MATH SCAND. 55 (1984), 161-187

PROJECTIVE TENSOR PRODUCTS OF C*-ALGEBRAS

STEN KAIJSER and ALLAN M. SINCLAIR

1. Introduction.

In this paper we investigate some properties of the projective tensor product of two C*-algebras, bilinear forms on C*-algebras, and hermitian elements in the projective tensor product of two unital Banach algebras. We shall show that there are various natural relationships between these ideas.

G. Pisier [17] generalized Grothendieck's inequality from commutative C*-algebras to non-commutative C*-algebras that satisfy a suitable approximability hypothesis. This hypothesis on the approximability of the bilinear form by suitable finite rank bilinear forms was removed by U. Haagerup [8], who also fond the best value of the constant in the non-commutative case. Haagerup's proof falls into two parts. In the first half he obtains a constant, 5/2, in the inequality that is not best possible, and in the "end" of the proof he obtains the best value by a complex analysis variational technique. In Theorem 2.1 we give a new proof of the first part of Haagerup's theorem [8, Theorem 1.1], and thus also of Pisier's result [17, Corollary 2.2]. Since a $C(X)$-space is a commutative C*-algebra, the Haagerup-Pisier result contains Grothendieck's "fundamental theorem on the metric theory of tensor products". We believe that our proof of the Haagerup-Pisier result is also new as a proof of Grothendieck's theorem.

In one way or another all proofs of Grothendieck's inequality are based on the following fact. If μ is a probability measure, if $q>2$, and if E is a subspace of $L^q(\mu)$ on which the L^q-norm and the L^2-norm are equivalent, then every $f \in E$ can be decomposed $f=g+h$ in such a way that $g \in L^\infty(\mu)$ is the "significant part" of f in the sense that the L^2-norm of $h \in L^2(\mu)$ is small. In the original proof h is simply forgotten, in the proofs based on the disc algebra one exploits the possibility of choosing g either analytic or antianalytic, and in proofs based on interpolation this decomposition is only implicit. For the original proof see Grothendieck [7], Lindenstrauss and Pełczynski [13], Lindenstrauss and Tsafriri [14]; for proofs using the disc algebra see Fournier [3] and Pełczynski [16]; and for proofs using interpolation see Krivine [11], Maurey [15], and Pisier [17]. Once one realises that the decomposition is a common feature of

Received June 20, 1983.

these proofs, then a simple way to obtain the decomposition is to truncate the function f at a suitable height, and this is what is done in our proof. The proofs based on interpolation really use the fact that this truncation can be done at all heights. One would expect that with this additional information proofs based on interpolation should give better estimates of the constants than those based on a single truncation. It is therefore somewhat surprising that in the non-commutative case our proof gives an estimate of the constant ($\leqq 81/8$) which is slightly better than Pisier's estimate ($\leqq 12$). A comparison of our proof with the proofs of Pisier and Haagerup shows certain similarities and differences. The same basic facts about C*-algebras are used in all proofs but like Haagerup our proof relies on C*-algebra techniques more heavily than Pisier's proof. We use spectral theory through the functional calculus for a hermitian element in a C*-algebra. Haagerup's detailed study of the imaginary part of the bilinear forms [8, Lemmas 3.2 and 3.3] is avoided in our proof, though his Lemma 3.1 is crucial in our proof. The basic version of the proof was circulated in 1980 in a preprint (Kaijser [10]). The new ingredient that gives Haagerup's result [8, Theorem 1.1] is the use of hermitian elements in the projective tensor products.

This use of hermitians suggests studying the nature of hermitian elements in the projective tensor product of two unital Banach algebras. Recall that an element h in a unital Banach algebra A is *hermitian* if and only if the *numerical range*

$$V(h) = \{f(h) : f \in A^*, \ \|f\| = f(1) = 1\}$$

is contained in the real line. If x and y are hermitian elements in unital complex Banach algebras A and B, respectively, then $x \otimes 1 + 1 \otimes y$ is a hermitian element in the projective tensor product $A \hat{\otimes} B$ of A and B. This is a trivial consequence of the isometric embeddings of A and B into $A \hat{\otimes} B$ given by $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$. In Theorem 3.1 we show that all the hermitians in $A \hat{\otimes} B$ are of this type provided A or B has the approximation property. The underlying reason for this result is that the unital Banach algebra $A \hat{\otimes} B$ has very many states and these states restrict the structure of the hermitian elements.

This theorem on hermitians in $A \hat{\otimes} B$ is closely related to the result that each hermitian operator T defined on $B(H)$, the space of all continuous linear operators on a Hilbert space H, may be written in the form $T = L_h + R_k$, where L_h is left multiplication on $B(H)$ by the hermitian element h and R_k is right multiplication by the hermitian element k [18, Remark 3.5]. The reason for the close relationship between these two identifications of hermitian operators is that for compact Hausdorff spaces Ω and Ψ the projective tensor algebra $C(\Omega) \hat{\otimes} C(\Psi)$ has a natural norm reducing bicontinuous monomorphism π into $B(B(H))$ for a suitable Hilbert space H. In Theorem 4.1 the map π is constructed and its bicontinuity is shown to be equivalent to Grothendieck's inequality. A more complicated non-commutative C*-algebra version of this theorem is proved in Theorem 5.2. As an application of the bicontinuity of π we show that the unital Banach subalgebra of $B(B(H))$ generated by the derivation formed from a hermitian element in $B(H)$ is isomorphic to a subalgebra of a projective tensor product of commutative C*-algebras (Corollary 4.6).

Let G br a (discrete) commutative group and let ψ be a faithful *-representation of $l^1(G)$ in $B(H)$ for some Hilbert space H. In 4.8 we show that if $\pi_1 \colon l^1(G) \to B(B(H))$ is defined by

$$(\pi_1(\alpha_g))T = \sum_{g \in G} \alpha_g \psi(g)^* T \psi(g)$$

for all $(\alpha_g) \in l^1(G)$ and $T \in B(H)$, then π_1 is a bicontinuous isomorphism with $\|\pi_1^{-1}\| \leqq K_G$. This result is a partial generalization to arbitrary representations of a theorem due to Størmer [20, Lemma 4.5] and Ghahramani [4, Theorem 5], who show that if ψ is the left regular representation of $l^1(G)$ on $l^2(G)$, then the representation π_1 above is isometric. Størmer [20] actually considers locally compact abelian groups, and Ghahramani [4] considers general locally compact groups; both replace $l^1(G)$ by the convolution measure algebra $M(G)$. Their methods of proof are different, and are different from our proof.

The relationship between the bicontinuity of π and Grothendieck's inequality was found independently by U. Haagerup (personal communication with the authors), and a special form of it occurs in M. Ljeskovac's thesis [12, Theorem 7.4]. We are grateful to G. Pisier for bringing to our attention reference [6] and the importance of Problem 3.3(b).

2. The Grothendieck-Pisier-Haagerup inequality.

Recall that the projective tensor norm $\|\cdot\|$ is defined on the algebraic tensor product $X \otimes Y$ of two Banach spaces X and Y by

$$\|u\| = \inf\left\{\sum_1^n \|x_j\| \cdot \|y_j\| : u = \sum_1^n x_j \otimes y_j\right\}$$

for all $u \in X \otimes Y$, and that the projective tensor product $X \hat{\otimes} Y$ is the completion of the normed space $(X \otimes Y, \|\cdot\|)$. Let A_h denote the set of hermitian elements in a C*-algebra A.

2.1. THEOREM. [Haagerup, 8].

(a) *There is a constant $K(\leqq 81/8)$ such that for each continuous bilinear form F on $A \times B$, where A and B are C*-algebras, there are states φ and ψ on A and B such that*

$$|F(a,b)| \leqq K\|F\|\varphi((a^*a+aa^*)/2)^{1/2}\cdot\psi((b^*b+bb^*)/2)^{1/2}$$

for all $a \in A$ and $b \in B$.

(b) *There is a constant $K(\leqq 81/8)$ such that for all $a_1,\ldots,a_n \in A_h$ and $b_1,\ldots,b_n \in B_h$, where A and B are C*-algebras*

$$\left\|\sum_1^n a_j\otimes b_j\right\| \leqq K\left\|\sum_1^n a_j^2\right\|^{1/2}\left\|\sum_1^n b_j^2\right\|^{1/2}.$$

We shall prove (b) in Lemmas 2.6 and 2.9, and shall sketch the equivalence of (a) and (b) in Remark 2.10(b) following Haagerup [8, Lemma 3.4]. The quantity on the right of the inequality in (b) may be used to define a tensor norm $\|\cdot\|_2$ on $A_h\otimes B_h$. In Lemma 2.6 we shall prove that the natural map ι from $A_h\hat{\otimes} B_h$ into the completion of $A_h\otimes B_h$ in this tensor norm $\|\cdot\|_2$ is onto. If either of the C*-algebras A or B satisfies the approximation property; then ι is one-to-one; thus Lemma 2.6 completes the proof of Theorem 2.1(b) in this case. In Lemma 2.9 we show that ι is one-to-one in general by using elementary properties of hermitian elements in a unital Banach algebra. If a is an element in C*-algebra, let the modulus of a be $|a|=((a^*a+aa^*)/2)^{1/2}$ (see Pisier [17, p. 397]). If A and B are C*-algebras, the norm $\|\cdot\|_2$ is defined on $A\otimes B$ by

$$\|u\|_2 = \inf\left\{\left\|\sum_1^n |a_j|^2\right\|^{1/2}\cdot\left\|\sum_1^n |b_j|^2\right\|^{1/2} : u = \sum_1^n a_j\otimes b_j\right\}$$

for all $u \in A\otimes B$, and $A\otimes_2 B$ denotes the completion of $A\otimes B$ in $\|\cdot\|_2$-norm. The following lemma ensures that the natural map ι from $A_h\hat{\otimes} B_h$ into $A_h\otimes_2 B_h$ is norm reducing. Lemma 2.9 shows that ι is one-to-one, and Lemma 2.6 shows that ι is onto and $\|\iota^{-1}\|\leqq 81/16$.

2.2. Lemma. *If A and B are C*-algebras then $4\|\cdot\|_2$ is an algebra norm on $A\otimes B$ satisfying $\|a\otimes b\|_2=\|a\|\cdot\|b\|$ for all $a \in A_h$ and $b \in B_h$. If $u \in A_h\otimes B_h$ then*

$$\|u\|_2 = \inf\left\{\left\|\sum_1^n a_j^2\right\|^{1/2}\cdot\left\|\sum_1^n b_j^2\right\|^{1/2} : u = \sum_1^n a_j\otimes b_j,\quad a_j \in A_h,\ b_j \in B_h\right\}.$$

Proof. We begin by proving the last equality. We define the involution $*$ on $A\otimes B$ by $(a\otimes b)^*=a^*\otimes b^*$, and observe that $u^*=u$ for all $u \in A_h\otimes B_h$.

Let $u \in A_h\otimes B_h$, and let

$$m = \inf\left\{\left\|\sum_1^n a_j^2\right\|^{1/2}\cdot\left\|\sum_1^n b_j^2\right\|^{1/2} : u = \sum_1^n a_j\otimes b_j,\quad a_j \in A_h,\ b_j \in B_h\right\}.$$

Then $\|u\| \leqq m$ because $|a|^2 = a^2$ for each hermitian element a in a C*-algebra. If

$$u = \sum_j^n (a_j + ic_j) \otimes (b_j + id_j)$$

with a_j, b_j, c_j, d_j hermitian, then

$$u = u^* = (u + u^*)/2 = \sum_1^n (a_j \otimes b_j - c_j \otimes d_j)$$

so that

$$\|\sum |a_j + ic_j|^2\| \cdot \|\sum |b_j + id_j|^2\| = \|\sum (a_j^2 + c_j^2)\| \cdot \|\sum (b_j^2 + d_j^2)\|$$

because $|a_j + ic_j|^2 = a_j^2 + c_j^2$. Thus $\|u\| \geqq m$, and $\|u\| = m$.

We now prove that $4\|\cdot\|_2$ is an algebra seminorm. By multiplying the first factor by a suitable $t > 0$ and the second factor by t^{-1}, we see that

$$(1)\quad \|u\|_2 = \inf\left\{\left\|\sum_1^n |a_j|^2\right\|^{1/2} \cdot \left\|\sum_1^n |b_j^2|\right\|^{1/2} : u = \sum_1^n a_j \otimes b_j,\ \left\|\sum_1^n |a_j|^2\right\| = \left\|\sum_1^n |b_j|^2\right\|\right\}$$

for all $u \in A \otimes B$. If $a_j, c_j \in A$ and $b_j, d_j \in B$ with

$$\|\sum |a_j|^2\| = \|\sum |b_j|^2\| \quad \text{and} \quad \|\sum |c_j|^2\| = \|\sum |d_j|^2\|,$$

then

$$\begin{aligned} &\|\sum |a_j|^2 + \sum |c_j|^2\|^{1/2} \cdot \|\sum |b_j|^2 + \sum |d_j|^2\|^{1/2} \\ &\leqq (\|\sum |a_j|^2\| + \|\sum |c_j|^2\|)^{1/2} \cdot (\|\sum |b_j|^2\| + \|\sum |d_j{}^2\|)^{1/2} \\ &= \|\sum |a_j|^2\|^{1/2} \cdot \|\sum |b_j|^2\|^{1/2} + \|\sum |c_j|^2\|^{1/2} \cdot \|\sum |d_j|^2\|^{1/2}. \end{aligned}$$

From this inequality and the equivalent definition of $\|\cdot\|_2$ given in (1), we see that $\|\cdot\|_2$ is a seminorm on $A \otimes B$.

Let $u = \sum a_j \otimes b_j$ and $v = \sum c_i \otimes d_i$. Then $uv = \sum a_j c_i \otimes b_j d_i$. From the standard C*-algebra inequality $c^* x c \leqq c^* c \|x\|$ for all hermitian x (see [17, p. 398]), we have

$$\begin{aligned} &\sum_{i,j} (c_i^* a_j^* a_j c_i + a_j c_i c_i^* a_j^*) \\ &\leqq \sum_i c_i^* c_i \left\|\sum_j a_j^* a_j\right\| + \sum_j a_j a_j^* \|\sum c_i c_i^*\| \\ &\leqq 2\left\|\sum_j (a_j^* a_j + a_j a_j^*)\right\| \cdot \left\|\sum_i (c_i^* c_i + c_i c_i^*)\right\|, \end{aligned}$$

because $x \leqq \|x+y\|$ and $\|x\| \leqq \|x+y\|$ for x and y positive elements in a C*-algebra.

Thus

$$\left\|\sum_{i,j} |a_j c_i|^2\right\| \leqq 4\left\|\sum_j |a_j|^2\right\| \cdot \left\|\sum_j |c_j|^2\right\|$$

so that

$$\|uv\|_2 \leqq 4\left\|\sum_j |a_j|^2\right\|^{1/2} \cdot \left\|\sum_j |b_j|^2\right\|^{1/2} \cdot \left\|\sum_i |c_i|^2\right\|^{1/2} \cdot \|\sum |d_i|^2\|^{1/2}.$$

Hence $\|uv\|_2 \leqq 4\|u\|_2 \cdot \|v\|_2$, and $4\|\cdot\|_2$ is an algebra seminorm on $A \otimes B$.

Let $a \in A_h$ and $b \in B_h$; then $\|a \otimes b\|_2 \leqq \| |a| \| \cdot \| |b| \| = \|a\| \cdot \|b\|$. Let f and g be states on A and B such that $|f(a)| = \|a\|$ and $|g(b)| = \|b\|$. If $a \otimes b = \sum a_j \otimes b_j$ with $a_j \in A_h$ and $b_j \in B_h$, then

$$\begin{aligned}\|a\| \cdot \|b\| &= |\sum f(a_j) g(b_j)| \\ &\leqq (\sum f(a_j)^2)^{1/2} (\sum g(b_j)^2)^{1/2} \\ &\leqq (\sum f(a_j^2))^{1/2} (\sum g(b_j^2))^{1/2}\end{aligned}$$

by the Cauchy Schwarz inequality

$$\leqq \|\sum a_j^2\|^{1/2} \|b_j^2\|^{1/2}.$$

Thus $\|a\| \cdot \|b\| \leqq \|a \otimes b\|_2$ and the proof is complete.

Is the $\|\cdot\|_2$-norm an algebra norm on $A \otimes B$ and is $\|a \otimes b\|_2 = \|a\| \cdot \|b\|$ for all $a \in A$ and $b \in B$?

Let $r_1, r_2, \ldots$ denote the sequence of Rademacher functions on $[0,1]$: note that the integrals in the following calculations are expectations over a finite probability space, because at each stage we are concerned only with $r_1, \ldots, r_n$, which are n independent random variables each taking values ± 1 with probability 1/2. The following Lemma is a C*-algebra version of a lemma used in Borel's proof of the strong law of large numbers, and is proved in Pisier [17, Lemma 1.1].

2.3. LEMMA. *If $a_1, \ldots, a_n$ are hermitian elements in a C*-algebra A, then*

$$\int \left|\sum_1^n r_j(t) a_j\right|^4 dt \leqq 3\left\|\sum_1^n a_j^2\right\| \cdot \left(\sum_1^n a_j^2\right).$$

Lemma 2.3 is used in the proof of the following truncation result. If x is a hermitian element in a C*-algebra and if f is a continuous function on the real line, then $f(x)$ denotes the element of the C*-algebra given by the action of f on x induced by the Gelfand-Naimark functional calculus. If $\tau > 0$, the truncation h_τ of the identity map, $\lambda(t) = t$ for all $t \in \mathsf{R}$, is defined by

$$h_\tau(t) = \begin{cases} \tau & \text{if } t \geqq \tau \\ t & \text{if } |t| \leqq \tau \\ -\tau & \text{if } t \leqq -\tau\,. \end{cases}$$

2.4. LEMMA. *Let $a_1, \ldots, a_n$ be hermitian elements in a C*-algebra, and let $a(t) = \sum_1^n r_j(t) a_j$ for all $t \in [0,1]$. If $\tau > 0$, then*

(i) $\displaystyle\int \{h_\tau(a(t))\}^2\,dt \leqq \sum_1^n a_j^2,$ *and*

(ii) $\displaystyle\int \{(\lambda - h_\tau)(a(t))\}^2\,dt \leqq \frac{3}{16\tau^2}\left\|\sum_1^n a_j^2\right\| \cdot \sum_1^n a_j^2\,.$

PROOF. (i) Since $h_\tau(t)^2 \leqq t^2$ for all $t \in \mathsf{R}$, we have $h_\tau(x)^2 \leqq x^2$ for all hermitian elements x. Thus

$$\begin{aligned}\int \{h_\tau(a(t))\}^2\,dt &\leqq \int a(t)^2\,dt \\ &= \sum_{i,j} a_i a_j \int r_i(t) r_j(t)\,dt \\ &= \sum_1^n a_j^2\,.\end{aligned}$$

(ii) The function $\lambda - h_\tau$ is skew-symmetric and satisfies

$$|(\lambda - h_\tau)(t)| = \begin{cases} t - \tau & \text{if } t \geqq \tau \\ 0 & \text{if } 0 \leqq t \leqq \tau\,. \end{cases}$$

Thus $|(\lambda - h_\tau)(t)| \leqq t^2/4\tau$ for all $t \in \mathsf{R}$ because

$$t^2/4\tau - (t - \tau) = (t - 2\tau)^2/4\tau \geqq 0 \qquad \text{for all } t \geqq \tau\,.$$

Hence $|(\lambda - h_\tau)(x)|^2 \leqq x^4/16\tau^2$ for all hermitian elements in a C*-algebra. Using this and Lemma 2.3 we have

$$\begin{aligned}\int \{(\lambda - h_\tau)(a(t))\}^2\,dt &\leqq \frac{1}{16\tau^2}\int a(t)^4\,dt \\ &\leqq \frac{3}{16\tau^2}\left\|\sum_1^n a_j^2\right\| \cdot \sum_1^n a_j^2\,.\end{aligned}$$

2.5. LEMMA. *Let A and B be C*-algebra and let $u \in A_h \otimes B_h$. If $\|u\|_2 < 1$, there are $v, w \in A_h \otimes B_h$ such that $u = v + w$, $\|v\| < 27/16$, and $\|w\|_2 < 2/3$.*

PROOF. By the equivalent definition of $\|u\|_2$ given in (1) of Lemma 2.2 there are $a_1, \ldots, a_n \in A_h$ and $b_1, \ldots, b_n \in B_h$ such that

$$u = \sum_1^n a_j \otimes b_j, \quad \left\|\sum_1^n a_j^2\right\| = \left\|\sum_1^n b_j^2\right\| = M < 1 .$$

As in Lemma 2.4 let

$$a(t) = \sum_1^n r_j(t) a_j \quad \text{and} \quad b(t) = \sum_1^n r_j(t) b_j$$

for all $t \in [0,1]$. Let $\tau > 0$, let

$$v = \int h_\tau(a(t)) \otimes h_\tau(b(t))\, dt ,$$

and let

$$w = \int \{(\lambda - h_\tau)(a(t)) \otimes h_\tau(b(t)) + a(t) \otimes (\lambda - h_\tau)(b(t))\}\, dt .$$

Then $v, w \in A_h \otimes B_h$ because the integrals are essentially finite averages over 2^n elements, and $u = v + w$ because

$$\begin{aligned} u &= \sum_{i,j} a_i \otimes b_j \int r_i(t) r_j(t)\, dt \\ &= \int a(t) \otimes b(t)\, dt . \end{aligned}$$

Using the definition of the projective norm and the observations that the integrals are finite averages, we have

$$\begin{aligned} \|v\| &\leqq \int \|h_\tau(a(t))\| \cdot \|h_\tau(b(t))\|\, dt \\ &\leqq \tau^2 , \end{aligned}$$

because $\|h_\tau(x)\| \leqq \tau$ for all hermitian x. Using the definition of $\|\cdot\|_2$ norm, we have

$$\|w\|_2 \leqq \left\|\int (\lambda - h_\tau)(a(t))^2\,dt\right\|^{1/2} \cdot \left\|\int h_\tau(b(t))^2\,dt\right\|^{1/2} + \left\|\int a(t)^2\,dt\right\|^{1/2} \cdot \left\|\int (\lambda - h_\tau)(b(t))^2\,dt\right\|^{1/2}$$

$$\leqq \frac{3^{1/2}}{4\tau}\|\textstyle\sum a_j^2\| \cdot \|\sum b_j^2\|^{1/2} + \|\sum a_j^2\|^{1/2} \cdot \frac{3^{1/2}}{4\tau}\|\sum b_j^2\|$$

$$\leqq \frac{3^{1/2}}{2\tau} M^{3/2}$$

by Lemma 2.4 and the choice of M. Taking $\tau = 3^{3/2}M^{1/2}/4$ gives $\|w\|_2 < 2/3$ and $\|v\| < 27/16$ as required. •

2.6. LEMMA. *Let A and B be C*-algebras and let $\iota\colon A_h \hat{\otimes} B_h \to A_h \otimes_2 B_h$ be the continuous linear operator defined by $\iota(a \otimes b) = a \otimes b$. If $u \in A_h \otimes_2 B_h$, then there is $z \in A_h \hat{\otimes} B_h$ with $\iota z = u$ and $\|z\| \leqq \frac{81}{16}\|u\|_2$. If A or B satisfy the approximation property, then ι is invertible with $\|\iota^{-1}\| \leqq 81/16$.*

PROOF. This lemma may be proved by a duality argument as in Kaijser [10, p. 6] or by the following standard elementary technique.

It is sufficient to show that for each $u \in A_h \otimes B_h$ with $\|u\|_2 < 1$, there is a $z \in A_h \hat{\otimes} B_h$ with $\|z\| < 81/16$ and $\iota z = u$. By induction using Lemma 2.5 we choose sequences $v_1, v_2, \ldots, w_0, w_1, w_2 \ldots$ in $A_h \otimes B_h$ such that $w_0 = u$ and $w_{n-1} = v_n + w_n$ with

$$\|v_n\| \leqq \tfrac{27}{16}\|w_{n-1}\|_2 \quad \text{and} \quad \|w_n\|_2 \leqq \tfrac{2}{3}\|w_{n-1}\|_2$$

for $n = 1, 2, \ldots$. Then

$$w_0 = u = \sum_1^n v_j + w_n, \quad \|v_n\| \leqq \tfrac{27}{16}(\tfrac{2}{3})^{n-1}, \quad \text{and} \quad \|w_n\| \leqq (\tfrac{2}{3})^n$$

for all n. If $z = \sum_1^\infty v_n$, then $z \in A_h \hat{\otimes} B_h$ with $\|z\| \leqq \frac{27}{16}(1 - \frac{2}{3})^{-1} = 81/16$ and $\iota z = u$.

If A or B satisfies the approximation property, then so does A_h or B_h as a real Banach space, because there is a continuous real projection from A onto A_h (and B onto B_h). Because $\|\cdot\|_2$ is a cross norm on $A_h \otimes B_h$, the map ι is one-to-one in this case.

2.7. LEMMA. *Let A and B be unital C*-algebras, and let F be a continuous bilinear form on $A \times B$ such that $\|F\| = F(1,1) = 1$. If $F(x, y)$ is real valued for all hermitian $x \in A$ and $y \in B$, then*

$$|F(a,b)| \leqq F(a^2, 1)^{1/2} \cdot F(1, b^2)^{1/2}$$

for all hermitian $a \in A$ and $b \in B$.

The proof of this useful elementary lemma is due to Haagerup [8, Lemma 3.1], and we omit the proof. The proof depends on expanding $F(\exp it\,a, \exp it\,b)$ as a power series in t and investigating the behaviour of the real part of the term in t^2 as t tends to zero using $\|\exp it\,a\| = 1 = \|\exp it\,b\|$ for all real t. From the identification between continuous bilinear forms on $A \times B$ and elements in the dual of $A \otimes B$, we see that the above lemma concerns a self-adjoint state F on $A \hat{\otimes} B$. A linear functional F on $A \hat{\otimes} B$ is self-adjoint if $F = F^*$, where $*$ is defined on $A \hat{\otimes} B$ by $(x \otimes y)^* = x^* \otimes y^*$, and $F^*(u) = \overline{F(u^*)}$ for all $u \in A \hat{\otimes} B$. The following corollary is just another way of wording Lemma 2.7 and is essentially Haagerup's Theorem [17, Theorem 1.1] for a self-adjoint state of $A \hat{\otimes} B$: the proof is part of [17, Lemma 3.6].

2.8. COROLLARY. *Let A and B be unital* C*-*algebras. If F is a self-adjoint state on $A \hat{\otimes} B$, then F is a continuous linear functional on $A \otimes_2 B$ with $\|F\|_2 \leqq 1$.*

PROOF. Let $x \in A$ and $y \in B$. By multiplying x by $e^{i\alpha}$ for suitable $\alpha \in \mathbf{R}$, we have $F(x, y) = \overline{F(x, y)} = F(x^*, y^*)$. If $x = a + ic$ and $y = b + id$ with a, b, c, d hermitian, then

$$F(x, y) = \{F(x, y) + F(x^*, y^*)\}/2 = F(a, b) - F(c, d)$$

so that

$$\begin{aligned} |F(x, y)| &\leqq \{F(a^2, 1)^{1/2} F(1, b^2)^{1/2} + F(c^2, 1)^{1/2} F(1, d^2)^{1/2}\} \\ &\leqq F(a^2 + c^2, 1)^{1/2} F(1, b^2 + d^2)^{1/2} \\ &= F(|x|^2, 1)^{1/2} F(1, |y|^2)^{1/2} . \end{aligned}$$

Hence

$$\begin{aligned} |F(\textstyle\sum x_j \otimes y_j)| &\leqq \textstyle\sum F(|x_j|^2, 1)^{1/2} F(1, |y_j|^2)^{1/2} \\ &\leqq (\textstyle\sum F(|x_j|^2, 1))^{1/2} \cdot (\textstyle\sum F(1, |y_j|^2))^{1/2} \\ &\leqq \|\textstyle\sum |x_j|^2\|^{1/2} \|\textstyle\sum |y_j|^2\|^{1/2} . \end{aligned}$$

This shows that $\|F\|_2 \leqq 1$.

In the proof of the following lemma we require the result that if h and ih^2 are hermitian elements of a unital Banach algebra, then $h = 0$. This may be deduced from the theorem that the norm of a hermitian element is its spectral radius [1, Theorem 11.17] or the result associated with states corresponding to the "end" of the numerical range of a hermitian element [2, Corollary 26.10]. In either case there is a state f on the Banach algebra such that $|f(h)| = \|h\|$ and $f(h^n) = f(h)^n$ for all $n \in \mathbf{N}$. Since $f(h)$ is real and $f(h^2)$ is imaginary by assumption, $\|h\|^2 = f(h)^2 = f(h^2) = 0$.

2.9. LEMMA. *If A and B are C*-algebras, the map $\iota: A\hat{\otimes}B \to A\otimes_2 B$ is one-to-one.*

PROOF. We adjoin identities to A and B if necessary. Note that $A\hat{\otimes}B$ is a closed subspace of $(A\oplus \mathbf{C}1)\hat{\otimes}(B\oplus \mathbf{C}1)$ because there is a natural projection from $A\oplus\mathbf{C}1$ onto A with kernel $\mathbf{C}1$ so that the map

$$A\hat{\otimes}B \to (A\oplus\mathbf{C}1)\hat{\otimes}(B\oplus\mathbf{C}1) \to (A\oplus\mathbf{C}1)/\mathbf{C}1\,\hat{\otimes}\,(B\oplus\mathbf{C}1)/\mathbf{C}1 \cong A\hat{\otimes}B$$

is the identity. Now let A and B be unital and let $u \in \operatorname{Ker}\iota$. Let F be a state on $A\hat{\otimes}B$ and let $G=(F+F^*)/2$. Then G is a self-adjoint state on $A\hat{\otimes}B$, so is a continuous linear functional on $A\otimes_2 B$ by Corollary 2.8. Hence $G\iota(u)=G(u)=0$ so that

$$F(u)+\overline{F(u^*)} = 0 \quad \text{and} \quad F(u+u^*)+\overline{F(u+u^*)} = 0\,.$$

Hence $i(u+u^*)$ is hermitian in $A\hat{\otimes}B$. Note that the involution $*$ defined on $A\otimes B$ by $(x\otimes y)^*=x^*\otimes y^*$ induces an isometric involution on $A\otimes_2 B$ because $|x^*|=|x|$ for all x. Thus $u^* \in \operatorname{Ker}\iota$. Now $(i(u+u^*))^2=-u^2-u^{2*}-uu^*-u^*u$ is i times a hermitian because the elements u^2+u^{2*}, uu^*, u^*u are self-adjoint in $A\hat{\otimes}B$ and are in $\operatorname{Ker}\iota$, since $\operatorname{Ker}\iota$ is an ideal in $A\hat{\otimes}B$ – recall $4\|\cdot\|_2$ is an algebra norm on $A\otimes B$. By the note before this lemma it follows that $u+u^*=0$ so that $u=0$ on replacing u by iu. This proves Lemma 2.9 and completes the proof of Theorem 2.1(b).

2.10. REMARKS. (a) In the proof of Lemma 2.9, we are essentially proving that each element in $\operatorname{Ker}\iota$ is the sum of a hermitian element plus i times a hermitian element. The Vidav-Palmer Theorem [1, Theorem 38.14] then ensures that $\operatorname{Ker}\iota$ is a C*-algebra. This and the observation that the square of a hermitian in $\operatorname{Ker}\iota$ is i times a hermitian provides another proof that ι is one-to-one.

(b) Here we remark on the equivalence of 2.1 (a) and (b). If $a_1,\dots,a_n \in A_h$, $b_1,\dots,b_n \in B_h$, and $u=\sum a_j\otimes b_j$, then there is a continuous bilinear form F on $A\times B$ such that $\|F\|=1$ and $F(u)=\|u\|$. If φ and ψ are the states given by Theorem 2.1(a), then

$$\begin{aligned}\|u\| &\leqq \textstyle\sum |F(a_j,b_j)|\\ &\leqq K\textstyle\sum \varphi(a_j^2)^{1/2}\psi(b_j^2)^{1/2}\\ &\leqq K(\textstyle\sum \varphi(a_j^2))^{1/2}\cdot(\textstyle\sum \psi(b_j^2))^{1/2}\end{aligned}$$

by the Cauchy-Schwarz inequality. This shows that 2.1(a) implies 2.1(b). That 2.1(b) implies 2.1(a) is Lemma 3.4 in Haagerup [8] and the argument moving from hermitians to general elements given in [8, Lemma 3.6] and Corollary 2.8.

3. Hermitian elements in the projective tensor product.

If A and B are unital Banach algebras, then $x\otimes 1$ and $1\otimes y$ are hermitian in $A\hat{\otimes}B$ for all hermitian $x\in A$ and $y\in B$ because $a\mapsto a\otimes 1: A\to A\hat{\otimes}B$ and $b\mapsto 1\otimes b: B\to A\hat{\otimes}B$ are norm reducing unital homomorphisms. In the following result we show that all hermitians in $A\otimes B$ are a sum of these two types provided the linear embedding of $A\otimes B$ into $C(\Omega\times\Psi)$ is one-to-one, where Ω is the state space of A and Ψ that of B. The philosophy behind the proof is that $A\hat{\otimes}B$ has so many states that the structure of the hermitian elements is severely restricted.

Recall that a *state* on a unital Banach algebra is a continuous linear functional f such that $\|f\|=f(1)=1$, and that the *state space* is the set of all states on the algebra.

3.1. THEOREM. *Let A and B be unital Banach algebras such that the canonical map from the projective tensor product $A\hat{\otimes}B$ into the injective tensor product $A\check{\otimes}B$ is one-to-one. If u is a hermitian element in the projective tensor product $A\hat{\otimes}B$, then there are hermitian elements x in A and y in B with $u=x\otimes 1+1\otimes y$.*

PROOF. Let Ω be the state space of A and Ψ be the state space of B each with the weak $*$-topology, and let $\theta: A\hat{\otimes}B\to C(\Omega\times\Psi)$ be defined by $\theta(a\otimes b)(\omega,\psi)=\omega(a)\psi(b)$. We begin by proving that

$$\theta u(\omega_1,\psi_1)+\theta u(\omega_2,\psi_2) = \theta u(\omega_1,\psi_2)+\theta u(\omega_2,\psi_1) \tag{1}$$

for all $\omega_1,\omega_2\in\Omega$ and $\psi_1,\psi_2\in\Psi$, where u is the hermitian element in $A\hat{\otimes}B$.

Suppose that equality (1) does not hold for some $\omega_1,\omega_2\in\Omega$ and $\psi_1,\psi_2\in\Psi$. Let F be the continuous linear function defined on $A\hat{\otimes}B$ by

$$F(v) = \frac{(1+i)}{4}\{\theta v(\omega_1,\psi_1)-i\theta v(\omega_1,\psi_2)-i\theta v(\omega_2,\psi_1)+\theta v(\omega_2,\psi_2)\} \tag{2}$$

for all $v\in A\hat{\otimes}B$. Then the imaginary part

$$\operatorname{Im}F(u) = \tfrac{1}{4}\{\theta u(\omega_1,\psi_1)-\theta u(\omega_1,\psi_2)-\theta u(\omega_2,\psi_1)+\theta u(\omega_2,\psi_2)\}$$

of $F(u)$ is non-zero. Once we have proved that $F(u)$ is a state on $A\hat{\otimes}B$ this shows that u is not hermitian, which is the required contradiction. Clearly $F(1\otimes 1)=1$. The matrix

$$\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

considered as an operator from the 2-dimensional l^∞-space l_2^∞ into the 2-dimensional l^1-space l_2^1 has norm equal to $2\sqrt{2}$. This may be easily observed by considering what this matrix does to the unit sphere in l^∞, which is partly a geometrical square. Alternatively one may evaluate the norm analytically by examining the behaviour of the matrix on the extreme points of l_2^∞. Regarded as an operator from l_2^∞ into l_2^1 the matrix

$$\frac{(1+i)}{4}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

has norm 1. From the definition of F in (2) and of θ, it follows that

$$F(a\otimes b) = \frac{(1+i)}{4}\begin{pmatrix}\omega_1(a)\\ \omega_2(a)\end{pmatrix}^T\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix}\psi_1(b)\\ \psi_2(b)\end{pmatrix}$$

so that $|F(a\otimes b)| \leqq \|a\|\cdot\|b\|$ for all $a \in A$ and $b \in B$. Hence F is a state on $A\hat{\otimes}B$ and we have shown that θu satisfies (1).

Fix $\omega_1 \in \Omega$ and $\psi_1 \in \Psi$, and define

$$P : A\hat{\otimes}B \to A : a\otimes b \mapsto \psi_1(b)a ,$$

and

$$Q : A\hat{\otimes}B \to B : a\otimes b \mapsto \omega_1(a)b .$$

Then $\|P\| \leqq 1$, $\|Q\| \leqq 1$, $P(1\otimes 1) = 1$. Using the state definition of numerical range it follows that P and Q map hermitian elements in $A\hat{\otimes}B$ into hermitian elements in A and B, respectively. This is a folklore result that norm reducing operators mapping the identity to the identity preserve hermitians: it follows from the observation that if ω is a state on A, then ωP is a state on $A\hat{\otimes}B$ so is real valued on hermitian elements. Let $x = Pu$ and $y = Qu - \theta u(\omega_1, \psi_1)1$. Then x and y are hermitian, and

$$\begin{aligned}\theta(u - x\otimes 1 - 1\otimes y)(\omega, \psi) \\ &= \theta u(\omega,\psi) - \theta u(\omega,\psi_1) - \theta u(\omega_1,\psi) + \theta u(\omega_1,\psi_1) \\ &= 0\end{aligned}$$

for all $\omega \in \Omega$ and $\psi \in \Psi$ by (1). The set of states on a unital Banach algebra generates the dual space as a linear space [2, Theorem 31.1], and thus θ is one-to-one because the natural map from $A\hat{\otimes}B$ into $A\check{\otimes}B$ is one-to-one. This completes the proof.

If A is a unital Banach algebra, let $U(A)$ denote the unitary group

$$\{u \in A : u^{-1} \in A,\ \|u\| = \|u^{-1}\| = 1\} ,$$

and let $U_0(A)$ denote the subgroup of $U(A)$ generated by the set $\{\exp ih : h \in A,$ h hermitian$\}$.

3.2. Corollary. *Let A and B unital Banach algebras. If A or B has the approximation property, then*

$$U_0(A\hat{\otimes}B) = U_0(A)\otimes U_0(B) = \{u\otimes v : u \in U_0(A),\ v \in U_0(B)\} .$$

The corollary follows directly from Theorem 3.1. If A and B are C*-algebras, then the embedding θ of the projective tensor product $A\hat{\otimes}B$ into the injective tensor product $A\check{\otimes}B$ is one-to-one by Haagerup [8, Proposition 2.2].

3.3. Problems. (a) If A and B are unital Banach algebras, are all hermitians in $A\hat{\otimes}B$ of the form $x\otimes 1+1\otimes y$, where $x \in A$ and $y \in B$ are hermitian?

(b) An element x in a unital Banach algebra A is called *hermitian equivalent* if there is an equivalent Banach algebra norm on A for which x is hermitian, or equivalently if the set $\{\|\exp itx\| : t \in \mathbf{R}\}$ is bounded. If A and B are unital Banach algebras such that the natural map from $A\hat{\otimes}B$ into $A\check{\otimes}B$ is one-to-one, and if A and B have no non-trivial projections, is each hermitian equivalent element in $A\hat{\otimes}B$ of the form $x\otimes 1+1\otimes y$, where x and y are hermitian equivalent elements in A and B?

Note that some restriction on A and B is required here. For example, if A and B are finite dimensional semisimple commutative Banach algebras of finite dimensions m and n, respectively, then $A\hat{\otimes}B$ is a C*-algebra in an equivalent norm, because it too is a finite dimensional commutative semisimple Banach algebra. The real linear dimensions of the real linear spaces of hermitian equivalent elements in A, B, and $A\hat{\otimes}B$ are m, n, and mn, respectively; note that the commutativity of the algebras ensures that the hermitian equivalent elements form a real linear space. However the dimension of the real linear space of elements of the form $x\otimes 1+1\otimes y$, where x and y are hermitian equivalent in A and B, respectively, is $m+n$.

Note that problem 3(b) is related to problem 13.19 of [6, p. 412] (and see also [22]).

(c) What additional hypotheses on A and B are required to ensure that

$$U(A\hat{\otimes}B) = U(A)\otimes U(B)?$$

(d) What are the hermitians in the injective tensor product $A\check{\otimes}B$, when $A\check{\otimes}B$ is a Banach algebra? Under what addition additional conditions on A and B does it follows that $(A\check{\otimes}B)_h = A_h\check{\otimes}B_h$?

4. Representations of continuous bilinear forms on $C(\Omega)$.

In this section we investigate a form of Grothendieck's inequality associated with the representation of $C(\Omega)\hat{\otimes}C(\Psi)$ as a closed subalgebra of $B(B(H))$ for suitable Hilbert space H. This result, Theorem 4.1, we prove in detail proving the required lemmas in full so that this section of the paper may be read without reference to general results on C*-algebras. In section 5 we rapidly study the corresponding problem for general C*-algebras. Theorem 4.1 is strong enough to bring out the close relationship between the structure of the hermitian elements in $C(\Omega)\hat{\otimes}C(\Psi)$ (Theorem 3.1) and in $B(B(H))$. We discuss this point in detail in Remark 4.5(a). Theorem 4.1 may also be used to show that the unital Banach subalgebras of $B(B(H))$ generated by *-automorphisms on $B(H)$ and by *-derivation are bicontinuously isomorphic to subalgebras of $C(\sigma(T))\hat{\otimes}C(\sigma(T))$, where T is the operator on H defining the automorphism or derivation (Corollary 4.6).

Recall that Grothendieck's inequality states that there is a constant K such that if F is a continuous bilinear form on $C(\Omega)\times C(\Psi)$, then there are probability measures μ and ν on the compact Hausdorff spaces Ω and Ψ, respectively, such that

$$|F(x,y) \leqq K\mu(|x|^2)^{1/2}\cdot \nu(|y|^2)^{1/2}$$

for all $x \in C(\Omega)$ and $y \in C(\Psi)$ (see [7] and [13]). The least such constant K_G over all F, Ω, Ψ is called (the complex) Grothendieck's constant. An equivalent form of the inequality is that there exists a constant K such that

$$\left\|\sum_1^n f_j\otimes g_j\right\|^2 \leqq K^2\left\|\sum_1^n |f_j|^2\right\|\cdot\left\|\sum_1^n |g_j|^2\right\|$$

for all $f_1,\ldots,f_n \in C(\Omega)$ and $g_1,\ldots,g_n \in C(\Omega)$. The least constants in these two forms of Grothendieck's inequality are the same. This is essentially proved in [13] and is in Grothendieck [7] in different notation.

4.1. THEOREM. *Let Ω and Ψ be compact Hausdorff spaces, and let θ and φ be (continuous) faithful unital *-representations of $C(\Omega)$ and $C(\Psi)$ on a Hilbert space H. Let*

$$\pi : C(\Omega)\hat{\otimes}C(\Psi) \to B(B(H))$$

be defined by $\pi(f\otimes g)T=\theta(f)T\theta(g)$ for all $f\in C(\Omega)$, $g \in C(\Psi)$, and $T\in B(H)$. Then π is a monomorphism from $C(\Omega)\hat{\otimes}C(\Psi)$ into $B(B(H))$ with $\|\pi\|\leqq 1$ and $\|\pi^{-1}\|\leqq K_G$, and K_G is the smallest constant K in general in the inequality $\|\pi^{-1}\| \leqq K$.

As we remarked we shall prove two little lemmas relating general states on $C(\Omega)$ with vector states arising from the representation in order to make the proof of Theorem 4.1 self-contained. Lemma 4.2 is an easy special case of a result of J. Glimm which is used in his C*-algebra Stone-Weierstrass Theorem

[5]. The idea is to approximate convex combinations of pure states on $C(\Omega)$ (=unit point masses on Ω) by states arising from orthogonal vectors in the Hilbert space. If for a state (=probability measure) μ on $C(\Omega)$ there is a unit vector ξ in the Hilbert space H such that $\mu(x)=\langle\theta(x)\xi,\xi\rangle$ for all $x \in C(\Omega)$, then μ is said to be a vector state on $C(\Omega)$ arising from the representation θ.

4.2. LEMMA. *Let Ω be a compact Hausdorff space and let θ be a unital faithful *-representation of $C(\Omega)$ on a Hilbert space H. Then the set of vector states arising from the representation θ is weak *-dense in the set of all states on $C(\Omega)$.*

PROOF. By the Krein-Milman Theorem the set of all convex combinations of pure states on $C(\Omega)$ is weak *-dense in the set of all states — a pure state is just an extreme point in the set of all states. Further a pure state on $C(\Omega)$ is the functional of evaluation at some fixed point of Ω. Let $\omega_1,\ldots,\omega_n \in \Omega$ and $\beta_1,\ldots,\beta_n>0$ with $\sum_1^n \beta_j=1$. Let $\varepsilon>0$ and $f_1,\ldots,f_m \in C(\Omega)$ with $\|f_k\|\leqq 1$ for all k. Then $\omega_1,\ldots,\omega_n, \beta_1,\ldots,\beta_n$ determine the convex combination of pure states which we shall approximate by a vector state within a neighbourhood determined by $\varepsilon, f_1,\ldots,f_n$. Let $\delta>0$ be small with $(3\delta-\delta^2)(1-\delta)^{-1}<\varepsilon$, and let V_j be disjoint neighbourhoods of ω_j. For each j choose g_j such that $0\leqq g_j\leqq 1$, $g_j(\omega_j)=1$, g_j is supported by V_j, and $\|(f_k-f_k(\omega_j)1)g_j\|<\delta$ for $k=1,\ldots,m$. The spectrum of g_j is real, and 1 is in the spectrum of g_j so 1 is in the approximate point spectrum of $\theta(g_j)$. Thus there is a $\zeta_j \in H$ with $\|\zeta_j\|=1$ and $\|(\theta(g_j)-1)\zeta_j\|<\delta$ for $j=1,\ldots,n$. Let

$$\xi_j = \theta(g_j)\zeta_j\cdot\|\theta(g_j)\zeta_j\|^{-1}$$

for each j. Then

$$\|(\theta(g_j)-1)\xi_j\| \leqq \delta(1-\delta)^{-1} .$$

Hence

$$\begin{aligned}
&|\langle\theta(f_k)\xi_j,\xi_j\rangle - f_k(\omega_j)|\\
&= |\langle(\theta(f_k)-f_k(\omega_j)1)\xi_j,\xi_j\rangle|\\
&\leqq 2\|f_k\|\cdot\|(\theta(g_j)-1)\xi_j\| + |\langle(\theta(f_k)-f_k(\omega_j)1)\theta(g_j)\xi_j,\xi_j\rangle|\\
&\leqq 2\|(\theta(g_j)-1)\xi_j\| + \|(f_k-f_k(\omega_j)1)g_j\|\\
&< 2\delta(1-\delta)^{-1}+\delta\\
&< \varepsilon
\end{aligned}$$

for all j and k. Further $\langle\xi_j,\xi_k\rangle=0$ for $j\neq k$, because ξ_j is in the range of the self-adjoint operator $\theta(g_j)$ and the supports of the functions g_j and g_k are disjoint.

Thus

$$\langle\theta(f_k)(\sum \beta_j^{1/2}\xi_j), (\sum \beta_j^{1/2}\xi_j)\rangle = \sum \beta_j\langle\theta(f_k)\xi_j, \xi_j\rangle$$

is within ε of $\sum_j \beta_j f_k(\omega_j)$ because $\sum \beta_j = 1$. This proves the lemma.

4.3. LEMMA. *Let Ω be a compact Hausdorff space and let θ be a unital faithful $*$-representation of $C(\Omega)$ on a Hilbert space H. For each state μ on $C(\Omega)$, each $\varepsilon > 0$, and each finite dimensional subspace M of $C(\Omega)$, there is unit vector ξ in H such that $\mu(|x|^2) \leqq (1+\varepsilon)\ \langle\theta(|x|^2)\xi, \xi\rangle$ for all $x \in M$.*

PROOF. Because M is finite dimensional there is a state ν on $C(\Omega)$ such that $x \in M$ and $\nu(|x|^2) = 0$ implies that $x = 0$: the state ν may be taken to be a strict convex combination of a finite number of pure states (=unit point masses) on $C(\Omega)$. Let

$$\mu_1 = (1+\varepsilon)^{-1/2}\mu + (1-(1+\varepsilon)^{-1/2})\nu .$$

The function $x \to \mu_1(|x|^2)^{1/2}$ is a norm on M and so is equivalent to the $\|\cdot\|_\infty$-norm M inherits as a subspace of $C(\Omega)$. Thus

$$S = \{x \in M : \mu_1(|x|^2) = 1\}$$

is a compact subset of M. Let $\delta > 0$ be such that $(1-3\delta)^{-1} = (1+\varepsilon)^{1/2}$.

Then there are $f_1, \ldots, f_n \in S$ such that

$$S \subseteqq \bigcup_1^n \{g \in M : \||f_j|^2 - |g|^2\| < \delta\} .$$

By Lemma 4.2 there is a $\xi \in H$ with $\|\xi\| = 1$ such that

$$\mu_1(|f_j|^2) \leqq \langle\theta(|f_j|^2)\xi, \xi\rangle + \delta$$

for $1 \leqq j \leqq n$. If $g \in S$, we choose f_j so that $\||f_j|^2 - |g|^2\| < \delta$ and then

$$\begin{aligned} 1 &= \mu_1(|g|^2) \\ &\leqq \mu_1(|f_j|^2) + \delta \\ &\leqq \langle\theta(|f_j|^2)\xi, \xi\rangle + 2\delta \\ &\leqq \langle\theta(|g|^2)\xi, \xi\rangle + 3\delta . \end{aligned}$$

Thus

$$\begin{aligned} \mu(|g|^2) &\leqq (1+\varepsilon)^{1/2}\mu_1(|g|^2) \\ &\leqq \frac{(1+\varepsilon)^{1/2}}{1-3\delta}\langle\theta(|g|^2)\xi, \xi\rangle \\ &\leqq (1+\varepsilon)\langle\theta(|g|^2)\xi, \xi\rangle . \end{aligned}$$

This proves the lemma.

4.4. PROOF OF THEOREM 4.1. The properties of π all follow directly from its definition except for π being an injection and $\|\pi^{-1}\| \leqq K_G$. To prove these properties it is sufficient to show that for an algebraic tensor $u \in C(\Omega) \otimes C(\Psi)$ and an $\varepsilon > 0$ there is a $T \in B(H)$ such that $\|T\| \leqq (1+\varepsilon)K_G$ and $\|\pi(u)T\| \geqq \|u\|$. By the Hahn-Banach Theorem and the identification of the dual of $C(\Omega) \hat{\otimes} C(\Psi)$ with the space of continuous bilinear forms, there is a bilinear form F on $C(\Omega) \times C(\Psi)$ with $\|F\| = 1$ and $F(u) = \|u\|$. By Grothendieck's inequality there are probability measures μ and ν on Ω and Ψ, respectively, such that

$$|F(x,y)| \leqq K_G \mu(|x|^2)^{1/2} \nu(|y|^2)^{1/2}$$

for all $x \in C(\Omega)$ and $y \in C(\Psi)$. If $u = \sum_1^n x_j \otimes y_j$, let M_1 be the linear span of $x_1, \ldots, x_n$ in $C(\Omega)$ and M_2 be the linear span of $y_1, \ldots, y_n$ in $C(\Psi)$. By Lemma 4.3 there are unit vectors ξ and η in H such that

$$\mu(|x|^2) \leqq (1+\varepsilon)^{1/2} \langle \theta(|x|^2)\eta, \eta \rangle$$

and

$$\nu(|y|^2) \leqq (1+\varepsilon)^{1/2} \langle \varphi(|y|^2)\xi, \xi \rangle$$

for all $x \in M_1$ and $y \in M_2$. Thus

$$|F(x,y)| \leqq K_G(1+\varepsilon)\|\theta(x)^*\eta\| \cdot \|\varphi(y)\xi\|$$

for all $x \in M_1$ and $y \in M_2$.

If $H_1 = \theta(M_1)^*\eta$ and $H_2 = \varphi(M_2)\xi$, then there is a linear operator T_0 from H_2 into H_1 such that $\|T_0\| \leqq (1+\varepsilon)K_G$ and $F(x,y) = \langle T_0\varphi(y)\xi, \theta(x)^*\eta \rangle$ for all $x \in M_1$ and $y \in M_2$. Let P be the orthogonal projection from H onto H_2, and let $T = T_0 P$. Then $\|T\| \leqq (1+\varepsilon)K_G$, and

$$\begin{aligned} \|u\| = F(u) &= \sum_1^n F(x_j, y_j) \\ &= \sum_1^n \langle \theta(x_j) T \theta(y_j)\xi, \eta \rangle \\ &= \langle (\pi(u)T)\xi, \eta \rangle \\ &\leqq \|\pi(u)T\| . \end{aligned}$$

This shows that π^{-1} exists and that $\|\pi^{-1}\| \leqq K_G$.

To show that the least constant K in Theorem 4.1 is Grothendieck's constant we shall prove that

$$\left\|\sum_1^n x_j\otimes y_j\right\|^2 \leqq K^2\left\|\sum_1^n |x_j|^2\right\|\cdot\left\|\sum_1^n |y_j|^2\right\|$$

for all $x_j \in C(\Omega)$ and $y_j \in C(\Psi)$. From this it follows that $K \geqq K_G$ because this is one of the standard equivalent forms of Grothendieck's inequality. Let $u = \sum_1^n x_j\otimes y_j$ and let $\varepsilon > 0$. Then there are $T \in B(H)$ and unit vectors $\xi, \eta \in H$ such that $\|T\| = 1$ and

$$\|u\| \leqq (K+\varepsilon)|\langle(\pi(u)T)\xi, \eta\rangle| .$$

Thus

$$\begin{aligned}\left\|\sum_1^n x_j\otimes y_j\right\|^2 &\leqq (K+\varepsilon)^2\left(\sum_1^n |\langle\theta(x_j)T\varphi(y_j)\xi, \eta\rangle|\right)^2\\ &\leqq (K+\varepsilon)^2\left(\sum_1^n \|\theta(x_j)^*\eta\|\cdot\|\varphi(y_j)\xi\|\right)^2\\ &\leqq (K+\varepsilon)^2\left(\sum_1^n \|\theta(x_j)^*\eta\|^2\right)\left(\sum_1^n \|\theta(y_j)\xi\|^2\right)\end{aligned}$$

by the Cauchy-Schwarz inequality,

$$\begin{aligned}&\leqq (K+\varepsilon)^2\left\langle\sum_1^n \theta(|x_j|^2)\eta, \eta\right\rangle\left\langle\sum_1^n \varphi(|y_j|^2)\xi, \xi\right\rangle\\ &\leqq (K+\varepsilon)^2\left\|\sum_1^n |x_j|^2\right\|\cdot\left\|\sum_1^n |y_j|^2\right\| .\end{aligned}$$

This completes the proof of Theorem 4.1.

4.5. REMARKS. (a) Let Ω and Ψ be compact Hausdorff spaces and let π be the representation of $C(\Omega)\hat{\otimes}C(\Psi)$ defined in Theorem 4.1 corresponding to the unital faithful *-representations θ and φ of $C(\Omega)$ and $C(\Psi)$ on a Hilbert space H. Since $\pi(1\otimes 1) = 1$ and $\|\pi\| = 1$, π maps the space of hermitian elements in $C(\Omega)\hat{\otimes}C(\Psi)$ into the space of hermitian elements in $B(B(H))$ (see the proof of Theorem 3.1). The hermitian operators on a unital C*-algebra are each a sum of left multiplication by a hermitian in the algebra and a hermitian derivation [18, Remark 3.5]. Because the derivations on $B(H)$ are all inner, the hermitians in $B(B(H))$ are of the form $L_h + R_h$, where h and k are hermitian elements in $B(H)$, and $L_h x = hx$ and $R_k x = xk$ for all $x \in B(H)$. Thus a hermitian element u in $C(\Omega)\hat{\otimes}C(\Psi)$ gives $\pi(u) = L_h + R_k$ for some hermitians h and k in $B(H)$. It is then straight forward to show that $h \in \theta(C(\Omega))$ and $k \in 0(C(\Psi))$. This gives the close relationship between hermitians in $C(\Omega\hat{)}C(\Psi)$ and $B(B(H))$ mentioned earlier.

(b) A corollary of Theorem 4.1 is that the unital Banach subalgebra of $B(B(H))$ generated by a hermitian is semisimple (see [9, Theorem 1]). This is because a hermitian in $B(B(H))$ is of the form $L_h + R_k$, where h and k are hermitians in $B(H)$, and so the unital Banach algebra generated by L_h and R_k is bicontinuously isomorphic to a subalgebra of $C(\sigma(h)) \hat{\otimes} C(\sigma(k))$ – take θ and φ in Theorem 4.1 to be the *-monomorphisms given by the Gelfand-Naimark Theorem. A consequence of this semi-simplicity is that there are hermitians in unital Banach algebras which cannot be bicontinuously represented in $B(B(H))$. The reason is that there are hermitians, which generated non-semi-simple Banach algebras (see [19]).

(c) Using the technique of the proof of Theorem 4.1 one can show that, if F is a continuous bilinear form on $C(\Omega) \times C(\Psi)$, then there exist cyclic representations θ of $C(\Omega)$ and φ of $C(\Psi)$ on Hilbert spaces H and K with unit cyclic vectors η and ξ, respectively, and a continuous linear operator T from K into H such that $\|T\| \leqq KG\|F\|$ and $F(x,y) = \langle \theta(x)T\theta(y)\xi, \eta \rangle$ for all $x \in C(\Omega)$ and $y \in C(\Psi)$. This result was found independently by U. Haagerup (personal communication to the authors) and M. Ljeskovac [12], and may be known to others.

(d) The following corollary of Theorem 4.1 identifies the unital Banach algebra generated by a *-derivation on $B(H)$, and there is a corresponding corollary for *-automorphisms. Corollary 4.6 implies that the unital Banach algebra generated by a *-derivation on $B(H)$ arising from a hermitian element T is completely determined by the spectrum of T (up to translation).

4.6. Corollary. *Let H be a Hilbert space and let T be a continuous hermitian operator on H with spectrum $\sigma(T)$. Let D be the derivation on $B(H)$ defined by $D(x) = Tx - xT$ for all $x \in B(H)$, and let A be the unital Banach subalgebra of $B(B(H))$ generated by D. There is a continuous unital isomorphism π from the closed subalgebra of $C(\sigma(T)) \hat{\otimes} C(\sigma(T))$ generated by $z \otimes 1 - 1 \otimes z$ onto A such that $\pi(z \otimes 1 - 1 \otimes z) = D$, $\|\pi\| = 1$, and $\|\pi^{-1}\| \leqq K_G$, where $z(t) = t$ for all $t \in \sigma(T)$.*

Proof. The Gelfand-Naimark continuous functional calculus for the hermitian element T in $B(H)$ gives a unital continuous faithful *-representation θ of $C(\sigma(x))$ in $B(H)$ with $\theta(z) = T$. The result follows directly from this and Theorem 4.1.

4.7. Problem. There is a natural norm reducing unital homomorphism χ from the extremal algebra $\mathrm{Ea}[-1,1]$ defined in numerical range (see [2, Section 24] for a definition) into $C[0,1] \hat{\otimes} C[0,1]$ such that the generator u of

Ea$[-1,1]$ is mapped to $z\otimes 1-1\otimes z$, because the numerical range of $z\otimes 1-1\otimes z$ is $[-1,1]$. Does χ have a continuous inverse?

The following corollary is a partial generalization of a result in F. Ghahramani [4] and E. Størmer [20, Proposition 4.6] concerning the representation of $L^1(G)$ in $B(B(L^2(G)))$ induced by the left regular representation of G on $L^2(G)$. In the corollary below the result is for discrete groups but the representation is any faithful representation rather than the left regular representation.

4.8. COROLLARY. *Let G be a commutative discrete group and let ψ: $l^1(G) \to B(H)$ be a faithful *-representation of $l^1(G)$ in $B(H)$. If π_1: $l^1(G) \to B(B(H))$ is defined by*

$$\pi_1((\alpha_g))T = \sum_{g\in G} \alpha_g \psi(\delta_g)^* T\psi(\delta_g)$$

for all $(\alpha_g) \in l^1(g)$ and $T \in B(H)$, where δ_g is the delta function, then π_1 is a norm reducing monomorphism with $\|\pi_1^{-1}\| \leqq K_G$.

PROOF. The idea is to use the natural isometry $l^1(G) \to A(\hat{G}) \to C(\hat{G})\hat{\otimes} C(\hat{G})$ studied by Varapoulos [21, p. 96] combined with the natural representation of $C(\hat{G})$ in $B(H)$ and Theorem 4.1. Here $\hat{G}$ denotes the dual group of G, and $\hat{G}$ is compact.

The norm reducing *-representation ψ induces a *-representation θ of $C(\hat{G})$ into $B(H)$ with the property that $\theta((\alpha_g)\hat{\ }) = \psi((\alpha_g))$ for all $(\alpha_g) \in l^1(G)$. The reason for this is that ψ is continuous in the spectral radius norm of $l^1(G)$, and that $C(\hat{G})$ is the completion of $l^1(G)$ in the spectral radius norm—an equivalent form is that $C(\hat{G})$ is the enveloping C*-algebra of $l^1(G)$. Let

$$\pi : C(\hat{G})\hat{\otimes} C(\hat{G}) \to B(B(H))$$

be the monomorphism defined in Theorem 4.1 from the faithful *-representation θ: $C(\hat{G}) \to B(H)$ by $\pi(f\otimes g)T = \theta(f)T\theta(g)$. Let

$$M : l^1(G) \cong A(\hat{G}) \to C(\hat{G})\hat{\otimes} C(\hat{G})$$

be the Varapoulos map defined by

$$(M(\alpha_g))(x,y) = (\alpha_g)\hat{\ }(x+y)$$

for all $x, y \in \hat{G}$ and $(\alpha_g) \in l^1(G)$, where $\hat{\ }$ is the Fourier transform from $l^1(\hat{G})$ into $C(G)$ (onto $A(G)$). The map M is initially defined into $C(\hat{G}\times\hat{G})$ but is in fact into $C(\hat{G})\hat{\otimes} C(\hat{G})$ and is isometric (see Varapoulos [21, p. 96] and [6, p. 309]). Let χ: $C(\hat{G}) \to C(\hat{G})$ be the dual of the inverse in the group: $\chi(f)(x) = f(-x)$

for all $f \in C(\hat{G})$ and all $x \in \hat{G}$. Then $\chi \otimes I$ is an isometry on $C(\hat{G}) \hat{\otimes} C(\hat{G})$, and so $M_1 : l^1(G) \to C(\hat{G}) \hat{\otimes} C(\hat{G})$ defined by $M_1 = (\chi \otimes I)M$ is isometric. Because $\|\pi^{-1}\| \leqq K_G$ by Theorem 4.1, the proof will be complete if we show that $\pi_1 = \pi M_1$. For $\delta_g \in l^1(G)$, we have

$$(M_1 \delta_g)(x, y) = \overline{x(g)} \cdot y(g) = \hat{\delta}_{-g} \otimes \hat{\delta}_g(x, y)$$

so that

$$\begin{aligned}(\pi M_1)(\delta_g)(T) &= \pi(\hat{\delta}_{-g} \otimes \hat{\delta}_g)T \\ &= \theta(\hat{\delta}_{-g}) T \theta(\hat{\delta}_g) \\ &= \psi(\delta_{-g})^* T \psi(\delta_g) \\ &= \pi_1(\delta_g) T\end{aligned}$$

for $T \in B(H)$.

4.9. Problem. (a) Is the map π_1 in Corollary 4.1 isometric?

(b) Let G be a locally compact group, let ψ be a strongly continuous unitary representation of G in $B(H)$, and let $\pi_1 : L^1(G) \to B(B(H))$ be defined by

$$\pi_1(f)T = \int_G f(x) \psi(x)^* T \psi(x) \, d\lambda(x)$$

for all $f \in L^1(G)$ and $T \in B(H)$, where λ is left Haar measure on G. Then π_1 is a norm reducing homomorphism, and let J denote the kernel of π_1. Under what conditions on G and ψ does π_1 drop to a bicontinuous monomorphism from $L^1(G)/J$ into $B(B(H))$?

5. Representation of bilinear forms on a C*-algebra.

The occurrence of the products xx^* and x^*x in the Grothendieck-Pisier-Haagerup inequality (Theorem 2.1) means that the homomorphism in the representation of a bilinear form on a commutative C*-algebra must be replaced by a Jordan homomorphism in general (Theorem 5.2). The action of the C*-algebra on the left of the Hilbert space is no longer sufficient and it is essential to consider action on the right too. There are several equivalent ways of wording this. A representation of a C*-algebra A in $B(H)$, for a Hilbert space H, turns H into a left Banach A-module with the involution in the algebra and module marching. The converse holds. An antirepresentation θ from A into $B(H)$ may be defined as a representation of the reversed algebra A^r, where the product $\circ$ in A^r is defined by $a \circ b = ba$, in $B(H)$; or we may regard H as a right Banach A-module with the involution in A matching that in $B(H)$.

Recall that a vector state φ on a C*-algebra A arising from a representation θ (or antirepresentation) of A on a Hilbert space H is of the form $\varphi(a) = \langle\theta(a)\xi, \xi\rangle$ for some unit vector $\xi \in H$. The hypothesis in the following lemma and theorem that each state on A may be weakly approximated by vector states arising from the representation θ (or antirepresentation) is satisfied if $\theta(A) \cap \mathrm{CL}(H) = \{0\}$ by Glimm's Lemma (see [5]), where CL (H) is the algebra of compact operators on H.

5.1. LEMMA. *Let A be a C*-algebra and let θ be a faithful representation of A on a Hilbert space H such that each state on A is in the weak *-closure of the set of vector states arising from the representation θ. If $\varepsilon > 0$, if M is a finite dimensional linear subspace of A, and if μ is a state on A, then there are unit vectors ξ and $\eta \in H$ such that*

$$\mu(x^*x) \leqq (1+\varepsilon)\langle\theta(x^*x)\xi, \xi\rangle \quad \textit{and}$$
$$\mu(xx^*) \leqq (1+\varepsilon)\langle\theta(xx^*)\eta, \eta\rangle \quad \textit{for all } x \in M .$$

The proof is the same as that of Lemma 4.3 provided that the modulus in the proof is interpreted as $(x^*x)^{1/2}$ when constructing ξ and as $(xx^*)^{1/2}$ when constructing η—rather than our usual modulus $((xx^*+x^*x)/2)^{1/2}$.

5.2. THEOREM. *Let A be a C*-algebra, let θ be a representation of A on a Hilbert space H, and let ψ be a representation of the reversed algebra A^r on a Hilbert space K. Let*

$$\pi\colon A \hat{\otimes} A \to B(B(H \oplus K))$$

be defined by

$$\pi(x \otimes y)T = (\theta \oplus \psi)(x)T(\theta \oplus \psi)(y)$$

*for all $x, y \in A$ and $T \in B(H \oplus K)$. Then π is a continuous Jordan homomorphism with $\|\pi\| \leqq 1$. If each state in A is in the weak *-closure of the set of vector states on A arising from θ and in the weak *-closure of those arising from ψ, then $\|\pi^{-1}\| \leqq 2$.*

PROOF. The direct sum in the statement is the Hilbert space direct sum so $\|\theta + \psi\| = 1$, and π is norm reducing. The map π is a Jordan homomorphism because θ is a representation and ψ is an anti-representation of A.

Now suppose that the state space of A is the weak *-closure of the set of vector states of A arising from θ and from ψ. Let $u \in A \otimes A$ with $\|u\| = 1$, and let $F \in (A \hat{\otimes} A)^*$ such that $F(u) = \|u\|$ and $\|F\| = 1$. By the Grothendieck-Pisier-Haagerup inequality (see Haagerup [8, Theorem 1.1] or Theorem 2.1) there are states φ_1, φ_2, ψ_1, ψ_2 on A such that

$$|F(x,y)| \leqq (\varphi_1(x^*x)+\varphi_2(xx^*))^{1/2}\cdot(\psi_1(y^*y)+\psi_2(yy^*)^{1/2}$$

for all $x, y \in A$. If $u=\sum_1^n x_j \otimes y_j$, let M_1 be the linear span of $x_1,\ldots,x_n$ and M_2 be the linear span of $y_1,\ldots,y_n$. By Lemma 5.1 there are unit vectors $\xi_1,\xi_2 \in H$ and $\eta_1,\eta_2 \in K$ such that

$$\varphi_1(x^*x) \leqq (1+\varepsilon)\langle\theta(xx^*)\xi_1,\xi_1\rangle\ ,$$

$$\varphi_2(xx^*) \leqq (1+\varepsilon)\langle\psi(x^*x)\eta_1,\eta_1\rangle\ ,$$

$$\psi_1(y^*y) \leqq (1+\varepsilon)\langle\theta(y^*y)\xi_2,\xi_2\rangle, \quad \text{and}$$

$$\psi_2(yy^*) \leqq (1+\varepsilon)\langle\psi(yy^*)\eta_2,\eta_2\rangle$$

for all $x \in M_1$ and $y \in M_2$.

Let

$$H_1 = \{(\theta\oplus\psi)(x)^*(\xi_1\oplus\eta_1) : x \in M_1\}\ ,$$

and

$$H_2 = \{(\theta\oplus\psi)(y)(\xi_2\oplus\eta_2) : y \in M_2\}\ .$$

Then

$$|F(x,y)| \leqq (\varphi_1(x^*x)+\varphi_2(xx^*))^{1/2}\cdot(\psi_1(y^*y)+\psi_2(yy^*))^{1/2}$$
$$\leqq (1+\varepsilon)\|(\theta\oplus\psi)(x^*)(\xi_1\oplus\eta_1)\|\cdot\|(\theta\oplus\psi)(y)(\xi_2\oplus\eta_2)\|$$

for all $x \in M_1$ and $y \in M_2$. Thus there is a continuous linear operator T_0 from H_2 into H_1 such that $\|T_0\| \leqq 1+\varepsilon$ and

$$F(x,y) = \langle T_0(\theta\oplus\psi)(y)(\xi_2\oplus\eta_2),\ (\theta\oplus\psi)(x^*(\xi_1\oplus\eta_1))\rangle$$

for all $x \in M_1$ and $y \in M_2$. Let P be the orthogonal projection from $H\oplus K$ onto H_1, and let $T=T_0P$. Then $\|T\| \leqq 1+\varepsilon$, and

$$\|u\| = F(u) = \sum_1^n F(x_j,y_j)$$
$$= \langle\pi(u)T(\xi_2\oplus\eta_2),(\xi_1\oplus\eta_1)\rangle$$
$$\leqq 2\|\pi(u)T\|\ .$$

Hence π^{-1} exists and $\|\pi^{-1}\| \leqq 2$. This completes the proof.

5.3. Remarks. (a) There is a result corresponding to Remark 4.5(d) that applies to general C*-algebras and its proof is a minor modification of part of the proof of Theorem 5.2. If F is a continuous bilinear form on a C*-algebra A, then there are cyclic representations θ_1 and θ_2 [anti-representations ψ_1 and

ψ_2] on Hilbert spaces H_1 and H_2 [K_1 and K_2] with unit cyclic vectors ξ_1 and ξ_2 [η_1 and η_2] and a continuous linear operator T: $H_1 \oplus K_1 \to H_2 \oplus K_2$ with $\|T\| \leqq \|F\|$ such that

$$F(x,y) = \langle (\theta_2 \oplus \psi_2)(x) T(\theta \oplus \psi)(y)(\xi_1 \oplus \eta_1), \xi_2 \oplus \eta_2 \rangle$$

for all $x, y \in A$.

(b) Let A and B be C*-algebras. We shall define four sets of bilinear forms LL, LR, RL, RR on $A \times B$ as follows. A continuous bilinear form F is in class LL, LR, RL, RR if and only if there are states φ on A and ψ on B such that

$$\text{(LL)} \quad |F(x,y)| \leqq \|F\| \varphi(x^*x)^{1/2} \psi(y^*y)^{1/2}$$

$$\text{(LR)} \quad |F(x,y)| \leqq \|F\| \varphi(x^*x)^{1/2} \psi(yy^*)^{1/2}$$

$$\text{(RL)} \quad |F(x,y)| \leqq \|F\| \varphi(xx^*)^{1/2} \psi(y^*y)^{1/2}$$

$$\text{(RR)} \quad |F(x,y)| \leqq \|F\| \varphi(xx^*)^{1/2} \psi(yy^*)^{1/2}$$

for all $x \in A$ and $y \in B$. An equivalent way to formulate this is in terms of representations, either cyclic ones or the universal representations.

Let θ_A and θ_B denote the universal representations of A and B on Hilbert spaces H_A and H_B, and let ψ_A and ψ_B denote the universal representations of the reversed algebras A^r and B^r on Hilbert spaces K_A and K_B, respectively. Then a continuous bilinear form F on $A \times B$ is in the respective class if and only if there is a continuous linear operator T with $\|T\| = \|F\|$ and unit vectors ξ and η such that

Class	T	$\xi \in$	$\eta \in$	$F(x,y) =$
RL	$H_B \to H_A$	H_B	H_A	$\langle \theta_A(x) T \theta_B(y) \xi, \eta \rangle$
RR	$K_B \to H_A$	K_B	H_A	$\langle \theta_A(x) T \psi_B(y) \xi, \eta \rangle$
LL	$H_B \to K_A$	H_B	K_A	$\langle \psi_A(x) T \theta_B(y) \xi, \eta \rangle$
LR	$K_B \to K_A$	K_B	K_A	$\langle \psi_A(x) T \psi_B(y) \xi, \eta \rangle$

for all $x \in A$ and $y \in B$.

Note that the compactness of the state spaces of A and B shows that the four classes are all closed in $(A \hat{\otimes} B)^*$.

Writing the operator T occuring in Theorem 5.2 and Remark (a) in matrix form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

the bilinear form F on $A \times A$ with $\|F\| = 1$ may be written in the form

$$F(x,y) = \langle \theta(x)T_{11}\theta(y)\xi, \xi\rangle + \langle \theta(x)T_{12}\psi(y)\eta, \xi\rangle + \langle \psi(x)T_{21}\theta(y)\xi, \eta\rangle + \langle \psi(x)T_{22}\psi(y)\eta, \eta\rangle$$

for all $x, y \in A$. Thus Haagerup's Theorem [8, Theorem 1.1] may be worded as follows: if F is a continuous bilinear form on $A \times A$ with $\|F\| = 1$ then there are four bilinear forms $F_{\mathrm{LL}} \in \mathrm{LL}$, $F_{\mathrm{LR}} \in \mathrm{LR}$, $F_{\mathrm{RL}} \in \mathrm{RL}$, $F_{\mathrm{RR}} \in \mathrm{RR}$ each of norm $\leqq 1$ such that

$$F = F_{\mathrm{LL}} + F_{\mathrm{LR}} + F_{\mathrm{RL}} + F_{\mathrm{RR}}\,.$$

5.4. PROBLEM. Under what additional conditions on the norms of F_{LL}, F_{LR}, F_{RL}, F_{RR} and the algebras A and B, is this decomposition unique in some fashion? Clearly the decomposition is highly non-unique for commutative algebras but this is because $\mathrm{LL} = \mathrm{LR} = \mathrm{RL} = \mathrm{RR}$.

REFERENCES

1. F. F. Bonsall and J. Duncan, *Complete normed algebras* (Ergeb. Math. Grenzgeb. 80), Springer-Verlag, Berlin - Heidelberg - New York, 1973.
2. F. F. Bonsall and J. Duncan, *Numerical ranges* II (London Math. Soc. Lecture Ser. 10), Cambridge University Press, Cambridge, 1973.
3. J. F. Fournier, *On a theorem of Paley and the Littlewood conjecture*, Ark. Mat. 17 (1979), 199–216.
4. F. Ghahramani, *Isometric representation of $M(G)$ on $B(H)$*, Glasgow Math. J. 23 (1982), 119–122.
5. J. Glimm, *A Stone-Weierstrass theorem for C*-algebras*, Ann. of Math. 72 (1960), 885–911.
6. C. C. Graham and O. C. McGhee, *Essays in commutative harmonic analysis* (Grundlehren Math. Wiss. 238), Springer-Verlag, Berlin - Heidelberg - New York, 1979.
7. A. Grothendieck, *Résumé de la théoriè métrique des produits tensoriels topologiques*, Bol. Soc. Brasil Mat. (1956), 1–79.
8. U. Haagerup, *The Grothendieck inequality for bilinear forms on C*-algebras*, (to appear).
9. H. Halpern, *Harmonic analysis of abelian inner automorphism groups of von Neumann algebras*, Math. Scand. 50 (1982), 79–99.
10. S. Kaijser, *A simple-minded proof of the Pisier-Grothendieck inequality*, Report No. 7, Dept. of Math. Uppsala University, 1980.
11. J. L. Krivine, *Théorèmes de factorisations dans les espace réticulés*, Sém. Maurey-Schwartz 1973–1974, exp. XXII–XXIII. École Polytechnique, Centre de Math., Paris.
12. M. Ljeskovac, *The projective tensor product of commutative Banach algebras*, Ph. D. Thesis, Univ. of Edinburgh, 1981.
13. J. Lindenstrauss and A. Pełczynski, *Absolutely p-summing operators in L_p-spaces and their applications*, Studia Math. 29 (1968), 275–326.
14. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* (Lecture Notes in Math. 338), Springer-Verlag, Berlin - Heidelberg - New York, 1977.
15. B. Maurey, *Une nouvelle démonstration d'un théorème de Grothendieck*, Sém. Maurey-Schwartz 1973–1973, exp. XXII, École Polytechnique, Centre de Math., Paris.
16. A. Pełczynski, *Banach spaces of analytic functions and absolutely summing operators*, (Expository lectures from the CBMS Regional Conference, Kent State Univ., Kent, Ohio,

1976. CBMS Regional Conf. Ser. in Math. 30), American Mathematical Society, Providence, R.I., 1977.

17. G. Pisier, *Grothendieck's theorem for non-commutative* C*-*algebras with an appendix on Grothendieck's constants*, J. Funct. Anal. 29 (1978), 397–415.
18. A. M. Sinclair, *Jordan homomorphisms and derivations on semi-simple Banach algebras*, Proc. Amer. Math. Soc. 24 (1970), 209–214.
19. A. M. Sinclair, *The Banach algebra generated by a Hermitian operator*, Proc. London Math. Soc. (3) 24 (1972), 681–691.
20. E. Størmer, *Regular Abelian Banach algebras of linear maps of operator algebras*, J. Funct. Anal. 37 (1980), 331–373.
21. N. T. Varopoulos, *Tensor algebras and harmonic analysis*, Acta Math. 119 (1967), 51–112.
22. N. T. Varopoulos, *On the endomorphism problem for tensor algebras*, Prakt. Akad. Athēnōn 47 (1972), 193–197.

MATEMATISKA INSTITUTIONEN
UPPSALA UNIVERSITET
THUNBERGSVÄGEN 3
S-752 38 UPPSALA
SWEDEN

AND

DEPARTMENT OF MATHEMATICS
JAMES CLERK MAXWELL BUILDING
KING'S BUILDINGS
EDINBURGH EH9 3JZ
SCOTLAND