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TRANSITION OPERATOR CHARACTERIZATIONS OF COMPACT AND MAXIMALLY ALMOST PERIODIC LOCALLY COMPACT GROUPS

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In Memory of Karel de Leeuw

In [3] arguments of Rota [7] and Schaefer [8] were combined with results on almost periodic semigroups of operators [1] to yield information on the point spectrum of unit modulus of a transition operator on $C(X)$. Here we note further consequences of the same line of argument, emphasizing more the eigenvectors, which easily yields characterizations of those families of transition operators that arise from the action of a compact transformation group. In particular, we characterize the indicated groups by the behavior of (what turns out to be) a generating set of translation operators, and also rotations, as transition operators on spheres.

We will consider the compact and locally compact settings in sequence in sections 1 and 2. Although the results were originally obtained by applying those of [1], the applications were all to the abelian case and have been abandoned in favor of more elementary arguments. Nevertheless, part of our original argument yields an apparently new fact about almost periodic semigroups (that, in the terms of [1], if the unitary subspaces span, the almost periodic compactification is a group of invertibles); this is included in a final section.

1. The Compact Case.

Here X will always be a compact Hausdorff space, $C(X)$ the usual space of continuous complex functions on X, and δ_x the unit point mass at x. $\mathfrak{T}$ will denote a set of *transition operators* on $C(X)$, i.e. non-negative operators fixing the constants. We shall call an eigenvector corresponding to an eigenvalue of modulus 1 a T^1-*eigenvector*, and shall consider principally the following properties of $\mathfrak{T}$ (and their variants):

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(a) *For each* $T \in \mathfrak{T}$ *the probability measures fixed by* T^* *have the union of their supports dense in* X.

(b) *There is a set of common* T^1*-eigenvectors of the elements of* $\mathfrak{T}$ *which separate the points of* X.

(c) $\mathfrak{T}$ *fixes only the constants.*

For example we characterize an element of the general orthogonal group on R^{n+1} as a transition operator on $C(S^n)$ satisfying (a) which leaves invariant a separating finite dimensional rotation invariant subspace while it is non-contracting there (cf. § 1.3, Theorem 3). As the reader will note, (a) is unnecessary in section 1, if we assume each $T \in \mathfrak{T}$ arises from a continuous self map φ of X; far less is needed in the result just cited (cf. Corollary 5).

Our Abelian result is the following:

THEOREM 1. *Each* $T \in \mathfrak{T}$ *arises from the action of an element of a fixed compact Abelian transformation group if and only if* (a) *and* (b) *hold. Moreover,* X *is itself a compact Abelian group* (*and* $\mathfrak{T}$ *a generating set of its translation operators*) *if* (*and only if*) (c) *also holds.*

Half of the result is standard: if G is a compact Abelian group acting as a transformation group on X (so $(g,x) \to g(x)$ is continuous) then the transition operators $f \to f \circ g$ on $C(X)$ satisfy (a), since each orbit carries an image of the Haar measure of G, fixed under our adjoint operators, while (b) follows from the fact that $C(G)$ is spanned by characters γ, so it contains an approximate identity of trigonometric polynomials. (Thus from continuity of $g \to f \circ g$ one has

$$\left\| f - \int \sum_1^n a_j \langle g, \gamma_j \rangle f \circ g \, dg \right\| < \varepsilon ,$$

while $\int \langle g, \gamma_j \rangle f \circ g \, dg$ is a common T^1-eigenvector.)

For the other direction we begin (as in [3]) with a simple observation on the common T^1-eigenvectors. Note that since $T \geqq 0$ and $T1 = 1$ for each $T \in \mathfrak{T}$ and $x \in X$ we have a probability measure m_x for which $Tf(x) = \int f \, dm_x$, $f \in C(X)$. Thus if $Th = \alpha h$, $\alpha \in \mathsf{C}$, with $|\alpha| = 1$, then

$$(1) \qquad |h(x)| = |Th(x)| = \left| \int h \, dm_x \right| \leqq \int |h| \, dm_x = T|h|(x)$$

so that $T|h| - |h| \geqq 0$. On the other hand, since each probability measure μ fixed by T^* has $\mu(T|h| - |h|) = 0$, while by (a) their supports are dense in X, we conclude that $T|h| = |h|$. From equality in the inequality (1) we now have h

constant on the support of m_x. Thus (b) clearly implies each m_x is a point mass $\delta_{\varphi(x)}$. And because $Tf(x)=f(\varphi(x))$ is continuous for each $f \in C(X)$, φ is a continuous map of X into itself.

Let $\{h_\alpha\}_{\alpha\in A}$ be our separating set of common $\boldsymbol{T}^1$-eigenvectors. Identifying X with its homeomorphic image in the product $\mathbf{C}^A$ under $x \to \{h_\alpha(x)\}_{\alpha\in A}$, the fact that

$$Th_\alpha(x) = h_\alpha(\varphi(x)) = \lambda_\alpha h_\alpha(x)$$

says that φ corresponds to a "rotation" of $\mathbf{C}^A$ on our image given by the element $\{\lambda_\alpha\}_{\alpha\in A}$ of the compact Abelian group $\boldsymbol{T}^A$. Thus the various elements T of $\mathfrak{T}$ correspond to those of a subset of $\boldsymbol{T}^A$, a subset which evidently generates our desired transformation group K in $\boldsymbol{T}^A$. The main assertion of Theorem 1 is now clear.

For the final assertion, note that (c) directly implies the action is minimal, so X consists of a unique orbit: fixing any $x_0 \in X$, $k \to k(x_0)$ maps K onto X. Because K is Abelian the isotropy subgroup $H_{x_0}=\{k \in K\colon k(x_0)=x_0\}$ is the same for any x_0, and thus fixes every $x \in X$, so consists only of the identity. Thus $k \to k(x_0)$ is a homeomorphism of K onto X, and our assertion is clear.

(Exactly as in [3] we could of course describe the point spectrum (or joint spectra) of unit modulus of elements of $\mathfrak{T}$.)

1.2. Our non-Abelian version merely replaces common $\boldsymbol{T}^1$-eigenvectors with the unitary subspaces of [1]: *a finite dimensional T-invariant subspace D of C(X) is a unitary subspace if $T \mid D$ is contained in a bounded* (*hence in a compact*) *group of operators on D whose identity is the identity operator on D.* (As is well known the group can be taken unitary, relative to some inner product on D [11, p. 70].) Thus we shall consider

(b′) *There is a set of unitary subspaces for $\mathfrak{T}$ which separate the points of X.*

THEOREM 2. *Each $T \in \mathfrak{T}$ arises from the action of an element of a fixed compact transformation group on X iff* (a) *and* (b′) *hold. Moreover, X is a homogeneous space of a compact group* (*and $\mathfrak{T}$ a generating set of left translations of the group acting on X*) *if* (*and only if*) (c) *also holds.*

To begin the proof, note that for each $T \in \mathfrak{T}$, each unitary subspace D is spanned by a set of $\boldsymbol{T}^1$-eigenvectors of T, since $T \mid D$ lies in a compact Abelian group on which there is an approximate identity of trigonometric polynomials. Thus the $\boldsymbol{T}^1$-eigenvectors of T separate the points of X by (b′). So we again have $Tf=f\circ\varphi$ all $f \in C(X)$, for some self map φ of X since each representing measure m_x is necessarily a point mass as earlier. Again φ is continuous.

Let $\{D_\alpha\}_{\alpha \in A}$ be a set of separating unitary subspaces for $\mathfrak{T}$. For each $T \in \mathfrak{T}$ and our corresponding map φ, $T \mid D_\alpha$: $f \to f \circ \varphi$ is given by a unitary matrix $M_\alpha = (m_{ij})$, if we select and fix an orthonormal basis $\{f_1^\alpha, \ldots, f_{n_\alpha}^\alpha\}$ for D_α relative to an appropriate inner product on D_α. Thus as before we can identify X with its image in the topological product $\boldsymbol{P}\mathbf{C}^{n_\alpha}$ under the map

$$x \to \{(f_1^\alpha(x), \ldots f_{n_\alpha}^\alpha(x))\}_{\alpha \in A} \, .$$

The fact that

$$Tf_i^\alpha = f_i^\alpha \circ \varphi = \sum_{j=1}^{n_\alpha} m_{ij}^\alpha f_j^\alpha$$

says that on the image φ corresponds to the natural action of an element of the (topological) product group $\boldsymbol{P}U(n_\alpha)$ under which the image is invariant. Our transformation group K is again the subgroup generated by the subset corresponding to $\mathfrak{T}$. Finally, given (c), we now only find X a homogeneous space K, completing our proof of Theorem 2.

Remark. For later use we should note that in both theorems the semigroup $\mathfrak{S}$ generated by $\mathfrak{T}$ has its image in K dense (since a closed subsemigroup of a compact group is a subgroup). Moreover, if A is the closed span of all the unitary spaces, K can easily be identified with a group of operators on A compact in the strong operator topology which is generated by $\mathfrak{S} \mid A$, and which thus coincides with the strong operator closure $(\mathfrak{S} \mid A)^-$ of $\mathfrak{S} \mid A$. Thus K is the almost periodic compactification of $\mathfrak{S} \mid A$ in the terminology of [1].

To characterize a compact group now from Theorem 2 only requires bringing in both left and right translations.

Corollary 1. *Suppose $\mathfrak{T} = \mathfrak{L} \cup \mathfrak{R}$ where $L \in \mathfrak{L}$ and $R \in \mathfrak{R}$ imply $LR = RL$. Then X is a compact group and $\mathfrak{L}$ (respectively $\mathfrak{R}$) is a generating set of its left (respectively right) translation operators iff* (a), (b′), *and* (c) *hold for each set $\mathfrak{L}$ and $\mathfrak{R}$.*

Here we obtain two compact groups $K_{\mathfrak{L}}$, $K_{\mathfrak{R}}$, while because of (c) X is a homogeneous space of each. But since any element of the first group commutes with any element of the second, if $h \in K_{\mathfrak{L}}$ fixes $x_0 \in X$, then $kx_0 = khx_0 = hkx_0$ for any $k \in K_{\mathfrak{R}}$. Since $K_{\mathfrak{R}} x_0 = X$ we conclude the isotropy subgroup in $K_{\mathfrak{L}}$ of any x_0 is trivial. Thus we can identify X as the underlying space of $K_{\mathfrak{L}}$, and $\mathfrak{L}$ as a generating family of left translations. Each element of $K_{\mathfrak{R}}$ then appears, as an operator, as right convolution with a probability measure m by a theorem of Wendel [12]; since it is now easy to identify the measure $m_x = k^* \delta_x$ of our proof for $k \in K_{\mathfrak{R}}$ as a translate of m, m is a point mass, so $\mathfrak{R}$ consists of a set of right translation operators.

Finally, in one special setting, Theorem 2 can be sharpened considerably.

COROLLARY 2. *Suppose X is also connected,* (a) *and* (b′) *hold, and some commuting subset $\mathfrak{T}_0$ of $\mathfrak{T}$ has the property* (c) *that only the constants are fixed under $\mathfrak{T}_0$. Then X is a compact Abelian group on which the elements of $\mathfrak{T}_0$ form a generating set of translation operators, while the elements of $\mathfrak{T}$ arise from translates of elements of an equicontinuous group of automorphisms of X (necessarily periodic when X is finite dimensional).*

Each of our unitary subspaces D for $\mathfrak{T}$ provided by (b′) is necessarily spanned by common T^1-eigenvectors for $\mathfrak{T}_0$, and thus (a), (b), and (c) hold for $\mathfrak{T}_0$, whence X is a compact Abelian group by Theorem 1. Now the compact group we obtain for $\mathfrak{T}$ from Theorem 2 can be taken as the group G of the proof of the main result of [4] (with X as the group H there), and precisely that proof yields our assertions about $\mathfrak{T}$. The corollary includes the following special case: if T is a transition operator on $C(G)$, for G a compact connected Abelian group, which (1) preserves Haar measure and (2) leaves invariant a set of finite dimensional translation invariant subspaces D of $C(G)$ which together separate G, then T arises from a translate of an automorphism. (Take $\mathfrak{T}_0$ the translations, $\mathfrak{T} = \mathfrak{T}_0 \cup \{T\}$.) But in fact if we start instead with a homeomorphism φ of G for which T: $f \to f \circ \varphi$ satisfies only (2) the same conclusion follows via the argument of [4] as soon as we recognize that because of (2) the $\mathfrak{T}$-invariant finite dimensional subspaces D generate $C(G)$ (as in the proof of Theorem 6 below).

1.3. We next note some variants of our conditions, and related results. First it is easy to see our single minimal orbit condition (c) can equally well be replaced in all our results by either

(c′) *Each probability measure fixed by all T^*, $T \in \mathfrak{T}$, has global support, or*

(c″) *For any non-zero $f \geqq 0$ in $C(X)$ there is a polynomial P in the elements of $\mathfrak{T}$ with positive coefficients for which Pf never vanishes.*

With $\mathfrak{S}$ the semigroup generated by $\mathfrak{T}$ one can replace (b′) by the seemingly weaker hypothesis.

(b″) *There is a separating set of invariant finite dimensional subspaces D for which $(\mathfrak{S} \mid D)^-$ contains exactly one idempotent, the identity I.*

Here $(\mathfrak{S} \mid D)^-$ can be taken as the strong operator closure or the closure of a corresponding semigroup of matrices since these coincide. Indeed the finite dimensionality of D implies $\mathfrak{S} \mid D$ is almost periodic, so the fact that I is the only idempotent in $(\mathfrak{S} \mid D)^-$ says its kernel is a group containing I, hence containing $\mathfrak{S} \mid D$, whence D is a unitary subspace [1, 4.11].

Alternatively in all the preceding, (b′) can be replaced by

(b‴) *There is a separating set of invariant finite dimensional subspaces D for which each $T \in \mathfrak{T}$ is non-contracting in the sense that $f \in D$, $0 \in \{T^n f\}^-$ imply $f=0$.*

Indeed this follows from a simple fact about operators of norm $\leqq 1$.

LEMMA 1. *Let $\mathfrak{T}$ be a subset of the unit ball of operators on a Banach space. Then a ($\mathfrak{T}$-) unitary subspace D is simply a finite dimensional $\mathfrak{T}$-invariant subspace on which each $T \in \mathfrak{T}$ is non-contracting.*

To see this, note that because D is finite dimensional the strong operator closure $\mathfrak{S} = \{T^n \mid D \colon n \geqq 1\}^-$ is a compact Abelian semigroup; thus it has a least ideal K which is a compact group, as is easily seen [10]. But the identity e of K must be the identity operator, since T is non-contracting on D: for $f \in D$ has $0 = e(f-ef) \in \{T^n(f-ef)\}^-$, so $f-ef=0$. Hence $\mathfrak{S} = e\mathfrak{S} \subset K \subset \mathfrak{S}$, and $\mathfrak{S} = K$ is a group of invertible operators. Since $\mathfrak{S}$ consists of operators of norm $\leqq 1$ it consists of isometries, so $T \mid D$ lies in the compact isometry of D, for each $T \in \mathfrak{T}$, whence D is $\mathfrak{T}$-unitary. Provided $\mathfrak{T}$ is a semigroup we can simply assume it is *bounded*: for then $\|k\| \leqq M$ for all $k \in K$ (for any T), so $T \mid D$ has an inverse of norm $\leqq M$ as above, and $(\mathfrak{T} \mid D)^-$ is a compact semigroup with inverses, so a group.

One way to obtain such invariant D is to assume $\mathfrak{T}$ commutes with a fixed compact operator. In analogy with our earlier usage, we call a semigroup $\mathfrak{S}$ of operators *non-contracting* if $0 \in (\mathfrak{S}f)^-$ implies $f=0$.

COROLLARY 3. *Suppose $\mathfrak{T} = \mathfrak{L} \cup \mathfrak{R}$, where $L \in \mathfrak{L}$ and $R \in \mathfrak{R}$ imply $LR = RL$, and each of $\mathfrak{L}$, $\mathfrak{R}$, satisfies* (a) *and* (c). *Suppose each T in $\mathfrak{T}$ is non-contracting while there is a compact operator $H_{\mathfrak{L}}$ (respectively $H_{\mathfrak{R}}$) commuting with each $T \in \mathfrak{L}$ (respectively $\mathfrak{R}$) whose generalized eigenspaces separate X.*

Then X is a compact metric group and $\mathfrak{L}$ (respectively $\mathfrak{R}$) is a generating set of its left (respectively right) translation operators.

(X is necessarily metric since we have a countable set separating X.)

Such compact operators H are obtained of course in the group setting as convolutions with appropriate central L^1 functions. Since H is compact, we know that for each of its non-zero eigenvalues λ we have a least integer n for which the finite dimensional null spaces D of $(\lambda I - H)^n$ and that of $(\lambda I - H)^{n+1}$ coincide; we are assuming X is separated by such generalized eigenspaces D. The point here of course is that such D are $\mathfrak{S}$ invariant, since H commutes with the elements of $\mathfrak{S}$, and now Lemma 1 implies these are unitary subspaces. Since their span separates the points of X in each case, we have (b′) in Corollary 1, so that result identifies X as our compact group.

In particular, we can apply the same sort of argument to specific subsets of $\mathbf{R}^n$. Let S^n be the n sphere in $\mathbf{R}^{n+1}$.

THEOREM 3. *Suppose T is a transition operator on $C(S^n)$ which satisfies* (a) *and leaves invariant a finite dimensional rotation invariant subspace D which separates S^n while T is non-contracting on this subspace. Then T arises from an element of the general orthogonal group on* $\mathbf{R}^{n+1}$.

For example, D could be the space of homogeneous polynomials of degree k $(\geqq 1)$. More generally, our proof applies equally well to show that *for a compact Riemannian globally symmetric space M of rank* 1 (so two point homogeneous [5, p. 355]) in place of S^n (and an isometry invariant subspace rather than rotation invariant) *T arises from an isometry.* As will be noted, the crucial fact is that the (isometry) orbit space is then an interval. One consequence for such M is that, *if I denotes its group of isometries, any I-invariant metric defining the topology of M yields the same isometry group.* Indeed, if I' is the (a priori larger) new isometry group, the compactness of I' shows we have a probability measure on M which is I'-invariant (the image of Haar measure) and that $C(M)$ is spanned by finite dimensional I'-invariant subspaces as usual. These are I-invariant of course, and finitely many suffice to separate M by a simple argument using compactness. We have only to note that those points not separated from $m_0 \in M$ by an invariant D form a compact submanifold invariant D form a compact submanifold M_D, so separation allows us to find a D_1 so that M_{D+D_1} has lower dimension. Thus we can find a D with M_D zero dimensional, hence finite, and so a D with $M_D = \{m_0\}$. Their span now provides our D, and for $\varphi \in I'$ if $Tf = f \circ \varphi$, our result shows T arises from an element of I, whence $\varphi \in I$.

For the proof of Theorem 3, note that $T|D$ non-contracting implies D is a unitary subspace for $\{T^j : j \geqq 1\}$ by Lemma 1. Thus D is spanned by T^1-eigenvectors of T exactly as in Theorem 2, and these separate S^n since D does. So just as there $Tf = f \circ \varphi$, $f \in D$, for a self map φ of S^n which lies in a compact transformation group acting on S^n, so φ is in fact a homeomorphism. Moreover, as we saw in the proof of Lemma 1, $T|D$ lies in the compact isometry group of the subspace D of $C(S^n)$.

Consequently, the subgroup of the group of isometries on D generated by homeomorphisms of S^n contains a compact subgroup K generated T and the operators arising from O_{n+1}. Let G be the corresponding group of homeomorphisms of S^n. Since K is compact in the uniform topology on operators on D the same is true of the isometric image group K^* of adjoint operators; identifying S^n with its homeomorphic image in D^*, $K^* \mid S^n$ clearly corresponds to our group of homeomorphisms G, and compactness of $K^* \mid S^n$ to the compactness of G in the topology of uniform convergence on S^n. Because of that compactness, G is a compact topological group, which acts as a transformation group on S^n.

Now as a homogeneous space of the compact homeomorphism group G, S^n carries a G-invariant metric ϱ defining its topology. But for $p_0 \in S^n$ the component of p_0 in the open ϱ ball about p_0 of (great circle) radius $r > 0$ has its boundary ∂ precisely the boundary of a Euclidean ball about p_0 in S^n: for if $p \in \partial$ then ∂ contains the orbit of p under the isotropy subgroup of p_0 in O_{n+1}, a replica of S^{n-1} and just the boundary of a Euclidean ball about p_0. Indeed because of this ∂ is just the union of such boundaries b Euclidean balls; since each b seprates S^n into exactly two components and all distinct b are pairwise disjoint we conclude ∂ consists of precisely one b, and our component is just an open Euclidean ball. Because our component is invariant under the isotropy subgroup G_0 of p_0 in G, we conclude that the elements of G_0 send open Euclidean balls $B_{R(r)}(p_0)$ of radius $R(r)$ centered at p_0 onto themselves. where R is a strictly increasing function possibly having jumps but $\downarrow 0$ as $r \downarrow 0$. Since we see G_0 also must send $\partial B_{R(r)}(p_0)$ onto itself and each element of G_0 sends $B_{R(r')}(p)$ onto $B_{R(r')}(g(p))$, for $p \in \partial B_{R(r)}(p_0)$ we have

$$B = B_{R(r)}(p_0) \cup \bigcup_{g \in G_0} B_{R(r')}(g(p))$$

mapped onto itself by each element of G_0.

Now B is just the open Euclidean ball of (great circle) radius $R(r) + R(r')$. Thus we similarly conclude that $g \in G_0$ sends any open ball centered at p_0, whose radius lies in the N-fold sum of the range of R, onto itself; indeed since an increasing union of such open balls is again mapped by $g \in G_0$ onto itself, we have the same conclusion for any radius which is the sum of a series ΣR_j, with each R_j in the range of R. Since we can find a sequence $R'_j \downarrow 0$ in the range of R, and any positive real is the sum of such a series (allowing repetitions), we conclude each $g \in G_0$ sends $B_r(p_0)$ onto itself for any r. Evidently then $g \in G$ sends $B_r(p)$ onto $B_r(g(p))$, and thus in particular our φ giving rise to T lies in O_{n+1}.

In case $D = P_k$, the space of polynomials of degree $\leq k$ (for $k \geq 1$), there is a simpler algebraic argument available to finish the proof after one obtains φ.

For then φ is a polynomial map and it suffices to see P_1 is invariant. Indeed for x_i the ith coordinate function, $x_i \circ \varphi$ is a polynomial of degree $\leqq k$ on S^n, as is $x_i^k \circ \varphi = (x_i \circ \varphi)^k$, and one can deduce that $x_i \circ \varphi$ coincides with a first degree polynomial on S^n from the fact that S^n is a hypersurface defined by a polynomial whose highest homogeneous part is irreducible, arguing with degrees. Of course almost all the argument is unnecessary if one can easily conclude P_1 is invariant; for example, this is the case if T is a transition operator on $C(S^n)$ satisfying (a) which commutes with the spherical Laplacean Δ_s and is non-contracting, for then the eigenspaces of Δ_s, the surface spherical harmonics, are T-invariant, and P_1 is one. Another instance is contained in

COROLLARY 4. *Suppose $X \subset \mathbf{R}^n$ is compact and not contained in the zero set of any non-zero complex polynomial. Further suppose T is a transition operator on $C(X)$ satisfying* (a) *which leaves P invariant and is non-contracting there. Then, modulo an affine change of coordinates, T arises from an orthogonal transformation.*

Here we obtain our homeomorphism φ exactly as in Theorem 3, and because T is consequently multiplicative while $p \to p \mid X$ is a $1-1$ map on P_k, we can easily argue in terms of degree to see P_1 is invariant (and thus T-unitary) since if $Tx_i = x_i \circ \varphi \in P_k$ has degree >1, $(x_i \circ \varphi)^k \in P_k$ has degree $>k$. Thus $x_i \circ \varphi = \sum_j c_{ij} x_j + d_i$ (with real coefficients). Since $T \mid P_1$ is (real and) unitary when P_1 is taken in the appropriate inner product, if we use the Gram-Schmidt process to obtain a real orthonormal basis $1, p_1, \ldots, p_n$ from $1, x_1, \ldots, x_n$ then, since $1 \perp p_i$, $\operatorname{span}_{\mathbf{R}} \{p_1, \ldots, p_n\}$ is invariant and $p_1, \ldots, p_n$ provide the desired affinely equivalent coordinates relative to which T is an orthogonal transformation.

Finally we note the simplification possible, when T is known to arise from a self map φ of S^n in Theorem 3. As a consequence of its proof we have

COROLLARY 5. *Suppose φ is a continuous map of S^n onto itself, and, for some finite dimensional rotation invariant subspace D of $C(S^n)$ other than the constants we have $D \circ \varphi \subset D$. Then φ coincides on S^n with an element of the general orthogonal group O_{n+1} on $\mathbf{R}^{n+1}$.*

Because φ is onto, $f \to f \circ \varphi$ is non-contracting on all of $C(S^n)$, and since D is non-trivial it separates the points of some non-trivial homogeneous space O_{n+1}/H of O_{n+1}. If this homogeneous space is S^n itself we are done by just the proof of Theorem 3; if not, H, which must contain the isotropy subgroup O_n of some north pole N of S^n, can only be generated by that subgroup and an element of O_{n+1} which interchanges north and south poles (as is well known and easily seen by considering the orbit of N under H), and O_{n+1}/H is $\mathbf{RP}^n$. Since $D \circ \varphi \subset D$, φ thus maps antipodal pairs to antipodal pairs, and the induced map φ on $O_{n+1}/H = \mathbf{RP}^n$ is now seen to arise from an element of O_{n+1} by exactly the proof of Theorem 3. Of course it is immediate that φ itself arises from the same element.

1.4. Returning to our conditions, we should note that as long as (a) holds, we can of course make an assertion without any separation condition, but only about the closed span B of the unitary subspaces; B is the pullback of $C(Y)$ for a quotient space Y of X, and our assertions apply to $T \mid B$. (Of course B may reduce to the constants.) As noted in [2, footnote 1] a transition operator T yields as well defined operator on $C(F_T)$, where F_T is the closure of the union of the supports of all fixed μ; thus if (a) fails we can make some assertion about any subset of $\mathfrak{T}$ for which all F_T coincide.

Next, we should probably note that (a) is precisely equivalent to the condition

(a′) For $T \in \mathfrak{T}$ and $f \in C(X)$, $0 \in \mathfrak{C}\{T^n|f|: n \geqq 1\}$ implies $f=0$.

Here $\mathfrak{C}$ is the closed convex hull. Indeed if (a) holds and 0 lies in the convex hull, we have

$$\sum_{n=1}^{N} \lambda_n T^n|f| < \varepsilon \quad \text{ for } \lambda_n \geqq 0$$

summing to 1, whence for a fixed μ we obtain $\mu(|f|) = \sum \lambda_n \mu(T^n|f|) < \varepsilon$, so (a) implies $f=0$ on X. Conversely assuming (a′), if $f \geqq 0$ vanishes on the set F_T, $f \in C(X)$, and ν is any probability measure, then since any w^* cluster point μ of $\{1/N \sum_{n=1}^N T^{*n}\nu\}$ is necessarily fixed by T^*, so $\mu(f)=0$, we have $\nu(1/N \sum_{n=1}^N T^n f) \to 0$, whence $1/N \sum_{n=1}^N T^n f \to 0$ weakly in $C(X)$. By Mazur's theorem then $0 \in \mathfrak{C}\{T^n f: n \geqq 1\}$, so $f=0$ by (a′). Thus $F_T = X$ as (a) asserts.

Alternatively, (a) and (a′) are equivalent to

(a″) For $T \in \mathfrak{T}$ and $f \in C(X)$, $\|1/N \sum_{n=1}^N T^n|f|\|_\infty \to 0$ implies $f=0$.

Indeed $0 \in \mathfrak{C}\{T^n|f|\}$ is equivalent to the condition in (a″) since $\varepsilon > \sum_{n=1}^k \lambda_n T^n|f|$ (as above) implies

$$\varepsilon > \frac{1}{N} \sum_{n=1}^{N} T^j\left(\sum_{n=1}^{k} \lambda_n T^n|f|\right) \geqq \frac{1}{N} \sum_{j=k+1}^{N-k} T^j|f| \geqq \frac{1}{N} \sum_{j=1}^{N} T^j|f| - \varepsilon$$

if $2k\|f\| < \varepsilon N$.

Thus we can interpret (a)–(a″) as asserting that for the Markov proces associated with any $T \in \mathfrak{T}$, for each non-void open U in X there is a starting point x from which we return to U with positive expected frequency.

1.5. Both (b) and (c) are assertions about entities with a common property for all T in $\mathfrak{T}$; when $\mathfrak{T}$ is a finite commuting set and (a) holds, (b) can be replaced by individual restrictions.

THEOREM 4. *Let $\mathfrak{T}$ be a finite commuting set of transition operators satisfying* (a) *and suppose that*

(b*) *for each $T \in \mathfrak{T}$ the subspace A_T of elements of $C(X)$ almost periodic under $\{T^n: n \geqq 1\}$ separates X, while T is non-contracting on A_T.*

Then T arises from the action of a compact Abelian transformation group on X, and if (c) *also holds X is a compact Abelian group.*

$A_T = \{f \in C(X): \{T^n f: n \geqq 1\}^-$ is compact$\}$ is of course closed and T invariant. Because the strongly closed orbit $\{T^n f: n \geqq 1\}^-$ of $f \in A_T$ is compact, the strong operator closure of $\{T^n | A_T: n \geqq 1\}$ is a compact jointly continuous Abelian semigroup, as we can see by identifying it with the closure of the image of $\{T^n | A: n \geqq 1\}$ in the topological product of orbits $\boldsymbol{P}_{f \in A_T} \{T^n f: n > 1\}^-$. Consequently, its least ideal is a compact group K_T, exactly as in Lemma 1, whose proof in fact shows the identity of K_T is the identity operator on A by non-contraction. Because of this A_T is spanned by $\boldsymbol{T}^1$-eigenvectors of T via the argument for the standard half of Theorem 1 above (with kf in place of $f \circ g$). So the $\boldsymbol{T}^1$-eigenvectors for T separate X.

In fact we now know T arises from a map φ of X as in Theorem 1, so the $\boldsymbol{T}^1$-eigenvectors for T form a multiplicative semigroup: $Tf_i = \lambda_i f_i$, $i = 1, 2$ implies

$$Tf_1 f_2 = (f_1 f_2) \circ \varphi = (f_1 \circ \varphi)(f_2 \circ \varphi) = \lambda_1 \lambda_2 f_1 f_2 .$$

Since $T\bar{f} = \overline{Tf}$ we conclude the closed span of the $\boldsymbol{T}^1$-eigenvectors of T is a self-adjoint algebra separating X, which must therefore be $C(X)$ by Stone-Weierstrass.

Hence $A_T = C(X)$ for each $T \in \mathfrak{T}$, and, since the identity operator of K_T is now the identity operator on $C(X)$ and $k \to kf$ is strongly continuous for $f \in C(X)$, we can in fact approximate f by $\boldsymbol{T}^1$-eigenvectors using an approximate identity of trigonometric polynomials on K_T: for $\varepsilon > 0$ and certain $a_j \in \mathbf{C}$, $\hat{k}_j \in \hat{K}_T$ we have

$$\left\| f - \sum_1^m a_j \int_{K_T} \langle k, \hat{k}_j \rangle k f \, dk \right\| < \varepsilon;$$

thus those non-zero elements of the form $\int_{K_T}\langle k,\hat{k}\rangle kf\,dk$ yield $\boldsymbol{T}^1$-eigenvectors for T which span $C(X)=A_T$. We note this only because we can similarly assert that for $T_1, T_2 \in \mathfrak{T}$, non-zero elements of the form

$$\int_{K_{T_1}} \langle k_1,\hat{k}_1\rangle k_1\left(\int_{K_{T_2}} \langle k_2,\hat{k}_2\rangle k_2 f\,dk_2\right)dk_1$$

$$= \int_{K_{T_1}}\int_{K_{T_2}} \langle k_1,\hat{k}_1\rangle\langle k_2,\hat{k}_2\rangle k_1k_2 f\,dk_2\,dk_1$$

span $C(X)$, and are common $\boldsymbol{T}^1$-eigenvectors for T_1 and T_2 since the elements of K_{T_1} and K_{T_2} commute. Evidently then $C(X)$ is spanned by common $\boldsymbol{T}^1$-eigenvectors for $\mathfrak{T}$ since $\mathfrak{T}$ is finite. But now (b) holds so we are done.

2. The locally compact case.

Henceforth X will be a locally compact Hausdorff space, $C(X)$ the space of bounded continuous functions, and $C_0(X)$ the subspace of functions vanishing at infinity. In order to accomodate to the locally compact setting we shall have to use hypotheses on a semigroup $\mathfrak{S}$ of transition operators (rather than a set) leaving $C_0(X)$ invariant. Then $Tf(x)=\int f\,dm_x$, with $m_x=m_x^T$ a probability measure for which $x \to m_x$ is w^* continuous into $C_0(X)^*$, while in addition $x \to \infty$ in X implies $m_xK \to 0$ for any compact K in X.

Evidently, we can no longer use finite invariant measures, and our replacement for (a) will require that we assume $\mathfrak{S}$ has the property that $\mathfrak{S}T \subset T\mathfrak{S}$ for any $T\in\mathfrak{S}$. This insures that we can regard $\mathfrak{S}$ as a directed set by taking $T_1 < T_2$ if $T_2 \in \mathfrak{S}T_1$ (since then T and S in $\mathfrak{S}$ have TS ($=S'T$ for $S' \in \mathfrak{S}$) as a common upper bound). Now call a closed subset F of X *uniformly absorbing on compacta* if, for each compact K in X, $m_x^TF \xrightarrow[\mathfrak{S}]{} 1$ uniformly for $x \in K$ (that is, $m_x^TF > 1-\varepsilon$ for $x\in K$, $T\in\mathfrak{S}T_0$). Our replacement for (a) simply precludes the existence of any such sets save X itself. Our result, in its Abelian form, is

THEOREM 5. *Let X be a locally compact Hausdorff space and $\mathfrak{S}$ a semigroup of transition operators on $C(X)$ leaving $C_0(X)$ invariant with $\mathfrak{S}T\subset T\mathfrak{S}$, all $T\in\mathfrak{S}$. Suppose*

(α) *X has no proper closed subsets uniformly absorbing on compacta.*
(β) *The common $\boldsymbol{T}^1$-eigenvectors for $\mathfrak{S}$ separate X.*
(γ) *The only bounded lower semicontinuous functions fixed by all $T\in\mathfrak{S}$ are the constants, and $f\geqq 0$, $0\neq f\in C(X)$, $x_0\in X$, imply $\sup_{T\in\mathfrak{S}} Tf(x_0)>0$.*
(δ) *$\mathfrak{S}\,|\,C_0(X)$ is weakly almost periodic* [1].

Then X is a locally compact abelian group and $\mathfrak{S}$ a generating set of its translation operators.

"Generating" is in the sense of the strong operator topology on $C_u(X)$, the uniformly continuous functions. The converse is essentially trivial. Here (α)–(γ) correspond to (a)–(c) of course, noting that the latter half of (γ) merely says the union of the supports of the various m_x^T is dense in X, which evidently was an immediate consequence of (c″), while (δ) represents a new feature of the locally compact setting. Whether it is essential is unclear, but certainly both (α) (in some form) and (β) are; for example for X the additive non-negative reals and $\mathfrak{S}$ its translation operators, only (α) fails.

Actually, as we shall see after the proof, (δ) is implied here by the weaker hypothesis

(δ') *There is a non-zero f in $C_0(X)$ for which $\mathfrak{S}f$ is conditionslly weakly compact.*

2.1. The proof begins exactly as in Theorem 1. Let f be a common $\boldsymbol{T}^1$-eigenvector, $T \in \mathfrak{S}$, and $Tf = \lambda f$, $|\lambda| = 1$. Then

$$(1) \qquad |f(x)| = |Tf(x)| = \left| \int f\, dm_x \right| \leqq \int |f|\, dm_x = T|f|(x)$$

so $|f| \leqq T|f|$ for every $T \in \mathfrak{S}$. Set $h = \sup_{S \in \mathfrak{S}} S|f|$, so h is lower semi-continuous and bounded; we shall next show h is fixed under all T in $\mathfrak{S}$, hence constant by (γ).

From non-negativity of each $S \in \mathfrak{S}$ and $|f| \leqq T|f|$ we have

$$(2) \qquad S|f| \leqq ST|f|, \qquad S, T \in \mathfrak{T}\,.$$

Thus

$$h = \sup_{S \in \mathfrak{S}} S|f| \leqq \sup_{S \in \mathfrak{S}} ST|f| \leqq \sup_{S \in \mathfrak{S}} TS|f|$$

since $\mathfrak{S}T \subset T\mathfrak{S}$. But $TS|f| \leqq Th$, so, continuing,

$$h \leqq \sup_{s \in \mathfrak{S}} TS|f| \leqq Th\,.$$

On the other hand by a well known property of the integral

$$Th = T \sup_{S \in \mathfrak{S}} S|f| = \sup_{S \in \mathfrak{S}} TS|f| \leqq h$$

since $T\mathfrak{S} \subset \mathfrak{S}$; thus $Th = h$ for all $T \in \mathfrak{S}$ and h is constant by (γ). Evidently, if $c = \sup|f|$, so $c > 0$, then $c = \sup h$ and $h \equiv c$.

Now $T \to T|f|$ is an increasing net since $T_1 \prec T_2$ says $T_2 = ST_1 = T_1S'$ for some S, S' in $\mathfrak{S}$ (again since $\mathfrak{S}T_1 \subset T_1\mathfrak{S}$), and thus $T_1|f| \leqq T_1S'|f| = T_2|f|$ by (2). So because $T|f| \uparrow c$, a constant, convergence is uniform on compacta by Dini's theorem. Thus (α) implies the closed set $F = \{x : |f(x)| \geqq c - \varepsilon\}$ cannot be proper since

$$T|f|(x) = \int |f|\, dm_x^T \leqq (c-\varepsilon)m_x^T(X \setminus F) + cm_x^T(F)$$

now implies $m_x^T(F) \xrightarrow[\mathfrak{S}]{} 1$ uniformly on compacta. We conclude $|f| = c$, yielding equality in (1). But that, again by (β), implies the closed support of m_x^T is a set of constancy of f, and m_x^T is a point mass as before, so T arises from a self map φ of X and our common $\boldsymbol{T}^1$-eigenvectors form a multiplicative semigroup. Because $T\bar{f} = \overline{Tf}$ their closed span A is a closed separating self-adjoint subalgebra of $C(X)$.

Because A is spanned by common $\boldsymbol{T}^1$-eigenvectors $\mathfrak{S} \mid A$ is an almost periodic semigroup of operators (necessarily Abelian for the same reason), whose closure $K = (\mathfrak{S} \mid A)^-$ in the strong operator topology is a jointly continuous compact Abelian semigroup as in Theorem 4. Since K is Abelian it has a least ideal which is a compact group (as is well known and easily proved [10]). Now the identity of K is an idempotent operator E which necessarily has each common $\boldsymbol{T}^1$-eigenvector as a $\boldsymbol{T}^1$-eigenvector (as all elements of K clearly must), so E fixes each common $\boldsymbol{T}^1$-eigenvector. Hence E is the identity on A. Consequently $K = KE$ is its own least ideal, so a compact group.

2.2. Now A can be identified as $C(Y)$ for a compact Hausdorff space Y into which X maps in a continuous $1-1$ fashion since A separates X, while the group K of norm decreasing non-negative maps of $C(Y)$ must be a group of isometries since the identity operator is its identity. Thus each element of K is of the form

$$f \to f \circ g, \quad f \in A\,,$$

where g is a self-homeomorphism of Y. Henceforth, let G denote the group of homeomorphisms g we so obtain, and S those corresponding to elements of $\mathfrak{S} \mid A$.

2.3. Since we can identify measures on X with their images on Y, if $x \in X$, and $g \in S$ gives rise to an element T of $\mathfrak{S} \mid A \subset K$, we can evidently identify our m_x^T as the point mass at $g(x)$; thus such g map X into itself, and indeed continuously since $x \to m_x^T$ was w^* continuous into $C_0(X)^*$. Moreover $g \mid X$ even extends continuously as a self map of X_∞ (the one point compactification) into itself since $x_\delta \to \infty$ in X implies $g(x_\delta) \to \infty$ because $C_0(X) \circ g = TC_0(X)$

$\subset C_0(X)$. Thus we can extend g to a $1-1$ continuous map of X_∞ into itself, and so to a homeomorphism. In fact the map is also onto since $m_x^{T'}(g(X_\infty)\cap X)=1$ for all x, T', since $g'(x) \in g'g(X_\infty)\subset g(X_\infty)\subset g(X_\infty)$, so $g(X_\infty)\cap X$ is precisely X by (α). We conclude that $g \mid X$ is a self homeomorphism of X (as now $g^{-1} \mid X$ also must be). Consequently the subgroup

$$H = \{g \in G\colon g \mid X \text{ is a self homeomorphism of } X\}$$

of G contains the set S which gives rise to our generating subsemigroup $\mathfrak{S} \mid A$ of $K=(\mathfrak{S} \mid A)^-$.

2.4. Note that for any $x_0 \in X$, Sx_0, and so Hx_0, is dense in X by the latter half of (γ).

Now we want to see $Hx_0=X$. Recalling that we have an isomorphism between the groups K and G, we can transfer the compact topology of K to G, and regard G as a compact topological group. Since the topology of K is the strong operator topology on $C(Y)=A$, that on G is precisely uniform convergence on Y of course. Moreover since $(\mathfrak{S} \mid A)^-=K$, $S^-=G$.

Now suppose that $x_1 \in X \setminus Hx_0$. We have a net $\{h_\delta\}$ in S with $h_\delta x_0 \to x_1$ in X, and we can of course assume $h_\delta \to g$ in G. Fix $h \in H$. Then there is an $f=f_h$ in $C_0(X)$ which has $f(h(x_1))=1$. Consequently

$$f(h_\delta(h(x_0))) = f(hh_\delta(x_0)) \to f(h(x_1)) = 1\,,$$

and since $f \in C_0(X)$ this implies that for $x=h(x_0)$ there is a compact $K_x \subset X$ for which $\{h_\delta(x)\}_{\delta \geqq \delta_x} \subset K_x$. But $h_\delta(x) \to g(x)$ in Y, so $\{h_\delta(x)\}$ can have at most one cluster point, $g(x)$, in K_x. Hence $x \in Hx_0$ implies $g(x) \in X$ and $h_\delta(x) \to g(x)$ in X. In particular, $g(x_0)=x_1$.

But for any $f \in C_0$, since $h_\delta \in S$ by (δ) $\{f \circ h_\delta\}$ has a weak, hence pointwise cluster point f' in $C_0(X)$, and since $f(h_\delta(x)) \to f(g(x))$ for x in Hx_0, $f \circ g=f'$ on Hx_0. So for every $f \in C_0(X)$, $(f \circ g) \mid Hx_0$ has a continuous extension to X, and that says $g \mid Hx_0$ also has a continuous extension, g^*, to X. The same argument can now be applied to g^{-1}: since Sx_1 is dense in X there is a net $\{h'_\delta\}$ in S with $h'_\delta x_1 \to x_0=g^{-1}x_1$, and for any cluster point g' of $\{h'_\delta\}$ in G, $g'gx_0=g'x_1=x_0$. Thus $g'ghx_0=hg'gx_0=hx_0$ for all h in H, so $g'g$ fixes each x in a dense subset of X, hence of Y, and $g'=g^{-1}$. So $h'_\delta \to g^{-1}$ by compactness, and now the preceding argument says $g^{-1} \mid Hx_1$ has a continuous extension g_* to X. But $g^{-1} \mid Hx_1=(g \mid Hx_0)^{-1}$, and thus the extension g_* is inverse to g^*, and g^* is a self homeomorphism of X. Moreover as we have seen, for any f in $C_0(X)$, $\{f \circ h_\delta\}$ has a weak cluster point f' coinciding on Hx_0 with $(f \circ g) \mid Hx_0=(f \circ g^*) \mid Hx_0$. So f' is unique, and necessarily the continuous function $f \circ g^*$. Thus $f \circ g^* \in C_0(X)$, and $f \circ h_\delta \to f \circ g^*$ weakly, hence pointwise on X, so $h_\delta \to g^*$

pointwise on X. Finally then $a \circ h_\delta \to a \circ g^*$ pointwise on X for $a \in A = C(Y)$. Since $a \circ h_\delta \to a \circ g$ uniformly, $a \circ g = a \circ g^*$ on X, and now we see $g^* = g \mid X$; similarly $g_* = g^{-1} \mid X$ so g lies in H. Hence $x_1 = g(x_0) \in Hx_0$ after all, and $X = Hx_0$.

Because H is Abelian the isotropy subgroup of x_0 is trivial ($h_0 x_0 = x_0$ implies $h_0 h x = h h_0 x_0 = h x_0$, so h_0 fixes every x in $Hx_0 = x$, whence h_0 is the identity map). Thus $h \to h(x_0)$ is $1-1$ and we can now transfer our group structure on H to X, setting, say, $h(x_0) \cdot h'(x_0) = hh'(x_0)$. Since each h is continuous on X our multiplication is continuous in the second factor: if $h'_\delta(x_0) \to h'(x_0)$ in X, $hh'_\delta(x_0) \to hh'(x_0)$ so $h(x_0) \cdot h'_\delta(x_0) \to h(x_0)h'(x_0)$. So X is now an Abelian locally compact separately continuous group, hence a locally compact Abelian group by Ellis' Theorem [2, 6, 9]. Of course each element of S amounts to a translation on our group, and we are done.

In order to see (δ') implies (δ) in the presence of (α)–(γ), note that once we know each $T \in \mathfrak{S}$ arises from a self map of X, hence is multiplicative, the elements f in $C(X)$ for which $\mathfrak{S}f$ is conditionally weakly compact (i.e., the $\mathfrak{S}$-weakly almost periodic elements,) form an algebra B closed under the taking of conjugates and absolute values. Indeed, this follows since conditional weak compactness in $C(X)$, (hence in $C(\beta X)$) amounts to conditional compactness in the topology of pointwise convergence on the Stone-Čech compactification βX.

Consequently, since the elements A are almost periodic, $B \cap C_0(X)$ contains the A-module $\mathfrak{S}f$ generates in $C_0(X)$ if f is our non-zero weakly almost periodic function given in (δ'). Since this A-module separates any pair of points not both in $(Tf)^{-1}(0)$ because A does, it must separate X if $\bigcap_{T \in \mathfrak{S}} (Tf)^{-1}(0) = \emptyset$. But $Tf(x) = 0$ for all $T \in \mathfrak{S}$ implies f vanishes on Sx which is dense in X as we saw in the beginning of 2.4.

Thus $B \cap C_0(X)$ is dense subalgebra of $C_0(X)$ by Stone-Weierstrass, and since the $\mathfrak{S}$-weakly almost periodic elements of $C_0(X)$ form a necessarily closed subspace, (δ) follows.

Evidently the group G in our proof is the Bohr compactification of X; the same is true in our next, more general, setting.

2.5. For maximally almost periodic locally compact groups the corresponding result again brings in the unitary subspaces of [1].

THEOREM 6. *Suppose X is a locally compact Hausdorff space and $\mathfrak{S}_{\mathfrak{L}}$ and $\mathfrak{S}_{\mathfrak{R}}$ are semigroups of transition operators on $C(X)$ leaving $C_0(X)$ imvariant, with $RL = LR$ for $L \in \mathfrak{S}_{\mathfrak{L}}$, $R \in \mathfrak{S}_{\mathfrak{R}}$, while $\mathfrak{S}_{\mathfrak{L}} L \subset L\mathfrak{S}_{\mathfrak{L}}$, $\mathfrak{S}_{\mathfrak{R}} R \subset R\mathfrak{S}_{\mathfrak{R}}$. Suppose*

(α) *X has no proper closed subsets uniformly absorbing on compacta for $\mathfrak{S}_{\mathfrak{L}}$ or $\mathfrak{S}_{\mathfrak{R}}$.*

(β) *There is a set of finite dimensional subspaces of $C(X)$ which separate X and each of which is both an $\mathfrak{S}_{\mathfrak{L}}$- and $\mathfrak{S}_{\mathfrak{R}}$-unitary subspace.*

(γ) *The only bounded lower semicontinuous functions fixed by all T in $\mathfrak{S}=\mathfrak{S}_{\mathfrak{L}}$ (or in $\mathfrak{S}_{\mathfrak{R}}$) are the constants, and in each case, $f\geqq 0, f\neq 0$ in $C(X)$, $x_0\in X$, imply $\sup_{T\in\mathfrak{S}} Tf(x_0)>0$.*

(δ) *$\mathfrak{S}_{\mathfrak{L}}|C_0(X)$ and $\mathfrak{S}_{\mathfrak{R}}|C_0(X)$ are weakly almost periodic.*

Then X is a (necessarily maximally almost periodic) locally compact group with $\mathfrak{S}_{\mathfrak{L}}$ and $\mathfrak{S}_{\mathfrak{R}}$ generating sets of left and right translation operators.

(Again the analogue of (δ') will suffice.)

Exactly as in Theorem 2 for each $T\in\mathfrak{S}_{\mathfrak{L}}$ (or $\mathfrak{S}_{\mathfrak{R}}$) we have the subspaces in (β) spanned by T^1-eigenvectors for T, so these eigenvectors span a self-adjoint algebra A_T separating X as in 2.1. Thus T arises from a $1-1$ map g of X into itself which is in fact onto and a homeomorphism exactly as in 2.3 (where $g(X_\infty)\cap X$ is now $\mathfrak{S}_{\mathfrak{R}}$-uniformly absorbing on compacta, so all of X).

Now consider the group of homeomorphisms of X all such g generate. Each common $\mathfrak{S}_{\mathfrak{L}}$- and $\mathfrak{S}_{\mathfrak{R}}$-unitary subspace D is necessarily invariant under the norm 1 maps these homeomorphisms induce on $C(X)$, and if D_1, D_2 are two such subspaces then the finite dimensional subspace

$$D = \operatorname{span}\{f_1 f_2 : f_i\in D_i,\ i=1,2\}$$

of $C(X)$ is clearly invariant under these maps, which form a group with the identity operator on D as its identity, and of course the same applies to the corresponding maps for $\mathfrak{S}_{\mathfrak{R}}$. Thus D is again a unitary space for $\mathfrak{S}_{\mathfrak{L}}$ and $\mathfrak{S}_{\mathfrak{R}}$, so the closed span A of all common $\mathfrak{S}_{\mathfrak{L}}$ and $\mathfrak{S}_{\mathfrak{R}}$-unitary subspaces is an algebra, and necessarily self-adjoint since $\bar{D}$ is unitary if D is.

We can of course now identify A and $C(Y)$ for a compact Hausdorff space Y into which X maps in a continuous $1-1$ fashion. As in Theorem 2 each element of $\mathfrak{S}_{\mathfrak{L}}$ (respectively $\mathfrak{S}_{\mathfrak{R}}$) arises from a homeomorphism of Y, and as in the remark following Theorem 2 have strong operator closures which are compact groups $K_{\mathfrak{L}}, K_{\mathfrak{R}}$. The corresponding compact groups $G_{\mathfrak{L}}, G_{\mathfrak{R}}$ (as in 2.2) of homeomorphisms, with semigroups $S_{\mathfrak{L}}, S_{\mathfrak{R}}$ corresponding to $\mathfrak{S}_{\mathfrak{L}}, \mathfrak{S}_{\mathfrak{R}}$, have trivial isotropy subgroups for any $y\in Y$, as in the proof of Corollary 1. Thus we can identify Y with each of these groups, and the induced map $G_{\mathfrak{L}}\to G_{\mathfrak{R}}$ is of course an anti-isomorphism. Now exactly as in 2.3 we have subgroups

$$H_{\mathfrak{L}} = \{g\in G_{\mathfrak{L}}\colon g|X \text{ is a self homeomorphism of } X\}$$

and $H_{\mathfrak{R}}$ of $G_{\mathfrak{L}}, G_{\mathfrak{R}}$ which contain the semigroups $S_{\mathfrak{L}}, S_{\mathfrak{R}}$, and as before $H_{\mathfrak{L}}x_0$ and $H_{\mathfrak{R}}x_0$ are dense in X for any $x_0\in X$. Because $h\in H_{\mathfrak{L}}, h'\in H_{\mathfrak{R}}$ imply h and h' commute we can argue just as in 2.4 that in fact $H_{\mathfrak{R}}x_0=X$; one has only to use h_δ in $S_{\mathfrak{L}}$ and h in $S_{\mathfrak{R}}$. In the same way any weak cluster point f' of $\{f\circ h_\delta\}$ (with $\{h_\delta\}$ in $S_{\mathfrak{L}}$ say) has $f\circ g=f'$ on $H_{\mathfrak{R}}x_0$ (where $h_\delta \to g$ in $G_{\mathfrak{L}}$), so one obtains the continuous extension g^* of $g\,|\,H_{\mathfrak{R}}x_0$ as before.

Thus our earlier arguments yield the fact that $H_{\mathfrak{L}}x_0=X=H_{\mathfrak{R}}x$. for any x_0 in X, and one can identify these groups with X, obtaining now a locally compact group with continuity of multiplication in the right factor as before; continuity in the left factor follows from the fact that the anti-isomorphism of $G_{\mathfrak{L}}$ onto $G_{\mathfrak{R}}$ send $H_{\mathfrak{L}}$ onto $H_{\mathfrak{R}}$ (so that the continuity of $h' \to hh'(x_0)$ for h' in $H_{\mathfrak{R}}$ provides that). Finally, that $H_{\mathfrak{L}}$ maps into $H_{\mathfrak{R}}$ (and vice versa) is a direct consequence of $H_{\mathfrak{L}}x_0=X=H_{\mathfrak{R}}x_0$ and the triviality of the $G_{\mathfrak{L}}$-, $G_{\mathfrak{R}}$-isotropy subgroups: if $g(x_0)=h(x_0)$ for $g \in G_{\mathfrak{L}}$, $h \in H_{\mathfrak{L}}$, $g^{-1}h(x_0)=x_0$, and then $g=h$, so the elements of $H_{\mathfrak{L}}$ (respectively $H_{\mathfrak{R}}$) are characterized as those sending an x_0 in X into X.

3.

We close with two facts concerning unitary subspaces which emerged from our original argument but were not used above, and may have independent interest in the context of [1]. First, if $\mathfrak{S}$ is a semigroup of operators on a Banach space B, *the sum of finitely many* ($\mathfrak{S}$-)*unitary subspaces is unitary.*

Indeed it is clear that a direct sum is unitary. So suppose D_1 and D_2 are unitary, so $D_1 \cap D_2$ is invariant. Relative to an inner product on D_2 which puts $\mathfrak{S}\,|\,D_2$ into the unitary operators the orthogonal complement $D_2 \ominus (D_1 \cap D_2)$ is also invariant, hence unitary. Thus $D_1+D_2=D_1 \oplus (D_2 \ominus (D_1 \cap D_2))$ is unitary.

One consequence of this is that B_p, the closed span in B of all unitary subspaces is just the closure of their union. Because of this we have

PROPOSITION 1. *Suppose $\mathfrak{S}$ is a uniformly bounded semigroup of operators on a Banach space B and the span of the $\mathfrak{S}$-unitary subspaces is dense in B. Then its strong operator closure $\mathfrak{S}^-$ is a compact group of invertibles in the strong operator topology.*

Let $\|T\| \leqq M$, all $T \in \mathfrak{S}$. For any unitary subspace D, because $\mathfrak{S}\,|\,D$ lies in a compact group whose identity is the identity operator on D, $(\mathfrak{S}\,|\,D)^-$ is a compact group of invertible operators on D, as a closed subsemigroup of a compact group. But for $\varepsilon>0$ and $f \in B$ there is a D and $f' \in D$ with $\|f-f'\|<\varepsilon$. Thus since $(T\,|\,D)^{-1} \in (\mathfrak{S}\,|\,D)^-$ has norm $\leqq M$ for any $T \in \mathfrak{S}$,

$$\|Tf\| \geqq \|Tf'\|-M\varepsilon \geqq \frac{1}{M}\|f'\|-M\varepsilon \geqq \frac{1}{M}\|f\|-\frac{\varepsilon}{M}-M\varepsilon\,.$$

So $M\|Tf\| \geqq \|f\| \geqq 1/M\|Tf\|$, and this holds in fact for $T \in \mathfrak{S}^-$. Hence each $T \in \mathfrak{S}^-$ is topological, with its necessarily closed range all of B since that contains each unitary subspace D. Thus $\mathfrak{S}^-$ consists of invertibles with inverses also bounded by M.

Now $\mathfrak{S}$ is an almost periodic semigroup of operators on B so $\mathfrak{S}^-$ is a compact semigroup when taken in the strong operator topology [1]. Thus $0 \in (\mathfrak{S}f)^- = \mathfrak{S}^- f$ implies $f=0$, which says [1, 4.9] $\mathfrak{S}^-$ has a unique minimal left ideal. On the other hand the same remarks apply to the semigroup $\mathfrak{S}^{-1} = \{T^{-1}: T \in \mathfrak{S}\}$ and $(\mathfrak{S}^{-1})^-$ since $\mathfrak{S}$-unitary subspaces are $\mathfrak{S}^{-1}$-unitary, trivially, and $0 \in (\mathfrak{S}^{-1}f)^- = (\mathfrak{S}^{-1})^- f$ implies $f=0$. So $(\mathfrak{S}^{-1})^-$ also has a unique minimal left ideal.

But the anti-isomorphism $T \to T^{-1}$ of $\mathfrak{S}$ onto $\mathfrak{S}^{-1}$ trivially extends to take $\mathfrak{S}^-$ onto $(\mathfrak{S}^{-1})^-$ (as does its inverse, so it extends as an anti-isomorphism): for multiplication is jointly continuous in the strong operator topology on bounded sets of operators, so that if $T_\delta \to T$, $T_\delta \in \mathfrak{S}$, and S is a cluster point of $\{T_\delta^{-1}\}$, then ST is a cluster point of $\{T_\delta^{-1} T_\delta\}$ and TS one of $\{T_\delta T_\delta^{-1}\}$ whence $S = T^{-1}$.

We now conclude from the uniqueness of the minimal left ideal in $(\mathfrak{S}^{-1})^-$ that $\mathfrak{S}^-$ also has a unique minimal right ideal, and hence that its kernel K is a compact group [1, 2.5]. By [1, 4.11 and 4.1] the identity E of K is the projection of B onto the closed span of the unitary subspaces, so the identity operator. Thus $\mathfrak{S}^- = E\mathfrak{S}^- \subset K \subset \mathfrak{S}^-$ so $\mathfrak{S}^- = K$ is a compact group, which consists of invertibles since its identity is the identity operator, completing our proof.

When $\mathfrak{S}$ consists of operators of norm $\leqq 1$ as earlier, our operators are all isometries and we can avoid the ideal structure of $\mathfrak{S}^-$ since we can immediately claim the group of isometries generated by $\mathfrak{S}$ is almost periodic on B, being uniformly bounded. Just this last property to appear easily in the more general case.

Finally, we should note an alternative argument in the general case. In terms of [1, 4.11(iv)] we are assuming $B = B_n$, and the first part of the preceding argument show $B_0 = \{0\}$, so (iv) requires showing $B_r \supset B_p = B$. But for $f \in \mathfrak{S}^- g$ and $\varepsilon > 0$ if we choose D so that it contains f', g' with $\|f - f'\| < \varepsilon$ and $\|g - g'\| < \varepsilon$, then since $f = Tg$, $T \in \mathfrak{S}^-$, $\|f - Tg'\| < M\varepsilon$ so $\|f' - Tg'\| < 2M\varepsilon$. Now there is an S in $\mathfrak{S}^-$ for which $S \mid D = (T \mid D)^{-1}$ since $\mathfrak{S}^- \mid D$ is compact and, containing $\mathfrak{S} \mid D$, must contain $(\mathfrak{S} \mid D)^-$, a group. So $\|Sf' - g'\| < 2M^2\varepsilon$, whence $\|Sf - g\| < M\varepsilon + 2M^2\varepsilon + \varepsilon$, and thus $g \in \mathfrak{S}^- f$ and we have shown any $f \in B$ lies in B_r as required.

In fact this last bit or argument immediately yields the following slight improvement of (iv) $\Rightarrow$ (ii) in [1, 4.11], eliminating the hypothesis that $B_r = B_p$. (The terminology is that of [1].)

PROPOSITION 2. *Suupose $\mathfrak{S}$ is a weakly almost periodic semigroup of operators on a Banach space B and the subset B_0 is a closed $\mathfrak{S}$-invariant subspace while $B = B_0 \oplus B_p$. Then the least ideal $K(\mathfrak{S}^-)$ of $\mathfrak{S}^-$, the weak operator closure of $\mathfrak{S}$, is a compact group (in either the weak or strong operator topology) whose identity E is the projection of $B_0 \oplus B_p$ onto B_p.*

Indeed the preceding proof shows $B_p \subset B_r$, and if $f \in B_r$, since $f = f_0 + f_p$ with $f_0 \in B_0$ and $f_p \in B_p \subset B_r$, $f_0 \in B_r \cap B_0$, whicn is $\{0\}$ just by definition. So $B_r \subset B_p$, and now (iv) of [1, 4.11] holds, so our proposition follows from (iv) implies (ii) of [1, 4.11]. That says $K(\mathfrak{S}^-)$ is a compact group in the weak operator topology; in fact it is a compact group in the strong operator topology, since $K(\mathfrak{S}^-) | B_p$ is compact in the (less fine) strong operator topology of $B(B_p)$ by [1, 8.2].

We should also note that $B_p \subset B_r$ always holds for a bounded semigroup $\mathfrak{S}$ (improving [1, 4.4], which asserts each unitary subspace $D \subset B_r$), as a consequence of the fact that a finite sum of unitary subspaces D is unitary. For if $\varepsilon > 0$ and $f \in B_p$, then there is an f' lying in some D with $\|f - f'\| < \varepsilon$, and, since $\mathfrak{S}^- | D \subset (\mathfrak{S} | D)^-$, while the latter is a group, for $T \in \mathfrak{S}^-$ there is a $T' \in \mathfrak{S}$ for which $\|T'Tf' - f'\| < \varepsilon$. Thus $\|T'Tf - f\| < 3M\varepsilon$ where M bounds $\mathfrak{S}$, whence $f \in B_r$.

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