$, *and if for some finely closed set* $F$

$$\int_0^\infty \operatorname{cap}\{(u>t)\cap F\}\,dt^2 < \infty , \tag{4.3}$$

*then* $u^F = \hat{R}_u^F \in W$ *and*

$$\|u^F\|^2 \leqq \text{const.} \int_0^\infty \operatorname{cap}\{(u>t)\cap F\}\,dt^2 . \tag{4.4}$$

PROOF. Since the topology of $X$ has a countable base, one can construct $G\mu$, $\mu \in \mathcal{E}$, such that $G\mu(x)>0$ for all $x \in X$. Then $u_n = u \wedge (nG\mu)$ is in $\mathcal{S} \cap W$ by (3.7). Hence, for some $\mu_n \in \mathcal{E}$, $u_n = G\mu_n$. Define

$$A_n^k = \{2^k < u_n \leqq 2^{k+1}\}, \quad k \in \mathbf{Z} , \tag{4.5}$$

and let $\lambda_k$ be the equilibrium measure for $A_n^k$. That is, $G\lambda_k = \hat{R}_1^{A_n^k}$ and

$$\operatorname{cap} A_n^k = I(\lambda_k) = \int_X d\lambda_k , \tag{4.6}$$

since $G\lambda_k = 1$ q.e. on $A_n^k$. (That this holds can be seen e.g. from Lemma 5.2 below.) Then

$$\begin{aligned} I(\mu_n) &\leqq 2\sum_k 2^k \int_{A_n^k} d\mu_n = 2\sum_k 2^k \int_{A_n^k} G\lambda_k\,d\mu_n \\ &\leqq 2\sum_k \left\{2^k I(\lambda_k)^{\frac12}\left(\int_{A_n^k} G\mu_n\,d\mu_n\right)^{\frac12}\right\} \\ &= 2\sum_k \left\{(2^{2k}\operatorname{cap} A_n^k)^{\frac12}\left(\int_{A_n^k} G\mu_n\,d\mu_n\right)^{\frac12}\right\} \\ &\leqq 2\left\{\sum_k 2^{2k}\operatorname{cap} A_n^k\right\}^{\frac12}\left\{\sum_k \int_{A_n^k} G\mu_n\,d\mu_n\right\}^{\frac12} \\ &= 2\left\{\sum_k 2^{2k}\operatorname{cap} A_n^k\right\}^{\frac12} I(\mu_n)^{\frac12} , \end{aligned}$$

so that

$$I(\mu_n) \leqq 4\sum_k 2^{2k}\operatorname{cap} A_n^k , \tag{4.7}$$

which follows after dividing by $I(\mu_n)^{\frac{1}{2}}$ (which is finite) and squaring both sides. Now

$$\sum_k 2^{2k} \operatorname{cap} A_n^k \leqq \sum_k 2^{2k} \operatorname{cap}(u_n > 2^k)$$

$$\leqq \sum_k 2^{2k} \operatorname{cap}(u > 2^k) \leqq \text{const.} \int_0^\infty \operatorname{cap}(u > t)\, dt^2 ,$$

so, since $\|u_n\|^2 = I(\mu_n)$, we get

$$\forall n : \ \|u_n\|^2 \leqq \text{const.} \int_0^\infty \operatorname{cap}(u > t)\, dt^2 .$$

Thus, the assumption (4.1) shows that $\|u_n\|^2$ is uniformly bounded. Since $u_n \uparrow u$, it follows, using (1.3) and (1.5), that

$$E_t(u) \leqq \text{const.} \int_0^\infty \operatorname{cap}(u > s)\, ds^2, \qquad t > 0 ,$$

by monotone convergence with respect to $n$. From monotone convergence again, but this time with respect to a sequence $t_n \downarrow 0$, we get

$$\|u\|^2 \leqq \text{const.} \int_0^\infty \operatorname{cap}(u > s)\, ds^2 ,$$

as $n \to \infty$, and this is (4.2).

To obtain (4.4) from (4.3), let $(\lambda_n)$ be a sequence in $\mathscr{E}$ with $G\lambda_n \uparrow G\mu$. Then

$$\int_X G\lambda_n\, d\nu \uparrow \int_X G\mu\, d\nu \qquad \text{for all } \nu \in E .$$

In particular $\int_X G\lambda_n\, d\nu^F \uparrow \int_X G\mu\, d\nu^F$, for all $\nu \in \mathscr{E}$, which gives

$$\text{(4.8)} \qquad \int_X G\lambda_n^F\, d\nu \uparrow \int_X G\mu^F\, d\nu, \qquad \forall \nu \in \mathscr{E} .$$

Now $\int_{A_n^k} d\lambda_n^F = \int_{A_n^k \cap F} d\lambda_n^F$, and the calculation preceding (4.7) carries over with the result

$$\text{(4.9)} \qquad I(\lambda_n) \leqq 4 \sum_k 2^{2k} \operatorname{cap}(A_n^k \cap F) ,$$

so, writing $u_n = G\lambda_n^F$, we get

$$\sup_n \|u_n\| \leqq \text{const.}$$

Thus $\int_X u^F\, d\nu < +\infty$ for any measure $\nu$ of finite energy. It follows from uniform integrability that $u_n \to u^F$ in $L^1(\nu)$ for any $\nu \in \mathscr{E}$, hence also for any signed measure with components in $\mathscr{E}$. Recalling that $\{Gv: v \in \mathscr{E} - \mathscr{E}\}$ is dense in $W$, and using that $(Gv \mid v) = \int_X v \, dv$ $(v \in W)$ for such measures $v$, we see that $u_n$ tends to $u^F$ weakly in $W$ as $n$ approaches infinity. By the Banach-Saks Theorem (see e.g. Rudin [22, Theorem 3.13]), there is a sequence of convex linear combinations of the $u_n$'s which is strongly convergent in $W$. A subsequence of this one gives us a new sequence which converges also q.e. Clearly the limit function has to be equal to $u^F$ q.e. Hence $u^F \in W$, and (4.4) should be clear from (4.9).

4.2. REMARKS.

1. The second part of Theorem 4.1 is an extension theorem in line with the results of [17]. (See also [9, Theorem 1].) This is so because $u^F = u$ q.e. on $F$ so $u^F$ may be viewed as an extension of $u \mid F$ from $F$ to the entire space $X$.

2. An alternative to the second part of the theorem is as follows (suggested by P. Sjögren). If $u = G\mu$, $\mu \in \mathscr{M}^+(F)$ and if (4.3) holds, then (4.4) holds with $u^F$ replaced by $u$ – provided that $\mu$ charges no polar set.

In this case one can choose $\lambda_n = \mu \mid \{G\mu \leqq n\}$. Then $\lambda_n \uparrow \mu$ because $\mu(\{G\mu = \infty\}) = 0$.

3. The proof of Theorem 4.1 is essentially the same as the proof of the first part of Theorem 2 in [17].

4. We return to the condition $\mu \leqq \mathrm{cap}$, from section 2.2.3. If this condition is satisfied for a given positive measure $\mu$, and if also $\mu(X) < \infty$, then $\mu$ is of finite energy. This can be seen as follows. Let $f \in W^+$ and assume $\|f\| = 1$. Then

$$\int_X f \, d\mu = \int_{\{f \leqq 1\}} f \, d\mu + \int_{\{f > 1\}} f \, d\mu \leqq \mu(X) + \int_X f^2 \, d\mu$$

$$= \mu(X) + \int_0^\infty \mu(f > t) \, dt^2 \leqq \mu(X) + \int_0^\infty \mathrm{cap}\,(f > t) \, dt^2$$

$$\leqq \mu(X) + \mathrm{const.} \, \|f\|^2 = \mathrm{const.}\,,$$

so $\mu \in W'$, and consequently $\mu \in \mathscr{E}$.

4.3. In [17] we mentioned (Remark 1, Section IV) that this technique can be used also in non-linear potential theory. (The phrase about extension theorems is, however, not entirely correct, which we take the opportunity to adjust now.) To illustrate this, let $K$ be a symmetric and positive kernel on $\mathrm{R}^d \times \mathrm{R}^d$, and define

$$K\mu(x) = \int K(x, y) \, d\mu(y), \qquad \mu \in \mathscr{M}^+(\mathrm{R}^d)\,.$$

We do not specify what $K$ must satisfy, but one could think of a Riesz kernel, $K(x,y)=|x-y|^{\alpha-d}$, $0<\alpha<d$. The *non-linear potential* of $\mu$ is defined by

$$V\mu = V_{K,p}\mu \equiv K\{(K\mu)^{p'-1}\}, \qquad \mu \in \mathcal{M}^+(\mathsf{R}^d),$$

where $1<p<\infty$ and $1/p+1/p'=1$. When $p=2$ and $K$ is the kernel $H$ mentioned in section 3.7, we get the Green potential $G\mu$.

The *energy* is defined as

$$I(\mu) = I_{K,p}(\mu) \equiv \int V\mu \, d\mu .$$

By Fubini's theorem, $I(\mu)=\int (K\mu)^{p'}dx$; hence, writing $f=(K\mu)^{p'-1}$, we have $f\in L^p$, so $V\mu=Kf$ is an $L^p$-potential (if $I(\mu)<\infty$).

Capacitary estimates for $L^p$-potentials have been obtained by Adams, Dahlberg, Hansson and Maz'ja. We refer to Adams [1] and Hansson [15] for results analogous to our Theorem 2.1. For the theory of capacities, we refer to [21] (in the case of Riesz potentials). Let $\mu\in\mathcal{M}^+(\mathsf{R}^d)$. Define

$$A_n = \{V\mu \in [2^n, 2^{n+1})\}, \qquad n\in \mathsf{Z},$$

and let $V\lambda_n$ be the corresponding capacitary potentials. Then, as in the above Theorem 4.1, with $C$ denoting a generic constant, with $c_{K,p}$ denoting $(K,p)$-capacity, and with $\mu_n=\mu\,|\,A_n$,

$$\begin{aligned}\int V\mu\,d\mu &\leqq C\sum_n 2^n\int V\lambda_n\,d\mu_n = C\sum_n 2^n\int K\mu_n\cdot(K\lambda_n)^{p'-1}dx\\ &\leqq C\sum_n 2^n\Big(\int (K\mu_n)^{p'}dx\Big)^{1/p'}\Big(\int (K\lambda_n)^{p(p'-1)}dx\Big)^{1/p}\\ &= C\sum_n\Big(\int (K\mu_n)^{p'}\Big)^{1/p'}(2^{np}c_{K,p}(A_n))^{1/p}\\ &\leqq C\Big\{\sum_n\int (K\mu_n)^{p'}dx\Big\}^{1/p'}\Big\{\sum_n 2^{np}c_{K,p}(A_n)\Big\}^{1/p}\\ &\leqq C\Big\{\sum_n I(\mu_n)\Big\}^{1/p'}\Big\{\int_0^\infty c_{K,p}(V\mu>t)\,dt^p\Big\}^{1/p}.\end{aligned}$$

Again,

$$\sum_n I(\mu_n) \leqq \sum_n\int_{A_n} V\mu\,d\mu = \int V\mu\,d\mu = I(\mu),$$

so

(4.10) $$I(\mu) \leqq C \int_0^\infty c_{K,p}(V\mu > t)\, dt^p \,.$$

It follows from (4.10) that if $\mu \in \mathscr{M}^+$, $\int_0^\infty c_{K,p}(V\mu > t)\, dt^p < \infty$, and if for some sequence $(\mu_n)$ with $I(\mu_n) < \infty$, $V\mu_n \uparrow V\mu$, then $I(\mu) < \infty$.

4.4. Let us, finally, see how the results from [17], mentioned in the introduction, follows from Theorems 2.1 and 4.1. In this case $W$ is the space of BLD-functions in $\mathbf{R}^d$, $d \geqq 3$, described in section 1.5, and $H(E)$ is the class of functions in $W$ which are (finely) harmonic (q.e.) in the (finely) open set $E$.

From [17, Theorem 1], we know that the map $f \to f(x)$, $f \in H(E)$, is bounded if, and only if, the harmonic measure for $E$ at $x$, $\delta_x^{\complement E}$, is of finite energy, or, equivalently, $G\delta_x^{\complement E} \in W$. The condition in [17, Theorem 2], for this to happen was

(4.11) $$\sum_n \frac{\operatorname{cap}(A_n(x) \setminus E)}{\operatorname{cap}(A_n(x))^2} < \infty$$

where $A_n(x)$ is the annulus $\{y\colon 2^{-n-1} < |x-y| \leqq 2^{-n}\}$. The $n$th term in (4.11) is comparable to $2^{2n(d-2)} \operatorname{cap}(A_n(x) \setminus E)$. Since

$$G\delta_x^{\complement E}(y) = G(x,y) = |x-y|^{2-d} \quad \text{for q.e. } y \in \complement E \,,$$

we see that $A_n(x) \setminus E$ is contained in the set where

$$2^{n(d-2)} < G\delta_x^{\complement E} \leqq 2^{(n+1)(d-2)} \,.$$

The capacity of the latter set is certainly less than that of the set $\{G\delta_x^{\complement E} > 2^{n(d-2)}\}$, so the series in (4.11) is dominated by the series

$$\sum_n 2^{2n(d-2)} \operatorname{cap}(\{G\delta_x^{\complement E} > 2^{n(d-2)}\})$$

which is finite if $G\delta_x^{\complement E} \in W$, according to Theorem 2.1.

Conversely, if (4.11) holds, then an argument similar to that used in Theorem 2.1 shows that

(4.12) $$\sum_n 2^{2n(d-2)} \operatorname{cap}(\{G\delta_x^{\complement E} > 2^{n(d-2)}\} \setminus E) < \infty \,.$$

The series in (4.12) is comparable to the capacitary integral $\int_0^\infty \operatorname{cap}(\{G\delta_x^{\complement E} > t\} \setminus E)\, dt^2$, so $G\delta_x^{\complement E} \in W$ according to the second part of Theorem 4.1.

## 5. The obstacle problem in $W$.

We will treat the obstacle problem, well known (in the case of the Sobolev space $W_0^{1,2}$) e.g. from the book [16] by Kinderlehrer and Stampacchia.

Inspired by D. R. Adams [2], our aim is to connect the obstacle problem to capacitary integrals.

5.1. Define, for $f\colon X \to [-\infty, +\infty]$,

$$K_f = \{v \in W:\ v \geqq f \text{ q.e.}\}\ , \tag{5.1}$$

and let $\theta$ be a given element from the dual $W'$ of $W$. The *obstacle problem* in $W$ is to find $u \in K_f$ such that

$$(u \mid v-u) \geqq \langle \theta, v-u \rangle, \qquad v \in K_f\ , \tag{5.2}$$

where $\langle .\,, . \rangle$ is the duality bracket of $W$ and $W'$, and where $f$ is the *obstacle*.— It is easily seen that this may be reduced to a homogenous problem after a change of obstacle. We will therefore concentrate our efforts to this case; that is: find $u \in K_f$ such that

$$(u \mid v-u) \geqq 0, \qquad v \in K_f\ . \tag{5.3}$$

We can assume that $f \geqq 0$.

5.2. Consider the set $K_f$ and assume $K_f \neq \emptyset$. It is closed, because if $(v_n) \subset K_f$ and $v = \lim_n v_n \in W$, then $v \geqq \inf_n v_n \geqq f$ q.e. Clearly $K_f$ is convex. A standard argument now provides us with a unique function $u \in K_f$ with minimal norm in this class. It is easily seen that this is the solution to (5.3), whenever such a thing exists.

Now, consider this function $u$. By variation of $\|u + t\varphi\|^2$, $t \geqq 0$, $\varphi$ arbitrary in $W^+$, one sees that $(u \mid \varphi) \geqq 0$, so by (3.6), $u \in \mathscr{S}$. By definition of $\hat{R}_f$, $u \geqq \hat{R}_f$, so (3.7) gives $\|\hat{R}_f\| \leqq \|u\|$.

By uniqueness, $u = \hat{R}_f$.

We have proved

LEMMA. *If $K_f \neq \emptyset$, the solution to* (5.3) *is the function $\hat{R}_f$, and it has minimal norm over the class $K_f$.*

5.3. Lemma 5.2 motivates a closer look at the map

$$[0, +\infty]^X \ni f \to \hat{R}_f \in \mathscr{S}\ . \tag{5.4}$$

(We note in passing that $\hat{R}_f$ may be finite q.e. although $K_f = \emptyset$. The obstacle $|x|^{-1}$ in the Newtonian case of $\mathsf{R}^3$ gives an example.) In [11], Fuglede studies capacities defined for functions rather than sets, and we will also use this approach. As is pointed out in [12], one need not assume that the capacities take their values in $[0, +\infty]$; in fact, a lattice such as $\mathscr{S}$ may replace $[0, +\infty]$.

(Consider the "ordinary" capacity $f \to \int_X \hat{R}_f \, d\mu$, and let $\mu$ vary over $\mathscr{E}$.) In particular, $\hat{R}_.$, defined by (5.4), is an *upper capacity* for which [11] is applicable. That this is an upper capacity means, in complete analogy with the notion "outer capacity", that

$$(5.5) \qquad \hat{R}_f = \bigwedge \{\hat{R}_v \colon v \text{ l.s.c.}, \ v \geqq f\} .$$

To see that (5.5) holds, let $\mathscr{F}$ be the class of all l.s.c. functions $\geqq 0$. Then

$$\hat{R}_f = \bigwedge \{v \in \mathscr{S} : v \geqq f\} = \bigwedge \{\hat{R}_v : v \in \mathscr{S}, \ v \geqq f\} \geqq \bigwedge \{\hat{R}_v : v \in \mathscr{F}, \ v \geqq f\} ,$$

because by assumption (A10), $\mathscr{S} \subset \mathscr{F}$. The opposite inequality is obvious, so (5.5) follows.

5.4. We define $\mathscr{H}_0$ as the class of functions $X \to [0, +\infty]$ which are u.s.c., finite and compactly supported. A function $f\colon X \to [0, +\infty]$ is *$\mathscr{H}_0$-capacitable* (with respect to $\hat{R}_.$) *if*

$$(5.6) \qquad \hat{R}_f = \sup \{\hat{R}_h : h \in \mathscr{H}_0, \ h \leqq f\} .$$

We say that a function is an *$\mathscr{H}_0$-Suslin function* if it can be obtained from the class $\mathscr{H}_0$ using Suslin's operation (A); see e.g. Choquet [6]. (Of course $\cap$ and $\cup$ should be replaced by $\wedge$ and $\vee$.) It is easy to show that

$$(5.7) \qquad f_n \uparrow f \Rightarrow R_{f_n} \uparrow \hat{R}_f .$$

From [11, Theorem 3.6 (a)] one also has (clearly the capacity cap considered here is locally finite)

$$(5.8) \qquad [h_n \in \mathscr{H}_0, \ h_n \downarrow h] \Rightarrow [\hat{R}_{h_n} \downarrow \hat{R}_h] .$$

(In fact the proof of the invoked result carries over mutatis mutandis to the situation considered here.) Thus, Choquet's theorem is valid: *All $\mathscr{H}_0$-Suslin functions are capacitable.*

5.5. Suppose $f \in \mathscr{H}_0$. Then $K_f \neq \emptyset$, so $\hat{R}_f = G\mu$ for some $\mu \in \mathscr{E}$ (by Lemma 5.2). Our first objective is to characterize the carrier of $\mu$. Write $u = \hat{R}_f$ and $E = \{u > f\}$. Then $E$ is open. Choose $\varphi \in C_0^+ \cap W$ supported in $E$. Then, if $t > 0$ is small enough, $u - t\varphi > f$, that is $u - t\varphi \in K_f$. Hence

$$\|u - t\varphi\|^2 \geqq \|u\|^2 ,$$

from which it follows that $(u \mid \varphi) \leqq 0$. On the other hand, $(u \mid \varphi) = \int_X \varphi \, d\mu \geqq 0$, so $(u \mid \varphi) = 0$ holds. It follows, writing $\varphi = \varphi^+ - \varphi^-$, that this is true for all $\varphi \in C_0 \cap W$, supported in $E$, hence also that $\operatorname{supp} \mu \subset X \setminus E$. Since $\hat{R}_f = R_f$ q.e., $\{u < f\}$ is polar, hence a null-set for $\mu$. Thus $\mu$ is carried by the set $\{u = f\}$.

We have proved

PROPOSITION. *If* $f \in \mathscr{H}_0$, *then* $\hat{R}_f = G\mu$, $\mu \in \mathscr{E}$, *where* $\mu$ *is carried by the set* $\{\hat{R}_f = f\}$.

5.6. We can now prove a variant of Adams' result hinted at above.

THEOREM. *For any* $\mathscr{H}_0$*-capacitable function* $f\colon X \to [0, +\infty]$, *the obstacle problem* (5.3) *has a solution if and only if* $\int_0^\infty \operatorname{cap}(f>t)\,dt^2 < \infty$. *In particular, this holds for any* $\mathscr{H}_0$*-Suslin, hence for any Borel function.*

We can state this result in a more suggestive way. For $f$ as above,

$$\hat{R}_f \in W \Leftrightarrow f \in L^2(X, \operatorname{cap}) . \tag{5.9}$$

PROOF. Assume that (5.3) has a solution, i.e. that $\hat{R}_f \in W$. Since $\hat{R}_f \geqq f$ q.e., Theorem 2.1 yields

$$\int_0^\infty \operatorname{cap}(f>t)\,dt^2 \leqq \int_0^\infty \operatorname{cap}(\hat{R}_f>t)\,dt^2 \leqq \text{const.}\, \|\hat{R}_f\|^2 < \infty .$$

For the converse, let $f_n \in \mathscr{H}_0$, $n=1, 2, \ldots$ be an increasing sequence with $\hat{R}_{f_n} \uparrow \hat{R}_f = u$. Put $u_n = \hat{R}_{f_n} = G\mu_n$, $\mu_n \in \mathscr{E}$. By Proposition 5.5, $\mu_n$ is carried by the set $F_n = \{u_n = f_n\}$, so that $u_n = G\mu_n^{F_n}$. Hence

$$\begin{aligned} \|u_n\|^2 &\leqq \text{const.} \int_0^\infty \operatorname{cap}(\{u_n>t\} \cap F_n)\,dt^2 \\ &= \text{const.} \int_0^\infty \operatorname{cap}(\{f_n>t\} \cap F_n)\,dt^2 \\ &\leqq \text{const.} \int_0^\infty \operatorname{cap}(f_n>t)\,dt^2 \\ &\leqq \text{const.} \int_0^\infty \operatorname{cap}(f>t)\,dt^2 , \end{aligned}$$

where the first inequality comes from use of Theorem 4.1.. Thus $(u_n)_{n \geqq 1}$ is uniformly bounded in $W$. From monotone convergence it follows that

$$E_t(u) = \lim_n E_t(u_n) \leqq \text{const.} \quad \text{for all } t>0 ,$$

since $u_n, u \in \mathscr{S}$ and then $(1-p_t)u_n \cdot u_n \uparrow (1-p_t)u \cdot u$, as $n$ approaches infinity. That $u \in W$ follows upon letting $t \downarrow 0$.

REMARKS.

1. Any quasi l.s.c. function is $\mathscr{H}_0$-capacitable, and the same goes for quasi

u.s.c. functions, provided they vanish sufficiently rapidly at infinity. See [11, Theorem 2.5, Lemma 4.6]. Thus the Choquet property (section 3.10) implies e.g. that all finely l.s.c. functions are $\mathscr{H}_0$-capacitable — in fact, finely l.s.c. q.e. suffices — so the theorem is valid for this class of functions.

2. One could also introduce the capacity of $f$ via $\operatorname{cap} f = \inf\{\|u\|, u \in K_f\}$. Clearly $K_f$ is non-void if and only if $\operatorname{cap} f < \infty$. This is the approach of Adams [1, 2]. As was pointed out in [11], the functional $f \to \operatorname{cap} f$ gives rise to a Banach space $\mathscr{L}(X, \text{cap})$, say, defining $\operatorname{cap} f = \operatorname{cap}(|f|)$ if $f$ is extended real-valued, and taking the quotient modulo those functions which vanish q.e. The above Theorem 5.6 actually shows that $\mathscr{L}(X, \text{cap})$ and $L^2(X, \text{cap})$ are equivalent Banach spaces, cf. [2, Theorem 1].

That $\hat{R}_f \in W \Leftrightarrow f \in \mathscr{L}(X, \text{cap})$ in the case of Newton potentials was proved in [11, section 6.7].

3. The estimates for the measure $\mu$ given in [2, Theorem 5 and Remark 4] (see also [16, Theorem 6.11]) carry over. In fact, some of the proofs are implicit in section 4.

4. In [16, p. 40 ff.], the obstacle problem is treated in the Sobolev space $W_0^{1,2}(\Omega)$, where $\Omega \subset \mathbf{R}^N$ is open and bounded. (The inner product is $(u \mid v) = \int_\Omega \nabla u \cdot \nabla v \, dx$.) There is one difference compared to our setup, though. In (5.2) and (5.3), the term $(u \mid v-u)$ is replaced by $a(u, v-u)$, where $a$ is a bilinear form on $W_0^{1,2}(\Omega)$, not necessarily symmetric.

One also assumes that $a$, or rather the differential operator associated to $a$, is *uniformly elliptic*. This may be expressed by the condition that $a(u,u)$ and $\|u\|^2$ be comparable, independently of $u \in W_0^{1,2}(\Omega)$. In case $a$ is symmetric, $W_0^{1,2}(\Omega)$, equipped with $a(.,.)$ as inner product, is a Dirichlet space in the sense of this article. See [14, p. 43] and also Dynkin [7, Chapter V].

We remark that under the uniform ellipticity-type condition, our theorem carries over to the non-symmetric case as well by use of a well-known perturbation argument. See [16, Lemma 2.2, p. 26].

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