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TAKESAKI'S DUALITY FOR A NON-DEGENERATE CO-ACTION

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Abstract.

Let δ be a non-degenerate co-action of a locally compact group G on a C*-algebra A. We can find an action $\hat{\delta}$ on a δ-crossed product $A \times_{\delta} G$ and show that a crossed product $(A \times_{\delta} G) \times_{\hat{\delta}} G$ is isomorphic to $A \otimes C(L^2(G))$ where $C(L^2(G))$ is the algebra of all compact operators on $L^2(G)$.

A and B are C*-algebras. We denote by $M(A)$ the multiplier algebra of A. If A is a concrete C*-algebra, we may define $M(A) = \{a \in A''; \ ab + ca \in A$ for $b, c \in A\}$. Following [1] we put

$$\tilde{M}(A \otimes B) = \{x \in M(A \otimes B) \ ; \ x(1 \otimes b) + (1 \otimes c)x \in A \otimes B \text{ for } b, c \in B\} ,$$

where the symbol $\otimes$ means the spatial tensor product.

Let G be a locally compact group. $L^2(G)$ is the Hilbert space of square integrable functions on G with a left Haar measure ds on G. The left and right regular representations of G on $L^2(G)$ are defined by

$$(\lambda(s)\xi)(t) = \xi(s^{-1}t)$$

$$(\varrho(s)\xi)(t) = \Delta^{-\frac{1}{2}}(s)\xi(ts)$$

for $s, t \in G$ and $\xi \in L^2(G)$, where Δ is the modular function of G. Let $C_r^*(G)$ be the C*-algebra generated by $\{\lambda(f); f \in L^1(G)\}$ where

$$\lambda(f) = \int_G f(s)\lambda(s)\,ds ,$$

which is called the reduced group C*-algebra of G. We define a unitary operator W on $L^2(G \times G)$ by

$$(W\xi)(s,t) = \xi(s, st)$$

for $\xi \in L^2(G \times G)$ and we set $\delta_G(x) = W^*(x \otimes 1)W = \operatorname{Ad} W^*(x \otimes 1)$ for $x \in C_r^*(G)$. Then we can show easily that δ_G is an isomorphism of $C_r^*(G)$ into $\tilde{M}(C_r^*(G) \otimes C_r^*(G))$. Since $\delta_G(C_r^*(G))(1 \otimes C_r^*(G))$ generates $C_r^*(G) \otimes C_r^*(G)$, for

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each approximate identity $\{e_i\}$ of $C_r^*(G)$, $\delta_G(e_i)$ converges to 1 in the strict topology of $M(C_r^*(G)\otimes C_r^*(G))$.

Let θ be a homomorphism of A into $M(B)$ satisfying that $\theta(u_i)$ converges to 1 in the strict topology of $M(B)$ for each approximate identity $\{u_i\}$ of A. Then θ extends uniquely to a homomorphism (also denoted by θ) of $M(A)$ into $M(B)$ ([8, Lemme 0.2.6]). The above δ_G has a property

$$(\delta_G\otimes\iota)\delta_G = (\iota\otimes\delta_G)\delta_G ,$$

where ι is the identity map of $C_r^*(G)$ (the above $\delta_G\otimes\iota$ and $\iota\otimes\delta_G$ are homomorphisms on $M(C_r^*(G)\otimes C_r^*(G))$).

Definition. Let δ be an isomorphism of A into $\tilde{M}(A\otimes C_r^*(G))$. The isomorphism δ is called a co-action of G on A if for each approximate identity $\{u_i\}$ of A, $\delta(u_i)$ converges to 1 in the strict topology of $M(A\otimes C_r^*(G))$ and $(\delta\otimes\iota)\delta=(\iota\otimes\delta_G)\delta$.

We define a linear map δ_u by $\delta_u(a)=L_u\delta(a)$ for $u\in B_r(G)\equiv C_r^*(G)^*$, $a\in A$ where L_u is the left slice map of u (see [2]). Since δ is a map into $\tilde{M}(A\otimes C_r^*(G))$, by [7, Theorem 2.1] δ_u is a linear map of A into A.

Lemma 1. *Let δ be a co-action of G on A. For $x=\delta_u(a)$, $a\in A$ and $u\in B_r(G)\cap K(G)$, we have*

$$\int_G \delta_{\varphi\lambda(s)^*}(x)\otimes\lambda(s)z\,ds = \delta(x)(1\otimes\lambda(\check{\varphi})z) \tag{1}$$

for $\varphi\in B_r(G)\cap K(G)$ and $z\in C_r^*(G)$, where $\check{\varphi}(s)=\varphi(s^{-1})$ and $\langle z,\varphi\lambda(s)^*\rangle =\langle\lambda(s)^*z,\varphi\rangle$ ($K(G)$ is the family of continuous functions on G with compact supports).

Proof. Both functions $s\in G\to\lambda(s)z\in C_r^*(G)$ and $s\in G\to\varphi\lambda(s)^*\in C_r^*(G)^*$ are norm-continuous. The integrand:

$$s\in G\to\delta_{\varphi\lambda(s)^*}(x)\otimes\lambda(s)z$$

is continuous in the norm topology of $A\otimes C_r^*(G)$, whose support is contained in a compact set $(\operatorname{supp} u)\cdot(\operatorname{supp}\varphi)^{-1}$. Hence $\int_G\delta_{\varphi\lambda(s)^*}(x)\otimes\lambda(s)z\,ds$ is contained in $A\otimes C_r^*(G)$. For $\omega\in A^*$, $\psi\in B_r(G)\cap K(G)$ and $z=\lambda(f)$, $\in K(G)$, we have

$$\left\langle\int_G \delta_{\varphi\lambda(s)^*}(x)\otimes\lambda(s)z\,ds,\ \omega\otimes\psi\right\rangle$$

$$=\int_G\langle\delta(x),\omega\otimes\varphi\lambda(s)^*\rangle\langle\lambda(s)z,\psi\rangle\,ds ,$$

since the function

$$s \in G \to \langle \lambda(s)z, \psi \rangle = \langle \lambda(s)\lambda(f), \psi \rangle$$

is continuous whose support is contained in a compact set $(\mathrm{supp}\,\psi)\cdot(\mathrm{supp}\,f)^{-1}$,

$$(2) \qquad = \left\langle \delta(x), \omega \otimes \int_G \langle \lambda(s)z, \psi \rangle \varphi \lambda(s)^* \, ds \right\rangle .$$

For $g \in L^1(G)$, we get

$$\begin{aligned}
&\left\langle \lambda(g), \int_G \langle \lambda(s)z, \psi \rangle \varphi \lambda(s)^* \, ds \right\rangle \\
&= \int_G \langle \lambda(s)z, \psi \rangle \langle \lambda(s)^* \lambda(g), \varphi \rangle \, ds \\
&= \iint_{G \times G} \langle \lambda(s)z, \psi \rangle g(t) \langle \lambda(s^{-1}t), \varphi \rangle \, dt ds \\
&= \iint_{G \times G} \langle \lambda(th)z, \psi \rangle g(t) \langle \lambda(h^{-1}), \varphi \rangle \, dt dh \qquad (s^{-1}t = h^{-1}) \\
&= \int_G \langle \lambda(g)\lambda(h)z, \psi \rangle \langle \lambda(h^{-1}), \varphi \rangle \, dh \\
&= \int_G \check{\varphi}(h) \langle \lambda(g)\lambda(h)z, \psi \rangle \, dh = \langle \lambda(g)\lambda(\check{\varphi})z, \psi \rangle \\
&= \langle \lambda(g), \lambda(\check{\varphi})z\psi \rangle .
\end{aligned}$$

Therefore we have $\int_G \langle \lambda(s)z, \psi \rangle \varphi \lambda(s)^* \, ds = \lambda(\check{\varphi})z\psi$. Hence

$$(2) = \langle \delta(x), \omega \otimes \lambda(\check{\varphi})z\psi \rangle = \langle \delta(x)(1 \otimes \lambda(\check{\varphi})z), \omega \otimes \psi \rangle .$$

Since $B_r(G) \cap K(G)$ is dense in the Fourier algebra $A(G)$ (see [2]), we obtain

$$(3) \qquad \int_G \delta_{\varphi\lambda(s)^*}(x) \otimes \lambda(s)z \, ds = \delta(x)(1 \otimes \lambda(\check{\varphi})z)$$

for $z = \lambda(f)$, $f \in K(G)$. Both sides in (3) are continuous with respect to z. Then we have the equation (3) for all $z \in C_r^*(G)$.

LEMMA 2. *Let δ be as above. The closure $I(A)$ of $\{\delta_\varphi(a);\, a \in A,\, \varphi \in A(G)\}$ is a C^*-subalgebra of A. Moreover for $x \in I(A)$ and $z \in C_r^*(G)$ the element $\delta(x)(1 \otimes z)$ is contained in $I(A) \otimes C_r^*(G)$.*

PROOF. Since $K(G)\cap A(G)$ is norm-dense in $A(G)$ and $\|\delta_\varphi\| \leqq \|\varphi\|$ for $\varphi \in B_r(G)$, $I(A)$ is the closure of $\{\delta_\varphi(a);\ a \in A,\ \varphi \in K(G)\cap A(G)\}$. Since $A(G)$ is a regular ring (see [2]), we can find, for $\varphi_1, \varphi_2 \in K(G)\cap A(G)$, φ_3 in $K(G)\cap A(G)$ with $\varphi_3 \equiv 1$ on a neighbourhood of $(\operatorname{supp}\varphi_1)\cdot(\operatorname{supp}\varphi_2)$ and $\operatorname{supp}\varphi_1 \cup \operatorname{supp}\varphi_2$. Then we obtain

$$\delta_{\varphi_3}(\delta_{\varphi_1}(x)\delta_{\varphi_2}(y)) = \delta_{\varphi_1}(x)\delta_{\varphi_2}(y)$$

$$\delta_{\varphi_3}(\delta_{\varphi_1}(x)+\delta_{\varphi_2}(y)) = \delta_{\varphi_1}(x)+\delta_{\varphi_2}(y)$$

for all $z, y \in A$. Therefore $I(A)$ is a C*-subalgebra of A. When we choose an approximate identity $\{\varphi_i\}$ of $L^1(G)$ in the set $K(G)\cap B_r(G) = K(G)\cap A(G)$, by the equation (1) we have

$$\lim_i \int_G \delta_{\varphi_i\lambda(s)^*}(x)\otimes\lambda(s)z\,ds = \lim_i \delta(x)(1\otimes\lambda(\check{\varphi}_i)z) = \delta(x)(1\otimes z)\,,$$

that is $\delta(x)(1\otimes z)$ is contained in $I(A)\otimes C_r^*(G)$ for $x=\delta_u(y)$, some $y \in A$ and $u \in K(G)\cap A(G)$. Therefore $\delta(x)(1\otimes z)$ is contained in $I(A)\otimes C_r^*(G)$ for $x \in I(A)$.

LEMMA 3. *Let δ be as above. The closed subspace $[\delta(I(A))(1\otimes C_r^*(G))]$ generated by $\delta(I(A))(1\otimes C_r^*(G))$ contains $I(A)\otimes C_r^*(G)$.*

PROOF. Take $x=\delta_u(y)$, $(y \in A,\ u \in A(G)\cap K(G))$, and by (1) we have, for $\varphi \in A(G)\cap K(G)$,

$$\delta(x)(1\otimes\lambda(\check{\varphi})) = \int_G \delta_{\varphi\lambda(s)^*}(x)\otimes\lambda(s)\,ds \qquad \text{(in the strict topology)}\,.$$

For $v \in A(G)\cap K(G)$, we obtain

$$\begin{aligned} &\iota\otimes L_v[(\iota\otimes\delta_G)\{\delta(x)(1\otimes\lambda(\check{\varphi}))\}] \\ &= \iota\otimes L_v\left(\int_G \delta_{\varphi\lambda(s)^*}(x)\otimes\lambda(s)\otimes\lambda(s)\,ds\right) \\ &= \int_G v(s)\delta_{\varphi\lambda(s)^*}(x)\otimes\lambda(s)\,ds\,. \end{aligned}$$

On the other hand, we get, for $\omega_1 \in A^*$, $\omega_2 \in C_r^*(G)^*$

$$\begin{aligned} &\langle \iota\otimes L_v[(\iota\otimes\delta_G)(\delta(x)(1\otimes\lambda(\check{\varphi})))], \omega_1\otimes\omega_2\rangle \\ &= \langle ((\iota\otimes\delta_G)\delta(x))(1\otimes\delta_G(\lambda(\check{\varphi}))), \omega_1\otimes\omega_2\otimes v\rangle \\ &= \left\langle (\delta\otimes\iota)\delta(x),\ \omega_1\otimes\left(\int_G \varphi(s)\lambda(s)\otimes\lambda(s)\,ds\right)(\omega_2\otimes v)\right\rangle \end{aligned}$$

$$= \int_G \check{\varphi}(s)\langle(\delta\otimes\iota)\delta(x),\ \omega_1\otimes\lambda(s)\omega_2\otimes\lambda(s)v\rangle\,ds$$

$$= \int_G \check{\varphi}(s)\langle\delta(\delta_{\lambda(s)v}(x))(1\otimes\lambda(s)),\omega_1\otimes\omega_2\rangle\,ds$$

$$= \left\langle \int_G \check{\varphi}(s)\delta(\delta_{\lambda(s)v}(x))(1\otimes\lambda(s))\,ds,\omega_1\otimes\omega_2\right\rangle .$$

Therefore

$$\begin{aligned}(4)\qquad & \iota\otimes L_v[(\iota\otimes\delta_G)(\delta(x)(1\otimes\lambda(\check{\varphi})))] \\ & = \int_G \check{\varphi}(s)\delta(\delta_{\lambda(s)v}(x))(1\otimes\lambda(s))\,ds = \int_G v(s)\delta_{\varphi\lambda(s)^*}(x)\otimes\lambda(s)\,ds .\end{aligned}$$

For $z \in C_r^*(G)$ we obtain,

$$\int_G \check{\varphi}(s)\delta(\delta_{\lambda(s)v}(x))(1\otimes\lambda(s)z)\,ds = \int_G v(s)\delta_{\varphi\lambda(s)^*}(x)\otimes\lambda(s)\,z ds .$$

Since the integrands in the above equation are norm-continuous,

$$\lim_v \int_G \check{\varphi}(s)\delta(\delta_{\lambda(s)v}(x))(1\otimes\lambda(s)z)\,ds = \delta_\varphi(x)\otimes z$$

in the norm topology, when the measure $v(s)ds$ tends to a Dirac measure at the identity of G. Then $[\delta(I(A))(1\otimes C_r^*(G))]$ contains $I(A)\otimes C_r^*(G)$.

REMARK. The restriction $\delta|_{I(A)}$ of δ to $I(A)$ is a co-action of G on $I(A)$.

LEMMA 4. *Let δ be as above. The closed linear* span $[\delta(A)(1\otimes C_r^*(G))]$ *is coincided with $I(A)\otimes C_r^*(G)$.*

PROOF. Without the condition $x=\delta_u(y)$ in the proof of the former equality in (4), we have, for $v,\varphi \in A(G)\cap K(G)$, $x \in A$,

$$\iota\otimes(\delta_G)_v(\delta(x)(1\otimes\lambda(\check{\varphi}))) = \int_G \check{\varphi}(s)\delta(\delta_{\lambda(s)v}(x))(1\otimes\lambda(s))\,ds .$$

Since $A\otimes C_r^*(G)$ contains $\delta(x)(1\otimes\lambda(\check{\varphi}))$, the norm closure of $\{(\iota\otimes(\delta_G)_v)(\delta(x)(1\otimes\lambda(\check{\varphi})));\ v \in A(G)\cap K(G)\}$ contains $\delta(x)(1\otimes\lambda(\check{\varphi}))$. The norm closure of

$$\left\{\int_G \check{\varphi}(s)\delta(_{\lambda(s)v}(x))(1\otimes\lambda(s)z)\,ds\ ;\ v \in A(G)\cap K(G)\right\}$$

contains $\delta(x)(1\otimes\lambda(\check{\varphi})z)$, $(z\in C_r^*(G))$. Since the element $\delta_{\lambda(s)v}(x)$ is in $I(A)$, Lemma 2 implies that $\int_G\check{\varphi}(s)\delta(\delta_{\lambda(s)v}(x))(1\otimes\lambda(s)z)\,ds$ is in $I(A)\otimes C_r^*(G)$, that is

$$\delta(x)(1\otimes\lambda(\check{\varphi})z)\in I(A)\otimes C_r^*(G)\,.$$

By taking φ as an approximate identity of $L^1(G)$, we get $\delta(x)(1\otimes z)\in I(A)\otimes C_r^*(G)$ for $x\in A$ and $z\in C_r^*(G)$. By Lemma 3, we have $[\delta(A)(1\otimes C_r^*(G))]=I(A)\otimes C_r^*(G)$.

THEOREM 5. *Let δ be as above. The following statements are equivalent.*

(i) $A=I(A)$,

(ii) $[\delta(A)(1\otimes C_r^*(G))]=A\otimes C_r^*(G)$,

(iii) $[\delta(A)(1\otimes C(L^2(G)))]=A\otimes C(L^2(G))$,

(iv) *δ is non-degenerate in Landstad's sence,* i.e. *for each non-zero linear functional ω in A^*, we can find $u\in B_r(G)$ with $(\omega\otimes u)\delta\neq 0$.*

PROOF. The equivalence of (i) and (ii) follows from Lemma 4. Since $C_r^*(G)\cdot C_0(G)$ generates $C(L^2(G))$, we have (ii) $\Rightarrow$ (iii). We shall prove (iii) $\Rightarrow$ (i). We define a rank one operator $\xi\otimes\eta^c$ with $(\xi\otimes\eta^c)(\zeta)=\langle\zeta,\eta\rangle\xi$ for ξ,η and $\zeta\in L^2(G)$. For ξ_i,η_i $(i=1,2)$, $\xi,\eta\in L^2(G)$ and elements p,q of the universal Hilbert space for A, we have

$$\begin{aligned}&\langle\{1\otimes(\xi_1\otimes\eta_1^c)\}\delta(a)\{1\otimes(\xi_2\otimes\eta_2^c)\}p\otimes\xi,\ q\otimes\eta\rangle\\ &=\langle\delta(a)(p\otimes\langle\xi,\eta_2\rangle\xi_2),q\otimes\langle\eta,\xi_1\rangle\eta_1\rangle\\ &=\langle\delta(a)(p\otimes\xi_2),q\otimes\eta_1\rangle\ \langle\xi,\eta_2\rangle\ \overline{\langle\eta,\xi_1\rangle}\\ &=\langle\delta_{\omega_{\xi_2,\eta_1}}(a)p,q\rangle\langle\langle\xi,\eta_2\rangle\xi_1,\eta\rangle\end{aligned}$$

where

$$\begin{aligned}\omega_{\xi_2\eta_1}(z)&=\langle z\xi_2,\eta_1\rangle\\ &=\langle\delta_{\omega_{\xi_2,\eta_1}}(a)\otimes(\xi_1\otimes\eta_2^c)(p\otimes\xi),q\otimes\eta\rangle\,.\end{aligned}$$

Then

$$[1\otimes(\xi_1\otimes\eta_1^c)]\delta(a)[1\otimes(\xi_2\otimes\eta_2^c)]=\delta_{\omega_{\xi_2,\eta_1}}(a)\otimes(\xi_1\otimes\eta_2^c)\,.$$

Since the family of finite rank operators on $L^2(G)$ generates $C(L^2(G))$, $\delta(A)(1\otimes C(L^2(G)))$ is contained in $I(A)\otimes C(L^2(G))$. Then by (iii), we have $A\otimes C(L^2(G))=I(A)\otimes C(L^2(G))$, which implies $A=I(A)$. If (ii) holds, for non zero functionals ω in A^* and u in $B_r(G)$, we can find a in A with $(\omega\otimes au)\delta\neq 0$. Suppose that $I(A)$ is a proper C*-subalgebra of A. We can find a non zero linear functional ω in A^* with $\omega(I(A))=0$. Then it follows from Lemma 4 and

[7, Theorem 2.1] that $(\omega\otimes u)\delta=0$ for all $u \in B_r(G)$, which is a contradiction with non-degeneracy of δ.

It is found in [4, Lemma 3.8] that a co-action δ of a discrete or amenable group G is automatically non-degenerate. Also a canonical co-action on a reduced crossed product for a C*-dynamical system is automatically non-degenerate. The author has been unable to prove the automatical non-degeneracy of δ for arbitrary locally compact group. For convenience of readers, we prove the automatical non-degeneracy for a discrete or amenable group. We prove the condition (i) in Theorem 5 in a slight different way.

PROPOSITION 6 ([4]). *Let G be a discrete or amenable group. A co-action δ of G on A is automatically non-degenerate.*

PROOF. By [7, Theorem 2.1], for $u \in B_r(G)$, we find a in A and $v \in B_r(G)$ with $u=av$. Then we have

$$\delta_u(g) = \delta_{av}(x) = L_v(\delta(x)(1\otimes a))$$

in $I(A)$ by Lemma 4. We have

$$(5)\qquad \begin{cases} \delta(\delta_u(x)) = \delta L_u(\delta(x)) = (\iota\otimes L_u)(\delta\otimes\iota)\delta(x) \\ \qquad = \iota\otimes L_u((\iota\otimes\delta_G)(\delta(x))) = \iota\otimes(\delta_G)_u(\delta(x)) . \end{cases}$$

Suppose that G is amenable, we take the identity $u_1(s)\equiv 1$ in $B_r(G)=B(G)$. Since $\iota\otimes(\delta_G)_{u_1}$ is an identity map of $M(A\otimes C_r^*(G))$, we have $\delta(\delta_{u_1}(x))=\delta(x)$ for $x \in A$, which implies $x=\delta_{u_1}(x) \in I(A)$ by the injectivity of δ. Suppose that G is discrete. Then $\delta(x)$ is contained in $A\otimes C_r^*(G)$. Therefore it is easy to prove that the closure of $\{\iota\otimes(\delta_G)_u(\delta(x));\ u \in B_r(G)\}$ contains $\delta(x)$. By (5), $\delta(I(A))$ contains $\delta(x)$ for $x \in A$, that is $x \in I(A)$. In the both cases we have $A=I(A)$.

Let $C^*(G)$ be the envelopping C*-algebra of $L^1(G)$, and U be the universal representation of G. We can define an isomorphism $\overline{\delta_G}$ of $C^*(G)$ into $\tilde{M}(C^*(G)\otimes C^*(G))$ such that

$$\overline{\delta_G}(U(f)) = \overline{\delta_G}\left(\int_G f(s)U(s)\,ds\right) = \int_G f(s)U(s)\otimes U(s)\,ds$$

for $f \in L^1(G)$. Moreover $(\iota\otimes\overline{\delta_G})\overline{\delta_G}=(\overline{\delta_G}\otimes\iota)\overline{\delta_G}$ and

$$[\overline{\delta_G}(C^*(G))(1\otimes C^*(G))] = C^*(G)\otimes C^*(G)$$

(see [3, Theorem 3.9]).

Let δ be an injective homomorphism of A into $\tilde{M}(A \otimes C^*(G))$ and $\delta(e_n)$ convergens 1 in the strict topology of $A \otimes C^*(G)$ for each approximate identity $\{e_n\}$ of A and $(\delta \otimes \iota)\delta = (\iota \otimes \overline{\delta_G})\delta$. Let π be a canonical homomorphism of $C^*(G)$ onto $C_r^*(G)$. Note that δ automatically satisfies the statements in Theorem 5 by the same proof as in the case of an amenable group G (Proposition 6). Set

$$\delta^1(x) = (\iota \otimes \pi)\delta(x) \quad \text{for } x \in A .$$

Since δ^1 is not in general injective, set

$$I = \operatorname{Ker} \delta^1 \quad \text{and} \quad \delta^r(\theta(x)) = (\theta \otimes \iota)\delta^1(x) \quad \text{for } x \in A ,$$

where θ is a canonical homomorphism of A onto A/I.

PROPOSITION 7. *The map δ^r is a non-degenerate co-action of G on A/I.*

PROOF. For $f \in L^1(G)$ and $x \in A$, we have

$$\begin{aligned} \delta^r(\theta(x))(1 \otimes \lambda(f)) &= (\theta \otimes \iota)\delta^1(x)(1 \otimes \lambda(f)) \\ = [(\theta \otimes \iota)(\iota \otimes \pi)\delta(x)](1 \otimes \lambda(f)) &= \theta \otimes \pi(\delta(x)(1 \otimes U(f))) , \end{aligned}$$

because of $\pi(U(f)) = \lambda(f)$. Then $\delta^r(\theta(x))(1 \otimes z)$ is contained in $A/I \otimes C_r^*(G)$ for $x \in A$ and $z \in C_r^*(G)$. Suppose $\delta^r(\theta(x)) = 0$ $(x \in A)$. Then for $\omega \in C_r^*(G)^*$, we have

$$\begin{aligned} 0 &= L_\omega(\delta^r(\theta(x))) = L_\omega((\theta \otimes \iota)\delta^1(x)) \\ &= \theta(L_\omega \delta^1(x)) = \theta(\delta^1_\omega(x)) . \end{aligned}$$

Therefore $\delta^1_\omega(x)$ is contained in I. Since

$$(\iota \otimes L_\omega)(\delta^1 \otimes \iota)\delta^1(x) = \delta^1(\delta^1_\omega(x)) = 0 \quad \text{for } \omega \in C_r^*(G)^* ,$$

we have $(\delta^1 \otimes \iota)\delta^1(x) = 0$. Since

$$\begin{aligned} (\delta^1 \otimes \iota)\delta^1 &= [\iota \otimes \pi \otimes \iota)(\delta \otimes \iota)][(\iota \otimes \pi)\delta] \\ &= (\iota \otimes \pi \otimes \pi)(\delta \otimes \iota)\delta = (\iota \otimes \pi \otimes \pi)(\iota \otimes \overline{\delta_G})\delta \\ &= (\iota \otimes \delta_G)(\iota \otimes \pi)\delta = (\iota \otimes \delta_G)\delta^1 \end{aligned}$$

because of $(\pi \otimes \pi)\overline{\delta_G} = \delta_G \circ \pi$, then we obtain $(\iota \otimes \delta_G)\delta^1(x) = 0$. Since $(\iota \otimes \delta_G)$ is an isomorphism of $\tilde{M}(A \otimes C_r^*(G))$ (see [1, Proposition 2.4]), we get $\delta^1(x) = 0$, that is δ^r is an isomorphism of A/I. We have, on A,

$$\begin{aligned} (\delta^r \otimes \iota)\delta^r(\theta(x)) &= [((\theta \otimes \iota)\delta^1) \otimes \iota](\theta \otimes \iota)\delta^1(x) \\ &= \{[(\theta \otimes \iota)\delta^1\theta] \otimes \iota\}\delta^1(x) = \{(\theta \otimes \iota)\delta^1 \otimes \iota\}\delta^1(x) \end{aligned}$$

$$= (\theta\otimes\iota\otimes\iota)(\delta^1\otimes\iota)\delta^1(x) = (\theta\otimes\iota\otimes\iota)(\iota\otimes\delta_G)\delta^1(x)$$

$$= (\iota\otimes\delta_G)(\theta\otimes\iota)\delta^1(x) = (\iota\otimes\delta_G)\delta^r(\theta(x)) .$$

Since $\delta(e_n)$ converges to 1 in the strict topology of $M(A\otimes C^*(G))$ for each approximate identity $\{e_n\}$ of A, it follows from [8, Lemme 0.2.6] that δ^r has the same property for A/I. Then we have proved that δ^r is a co-action of G on A/I. Also $\delta^r_u(\theta(x))=\theta(\delta_u(x))$ for $x\in A$ and $u\in A(G)$ and by the same proof as in the case of an amenable group G (Proposition 6), A is generated by $\{\delta_u(x); u\in A(G), x\in A\}$. Therefore $\{\delta^r_u(x); u\in A(G), x\in A/I\}$ generated A/I, that is δ^r is non-degenerate.

The isomorphism δ of A into $\tilde{M}(A\otimes C^*(G))$ (respectively $\tilde{M}(A\otimes C_r^*(G))$) satisfying $(\delta\otimes\iota)\delta=(\iota\otimes\overline{\delta_G})\delta$ (respectively $(\delta\otimes\iota)\delta=(\iota\otimes\delta_G)\delta$) is related with crossed product (respectively reduced crossed product).

Before we prove Takesaki's duality for a co-action, we need some notations and definitions. And we note that the discussion which we make below is the same which Landstad [5], Nakagami and Takesaki [6] and Van Heeswijck [9] do.

Let δ be a co-action of G on A and let $C_0(G)$ be the family of continuous functions on G vanishing at infinity. The crossed product $A\times_\delta G$ by δ is the C*-algebra generated by $\delta(A)(1\otimes C_0(G))$ in the full operator algebra $B(L^2(G,\mathscr{H}))$ ($\mathscr{H}$ is the universal Hilbert space for A and $C_0(G)$ acts as multiplication on $L^2(G)$). Let V be a unitary operator on $L^2(G\times G,\mathscr{H})$ satisfying

$$(V\xi)(s,t) = \Delta(t)^{\frac{1}{2}}\xi(st^{-1},t)$$

for $\xi\in L^2(G\times G,\mathscr{H})$ and Δ is the modular function of G. Set a dual action $\hat{\delta}$ of G,

$$\hat{\delta}(x) = V(x\otimes 1)V^*$$

for $x\in A\times_\delta G$. Then $\hat{\delta}(\delta(x))=\delta(x)\otimes 1$ $(x\in A)$ and $\hat{\delta}(1\otimes f)=1\otimes\alpha_G(f)$ $(f\in C_0(G))$, where

$$\alpha_G(f)(s,t) = f(st^{-1}) .$$

Therefore $\hat{\delta}$ is an isomorphism of $A\times_\delta G$ into $\tilde{M}(A\times_\delta G\otimes C_0(G))$ such that $\hat{\delta}(e_n)$ converges to 1 in the strict topology of $M(A\times_\delta G\otimes C_0(G))$ for each approximate identity $\{e_n\}$ of $A\times_\delta G$ and $(\hat{\delta}\otimes\iota)\hat{\delta}=(\iota\otimes\alpha_G)\hat{\delta}$. The crossed product $(A\times_\delta G)\times_{\hat{\delta}}G$ by the action $\hat{\delta}$ is the C*-algebra generated by $\hat{\delta}(A\times_\delta G)(1\otimes 1\otimes C_r^*(G))$. Set a co-action $\hat{\hat{\delta}}$ of G on $(A\times_\delta G)\times_{\hat{\delta}}G$,

$$\hat{\hat{\delta}}(x) = (1\otimes 1\otimes W^*)(x\otimes 1)(1\otimes 1\otimes W)$$

for $x\in(A\times_\delta G)\times_{\hat{\delta}}G$. Then $\hat{\hat{\delta}}$ is easily proved to be a non-degenerate co-action of G.

THEOREM 8. *Let δ be a non-degenerate co-action of G on A. The* C*-*algebra $(A \times_\delta G) \times_{\hat\delta} G$ is isomorphic to $A \otimes C(L^2(G))$, moreover its isomorphism transfers $\hat{\hat\delta}$ to $\tilde\delta$, where*

$$\tilde\delta(x) = (1 \otimes W)[(\iota \otimes \sigma)(\delta \otimes \iota)(x))]1 \otimes W^*$$

and σ is a flip map of $C_r^*(G) \otimes C(L^2(G))$ onto $C(L^2(G)) \otimes C_r^*(G)$.

PROOF. Let D be the C*-algebra generated by

$$S(1 \otimes W)\big((1 \otimes W^*)(\delta(A) \otimes 1)(1 \otimes W)(1 \otimes 1 \otimes C(L^2(G)))\big)(1 \otimes W^*)S^* ,$$

where S is a unitary operator defined by

$$(S\xi)(s,t) = \Delta(t)^{-\frac{1}{2}}\xi(s,t^{-1}) \qquad (\xi \in L^2(G \times G, \mathscr{H})) .$$

Then

$$\begin{aligned}(1 \otimes W^*)&(\delta(A) \otimes 1)(1 \otimes W)(1 \otimes 1 \otimes C(L^2(G)))\\ &= (\iota \otimes \delta_G)\delta(A)(1 \otimes 1 \otimes C(L^2(G)))\\ &= (\delta \otimes \iota)\delta(A)(1 \otimes 1 \otimes C(L^2(G)))\\ &= (\delta \otimes \iota)\big(\delta(A)(1 \otimes C(L^2(G)))\big) .\end{aligned}$$

Since δ is non-degenerate, by Theorem 5 (iii), $\delta(A)(1 \otimes C(L^2(G)))$ generates $A \otimes C(L^2(G))$. Then D is isomorphic to $A \otimes C(L^2(G))$. Therefore we have only to prove that D coincides $(A \times_\delta G) \times_{\hat\delta} G$. We prove easily the following facts:

$$(6)\quad \begin{cases} S(\delta(a) \otimes 1)S^* = \delta(a) \otimes 1 & (a \in A) \\ S(1 \otimes W)(1 \otimes 1 \otimes v(g))(1 \otimes W^*)S^* = 1 \otimes 1 \otimes \lambda(g) & (g \in L^1(G)) \\ \text{where } v \text{ is the right regular representation of } G & \\ S(1 \otimes W)(1 \otimes 1 \otimes f)(1 \otimes W^*)S^* = 1 \otimes \alpha_G(f) & (f \in C_0(G)) \\ C(L^2(G)) \text{ is generated by } \{f \cdot v(g);\, f \in C_0(G),\ g \in L^1(G)\}. & \end{cases}$$

By extending $\tilde\delta$ and $\hat{\hat\delta}$ to their multipliers, we have

$$(7)\quad \begin{cases} \tilde\delta(\delta(a)) = \delta(a) \otimes 1 & (a \in A) \\ \tilde\delta(1 \otimes f) = 1 \otimes f \otimes 1 & (f \in C_0(G)) \\ \tilde\delta(v(g)) = \displaystyle\int_G g(s)v(s) \otimes v(s)\, ds & (g \in L^1(G)) \end{cases}$$

and

$$(8)\qquad \begin{cases} \hat{\hat{\hat{\delta}}}(\delta(a)\otimes 1) = \delta(a)\otimes 1\otimes 1 & (a \in A) \\ \hat{\hat{\hat{\delta}}}(1\otimes\alpha_G(f)) = 1\otimes\alpha_G(f)\otimes 1 & (f \in C_0(G)) \\ \hat{\hat{\hat{\delta}}}(1\otimes 1\otimes\lambda(g)) = \int_G g(s)(1\otimes 1\otimes\lambda(s)\otimes\lambda(s))\,ds & (g \in L^1(G)) . \end{cases}$$

By (6), D is isomorphic to $(A\times_\delta G)\times_{\hat{\delta}} G$. By (7), (8) its isomorphism transfers $\hat{\hat{\hat{\delta}}}$ to $\tilde{\delta}$.

When G is a discrete or amenable group, Takesaki's duality by co-action of G holds true without non-degeneracy of δ. If G is compact, Landstad has already solved it in [5, Theorem 3].

REFERENCES

1. C. A. Akemann, G. K. Pedersen, and J. Tomiyama, *Multipliers of* C*-*algebras*, J. Funct. Anal. 13 (1973), 277–301.
2. P. Eymard, *L'algebre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181–236.
3. V. M. Iorio, *Hopf*-C*-*algebras and locally compact groups*, Pacific J. Math. 87 (1980), 75–96.
4. M. B. Landstad, *Duality theory for covariant systems*, Trans. Amer. Math. Soc. 248 (1979), 223–269.
5. M. B. Landstad, *Duality for dual* C*-*covariance algebras over compact groups*, preprint, University of Trondheim, 1978.
6. Y. Nakagami and M. Takesaki, *Duality for crossed products of von Neumann algebras* (Lecture Notes in Math. 731), Springer-Verlag, Berlin - Heidelberg - New York, 1979.
7. D. C. Taylor, *The strict topology for double centralizer algebras*, Trans. Amer. Math. Soc. 150 (1970), 633–643.
8. J. M. Vallin, C*-*algebres de Hopf spatiales* C*-*algebres de Kac*, preprint, Université d'Orléans.
9. L. Van Heeswijck, *Duality in the theory of crossed products*, Math. Scand. 44 (1979), 313–329.

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