MATH. SCAND. 55 (1984), 201–215

# TRIPLE PRODUCTS IN THE STEENROD ALGEBRA

JAMES H. FIFE

## 1. Introduction.

In this paper we shall study triple products in the Steenrod algebra. Let $p$ be a prime number, and let $\mathscr{A} = \mathscr{A}(p)$ be the mod-$p$ Steenrod algebra. If $\alpha, \beta, \gamma \in \mathscr{A}$ with $\beta\alpha = 0$ and $\gamma\beta = 0$, then the triple product $\langle \gamma, \beta, \alpha \rangle$ is defined and is an element of $\mathscr{A}/(\gamma\mathscr{A} + \mathscr{A}\alpha)$.

Triple products in the Steenrod algebra were introduced by Leif Kristensen in 1967 ([4]–[8]) to help describe the action of the Steenrod algebra on the cohomology of a two-stage Postnikov system. Since the early 1970's little seems to have been done with these triple products. In this paper we shall propose a new definition of triple products in the Steenrod algebra, based on some work of E. Spanier [10]. Using our definition, we are then able to obtain some new results and provide new proofs of some old results of Kristensen. Our first main result (Theorem 3.1) shows, for stable two-stage Postnikov systems $E$, how the action of $\mathscr{A}(p)$ on $H^*(E; \mathbf{Z}_p)$ is related to the triple product structure of $\mathscr{A}(p)$. Let $\chi$: $\mathscr{A}(p) \to \mathscr{A}(p)$ be the canonical antiautomorphism of the Steenrod algebra [11]. Our second main result (Theorem 4.2) asserts that if $\alpha, \beta, \gamma \in \mathscr{A}(p)$ are such that $\beta\alpha = 0$ and $\gamma\beta = 0$, then

$$\chi\langle \gamma, \beta, \alpha \rangle = \langle \chi(\alpha), \chi(\beta), \chi(\gamma) \rangle .$$

As corollaries of these two results, we are able to compute $\langle \gamma, \beta, \alpha \rangle$ in the case $p = 2$ and $\alpha$ or $\gamma = \mathrm{Sq}^1$ or $\mathrm{Sq}^2$.

Let $A$, $B$, $C$, and $D$ be pointed spaces, and let $f$: $A \to B$, $g$: $B \to C$, and $h$: $C \to D$ be maps such that the compositions $gf$ and $hg$ are null-homotopic. Then the triple product $\langle h, g, f \rangle$ is defined modulo a certain indeterminancy; it is a map from $\Sigma A$, the (reduced) suspension of $A$, into $D$. The definition and basic properties of $\langle h, g, f \rangle$ are due to Spanier [10], who generalized earlier work of Toda [12]. In the next section we shall generalize Spanier's constructions to the stable category, and in section 3 we shall use these results to define triple products in the Steenrod algebra and to compute $\langle \gamma, \beta, \alpha \rangle$ when $\alpha = \mathrm{Sq}^1$ or $\mathrm{Sq}^2$. Finally, in section 4 we shall use a result of E. Brown and M. Comenetz [3] to prove the theorem about the canonical antiautomorphism and to compute $\langle \gamma, \beta, \alpha \rangle$ when $\gamma = \mathrm{Sq}^1$ or $\mathrm{Sq}^2$.

Received April 28, 1983; in revised form June 23, 1983.

The results in this paper formed a portion of the author's doctoral dissertation, written at Yale University under Professor W. S. Massey. I would like to thank Professor Massey and Professor W. G. Dwyer for their considerable help.

## 2. Triple Products in the Stable Category.

Let $A$, $B$, $C$, and $D$ be spectra (in the sense of Adams [1]), and let $f\colon A \to B$, $g\colon B \to C$, and $h\colon C \to D$ be maps of degrees $n$, $m$, and $r$ respectively. Suppose $gf \cong *$ and $hg \cong *$; that is, $gf$ and $hg$ are null-homotopic. Note that $gf$ is a map of degree $n+m$ and $hg$ a map of $m+r$. We shall define the triple product $\langle h, g, f\rangle$. This will be a homotopy class of maps from $A$ to $D$ of degree $n+m+r+1$, defined modulo a certain indeterminancy.

First, let $E$ be a spectrum. There is a map $\sigma\colon \operatorname{Susp} E \to E$ of degree $-1$, defined as follows:

$$\sigma_n : (\operatorname{Susp} E)_n = S^1 \wedge E_n \approx E_n \wedge S^1 = \Sigma E_n \xrightarrow{(-1)^n \varepsilon_n} E_{n+1} .$$

(See [1, pp. 151–152] for the relevant definitions.) It is easy to see that $\sigma$ is a map of spectra. Note that the image of $\sigma$ is a cofinal subspectrum of $E$. Note also that if $E'$ is a cofinal subspectrum of $E$, then $\operatorname{Cone} E'$ and $\operatorname{Susp} E'$ are cofinal subspectra of $\operatorname{Cone} E$ and $\operatorname{Susp} E$, respectively.

Suppose that $f, g\colon \operatorname{Cone} E \to F$ are two maps of degree $r$ such that $f \mid E = g \mid E$. We define a map $\delta(f, g)\colon E \to F$ of degree $r+1$ in the following way: Let $E'$ be a cofinal subspectrum of $E$, and let $f'$ and $g'$ be functions from $\operatorname{Cone} E' \to F$ representing $f$ and $g$. Then for each $n$, $f'_n\colon I \wedge E'_n \to F_{n-r}$ and $g'_n\colon I \wedge E'_n \to F_{n-r}$ are such that $f'_n(1,x) = g'_n(1,x)$ for all $x \in E'_n$. Then, as in [10], we can define a function

$$d(f'_n, g'_n) : S^1 \wedge E_n \to F_{n-r}$$

by the rule

$$d(f'_n, g'_n)(t,x) = \begin{cases} f'_n(2t,x) & \text{if } 0 \leqq t \leqq \frac{1}{2} \\ g'_n(2-2t,x) & \text{if } \frac{1}{2} \leqq t \leqq 1 . \end{cases}$$

Furthermore, the following diagram commutes:

$$\begin{array}{ccc} \Sigma(S^1 \wedge E'_n) & \xrightarrow{\Sigma d(f'_n, g'_n)} & \Sigma F_{n-r} \\ {\scriptstyle 1 \wedge \varepsilon_n}\downarrow & & \downarrow{\scriptstyle \varphi_{n-r}} \\ S^1 \wedge E'_{n+1} & \xrightarrow[d(f'_{n+1}, g'_{n+1})]{} & F_{n-r+1} \end{array}$$

Thus we get a function $d(f', g')\colon \operatorname{Susp} E' \to F$, of degree $r$. Since $\operatorname{Susp} E'$ is cofinal in $\operatorname{Susp} E$, we therefore get a map $\delta_1(f, g) : \operatorname{Susp} E \to F$, of degree $r$.

Finally, since the image of $\sigma$: Susp $E \to E$ is cofinal in $E$, it follows that we have a map $\delta(f, g): E \to F$ of degree $r+1$, such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Susp} E & \xrightarrow{\delta_1} & F \\ \sigma\downarrow & \nearrow_{\delta} & \\ E & & \end{array} .$$

This operation $\delta$ has the following properties:

1. Let $f, g, h$: Cone $E \to F$ be three maps of degree $r$ such that $f \mid E = g \mid E = h \mid E$. Then

$$\delta(f, h) = \delta(f, g) + \delta(g, h) .$$

2. Let $f, g$: Cone $E \to F$ be two maps of degree $r$ such that $f \mid E = g \mid E$, and let $h$: $F \to X$ be another map. Then

$$\delta(hf, hg) = h\delta(f, g) .$$

3: Let $f, g$: Cone $E \to F$ be as above, and let $h$: $X \to E$ be any map. Then

$$\delta(f(\operatorname{Cone} h), g(\operatorname{Cone} h)) = \delta(f, g)h .$$

4. Let $f, g$: Cone $E \to F$ both be the constant map. Then $\delta(f, g)$ is the constant map.

Let $A$, $B$, $C$, and $D$ be spectra, and let $f$: $A \to B$, $g$: $B \to C$, and $h$: $C \to D$ be maps of degrees $n$, $m$, and $r$ respectively, such that $gf \cong *$ and $hg \cong *$. Since $gf \cong *$ there is a map $F$: Cone $A \to C$, of degree $n+m$, such that $F \mid A = gf$. Similarly, since $hg \cong *$, there is a map $G$: Cone $B \to D$, of degree $m+r$ such that $G \mid B = hg$. Consider the maps $hF$: Cone $A \to D$ and $G(\operatorname{Cone} f)$: $A \to D$. These are maps of degree $n+m+r$. Furthermore, $hF \mid A = hgf$ and $G(\operatorname{Cone} f) \mid A = hgf$. Thus there is a map

$$\delta(hF, G(\operatorname{Cone} f)) : A \to D$$

of degree $n+m+r+1$. Define $\langle h, g, f \rangle$ to be the homotopy class of this map. Thus $\langle h, g, f \rangle \in [A, D]_{n+m+r+1}$.

The definition of $\langle h, g, f \rangle$ depends on the choice of maps $F$ and $G$. However, as in the unstable case ([10, p. 268]), if $F_1$: Cone $A \to C$ and $G_1$: Cone $B \to D$ are other maps such that $F_1 \mid A = gf$ and $G_1 \mid B = hg$, then

$$\delta(hF_1, G_1(\operatorname{Cone} f)) = h_*(\alpha) + \langle h, g, f \rangle + f^*(\beta)$$

for some $\alpha \in [A, C]_{n+m+1}$ and $\beta \in [B, D]_{m+r+1}$; conversely, for each such $\alpha$ and $\beta$ there are maps $F_1$ and $G_1$ such that $F_1 \mid A = gf$ and $G_1 \mid B = hg$.

This triple product has the following properties:

THEOREM 2.1. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *be maps of spectra such that* $gf \simeq *$ *and* $hg \simeq *$. *Let* $E_f = B \cup_f \mathrm{Cone}\, A$ *be the mapping cone of* $f$ ([1, p. 154]) *and consider the following diagram*:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ & & & \searrow^{i} & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle a} \\ & & & & E_f & \xrightarrow[j]{} & A\,. \end{array}$$

(1) *There are maps* $b\colon E_f \to C$ *and* $a\colon A \to D$ *such that this diagram is homotopy commutative.*
(2) *The map* $a$ *in* (1) *is not unique; however, the homotopy class* $[a] \in [A, D]_*$ *is a well-defined element of the group* $[A, D]_*/(h_*[A, C]_* + f^*[B, D]_*)$.
(3) *As an element of this group,* $[a] = \langle h, g, f\rangle$.

PROOF. For (1), see [10, Lemma 3.1, p. 269]. For (2), let $b_1$ and $a_1$ be other maps such that the following diagram is homotopy commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ & & & \searrow^{i} & \uparrow{\scriptstyle b_1} & & \uparrow{\scriptstyle a_1} \\ & & & & E_f & \xrightarrow[j]{} & A\,. \end{array}$$

We shall show that $[a_1] = h_*[\bar{a}] + [a] + f^*[c]$ for some $[\bar{a}] \in [A, C]$ and $[c] \in [B, D]$.

Consider the following diagram:

$$\begin{array}{ccccc} [B, C] & \xleftarrow{i^*} & [E_f, C] & \xleftarrow{j^*} & [A, C] \\ & & \downarrow{\scriptstyle h_*} & & \\ & & [E_f, D] & \xleftarrow{j^*} & [A, D] & \xleftarrow{f^*} & [B, D]\,. \end{array}$$

Now $[b_1] \in [E_f, C]$, and $i^*[b_1] = [g] = i^*[b]$. Thus $i^*([b_1] - [b]) = 0$. Therefore $[b_1] - [b] = j^*[\bar{a}]$ for some $[\bar{a}] \in [A, C]$. So $[b_1] = [\bar{a}j] + [b]$.

Now $[a_1] \in [A, D]$, and a straightforward calculation shows that $j^*[a_1] = j^*([h\bar{a}] + [a])$. Thus $j^*([a_1] - [h\bar{a}] - [a]) = 0$. Therefore $[a_1] - [h\bar{a}] - [a] = f^*[c]$ for some $[c] \in [B, D]$. So $[a_1] = h_*[\bar{a}] + [a] + f^*[c]$.

Conversely, let $[\bar{a}] \in [A, C]$ and $[c] \in [B, D]$ be arbitrary, and let

$$[a_1] = h_*[\bar{a}] + [a] + f^*[c] \quad \text{and} \quad [b_1] = j^*[\bar{a}] + [b]\,.$$

Then $[b_1 i] = i^*[b_1] = [g]$, so $b_1 i \simeq g$. Also, another straightforward calculation shows that $[a_1 j] = [hb_1]$, so $a_1 j \simeq hb_1$. Thus the following diagram is homotopy commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ & & & {\scriptstyle i}\searrow & \uparrow{\scriptstyle b_1} & & \uparrow{\scriptstyle a_1} \\ & & & & E_f & \xrightarrow[j]{} & A\,. \end{array}$$

This proves (2).

For (3), let $F$: Cone $A \to C$ and $G$: Cone $B \to D$ be such that $F \mid A = gf$ and $G \mid B = hg$. Recall that $E_f = B \cup_f \text{Cone}\, A$. Then there is a map $b$: $E_f \to C$ such that $b \mid \text{Cone}\, A = F$ and $b \mid B = g$. Then we have the following diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ & & & {\scriptstyle i}\searrow & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle \delta(hF,\, G(\text{Cone}\, f))} \\ & & & & E_f & \xrightarrow[j]{} & A\,. \end{array}$$

The triangle is commutative because $b \mid B = g$. So to prove the theorem, it suffices to prove that the square is homotopy commutative.

To this end, it suffices to show that the following square is homotopy commutative:

$$\begin{array}{ccc} C & \xrightarrow{h} & D \\ {\scriptstyle b}\uparrow & & \uparrow{\scriptstyle \delta_1} \\ E_f & \xrightarrow[j]{} & \text{Susp}\, A \end{array}$$

For this, see [10, Theorem 3.3, pp. 269–270].

The proof of the following theorem is similar:

THEOREM 2.2. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *be as in Theorem* 2.1. *Let* $E_h$ *be the mapping cone of* $h$, *and consider the following diagram*:

$$\begin{array}{ccccccc} D & \xrightarrow{i} & E_h & & & & \\ {\scriptstyle a}\uparrow & & {\scriptstyle b}\uparrow & \searrow{\scriptstyle j} & & & \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C & \xrightarrow[h]{} & D\,. \end{array}$$

(1) *There are maps* $b$: $B \to E_h$ *and* $a$: $A \to D$ *such that this diagram is homotopy commutative.*

(2) *The homotopy class* $[a] \in [A, D]_*$ *is a well-defined element of the group* $[A, D]_*/(h_*[A, C]_* + f^*[B, D]_*)$.

(3) *As an element of this group,*

$$[a] = -\langle h, g, f \rangle\,.$$

The following corollary of Theorem 2.1 is analogous to Corollary 3.4 of [10]:

COROLLARY 2.3. *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {}_{h}\nwarrow \quad \swarrow_{g} & \\ & C & \end{array}$$

*be an exact triangle of spectra. Then the triple product* $\langle h, g, f\rangle$ *is defined and equals the homotopy class of the identity map* $A \to A$ *as an element of the group* $[A, A]_*/(h_*[A, C]_* + f^*[B, A]_*)$.

PROOF. Since

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {}_{h}\nwarrow \quad \swarrow_{g} & \\ & C & \end{array}$$

is an exact triangle, there is a homotopy equivalence $b\colon E_f \to C$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A \\ & & & {}_{i}\searrow & \uparrow{b} & & \uparrow{1} \\ & & & & E_f & \longrightarrow & A\,. \end{array}$$

The result now follows from Theorem 2.1.

The following theorem is analogous to Lemma 4.1 of [10].

THEOREM 2.4. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *be as in Theorem* 2.1, *and let* $A \xrightarrow{f_1} B \xrightarrow{g_1} C \xrightarrow{h_1} D$ *be such that* $f_1 \cong f$, $g_1 \cong g$, *and* $h_1 \cong h$. *Then* $g_1 f_1 \cong *$, $h_1 g_1 \cong *$, *and*

$$\langle h_1, g_1, f_1\rangle = \langle h, g, f\rangle\,.$$

This follows from Theorems 2.1 and 2.2; see [10, p. 272].

It follows from this theorem that if $\alpha \in [A, B]_n$, $\beta \in [B, C]_m$, and $\gamma \in [C, D]_r$ are such that $\beta\alpha = 0$ and $\gamma\beta = 0$, then we can define the triple product

$$\langle \gamma, \beta, \alpha\rangle \in [A, D]_{n+m+r+1}/\text{indeterminacy}$$

as follows: Let $f \in \alpha$, $g \in \beta$, and $h \in \gamma$. Then $gf \cong *$ and $hg \cong *$, so $\langle h, g, f\rangle$ is defined. Let $\langle \gamma, \beta, \alpha\rangle = \langle h, g, f\rangle$. By Theorem 2.4, $\langle \gamma, \beta, \alpha\rangle$ does not depend on the choice of $f$, $g$, and $h$.

THEOREM 2.5. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *be maps of spectra such that* $gf \cong *$ *and* $hg \cong *$. *Suppose one of* $f$, $g$, *or* $h$ *is the constant map. Then* $\langle h, g, f\rangle$ *is represented by the constant map* $A \to D$.

PROOF. See [10, Corollary 4.2, p. 272].

COROLLARY 2.6. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *be maps of spectra such that* $gf \simeq *$ *and* $hg \simeq *$. *Suppose* $f \simeq *$, $g \simeq *$, *or* $h \simeq *$. *Then* $\langle h, g, f \rangle$ *is represented by the constant map* $A \to D$.

PROOF. This follows from Theorems 2.4 and 2.5.

Finally, we show that triple products have the expected naturality property:

THEOREM 2.7. *Suppose we have the following homotopy commutative diagram of spectra and maps:*

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & D_1 \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle d}\downarrow \\ A_2 & \xrightarrow[f_2]{} & B_2 & \xrightarrow[g_2]{} & C_2 & \xrightarrow[h_2]{} & D_2 \end{array}$$

*such that* $g_1 f_1 \simeq *$, $h_1 g_1 \simeq *$, $g_2 f_2 \simeq *$, *and* $h_2 g_2 \simeq *$. *Then*

$$a^*\langle h_2, g_2, f_2 \rangle = d_*\langle h_1, g_1, f_1 \rangle \in [A_1, D_2]_*/(h_{2*}[A_1, C_2] + f_1^*[B_1, D_2]_*) .$$

PROOF. See [10, Theorem 4.3, pp. 272–273].

This theorem has the following corollary:

COROLLARY 2.8. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ *be maps of spectra.*

1. *Suppose* $kh \simeq *$ *and* $hg \simeq *$. *Then*

$$\langle k, h, g \rangle f = \langle k, g, gf \rangle .$$

2. *Suppose* $kh \simeq *$ *and* $hgf \simeq *$. *Then*

$$\langle k, h, gf \rangle = \langle k, hg, f \rangle .$$

3. *Suppose* $khg \simeq *$ *and* $gf \simeq *$. *Then*

$$\langle k, hg, f \rangle = \langle kh, g, f \rangle .$$

4. *Suppose* $hg \simeq *$ *and* $gf \simeq *$. *Then*

$$\langle kh, g, f \rangle = k\langle h, g, f \rangle .$$

*In each of these equations, the indeterminary is understood to be the maximum of the indeterminacies of both sides of the equations.*

PROOF. The various parts of this corollary follow from the theorem and the following commutative diagrams:

1.
$$\begin{array}{ccccccc} A & \xrightarrow{gf} & C & \xrightarrow{h} & D & \xrightarrow{k} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ B & \xrightarrow[g]{} & C & \xrightarrow[h]{} & D & \xrightarrow[k]{} & E \end{array}$$

2.
$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{hg} & D & \xrightarrow{k} & E \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{gf} & C & \xrightarrow{h} & D & \xrightarrow{k} & E \end{array}$$

3.
$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{kh} & E \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow[f]{} & B & \xrightarrow[hg]{} & D & \xrightarrow[k]{} & E \end{array}$$

4.
$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle k} \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C & \xrightarrow[kh]{} & E \end{array}$$

**3. Triple Products in the Steenrod Algebra.**

In this section we shall use the results of the previous section to define triple products in the Steenrod algebra. Let $\alpha, \beta, \gamma \in \mathscr{A}(p)$ be elements of degrees $n$, $m$, and $r$ respectively, and suppose $\beta\alpha = 0$ and $\gamma\beta = 0$. Regard $\alpha$ as an element of $[HZ_p, HZ_p]_{-n}$, $\beta$ as an element of $[HZ_p, HZ_p]_{-m}$, and $\gamma$ an element of $[HZ_p, HZ_p]_{-r}$. Then by the results of the previous section, the triple product $\langle\gamma, \beta, \alpha\rangle$ is defined and is an element of $[HZ_p, HZ_p]_{-n-m-r+1} = [HZ_p, HZ_p]_{-(n+m+r-1)}$ modulo indeterminacy. Thus $\langle\gamma, \beta, \alpha\rangle$ is an element of $\mathscr{A}(p)$ modulo indeterminacy of degree $n+m+r-1$. The indeterminacy is $\mathscr{A}\alpha + \gamma\mathscr{A}$. Thus $\langle\gamma, \beta, \alpha\rangle$ is a well-defined element of

$$\mathscr{A}(p)/(\mathscr{A}(p)\alpha + \gamma\mathscr{A}(p)) .$$

Let $\alpha \in \mathscr{A}(p)$. Regard $\alpha$ as a map $\alpha\colon HZ_p \to HZ_p$, and let $E_\alpha$ be the mapping cone of $\alpha$. We can think of $E_\alpha$ as the stable analog of a two-stage Postnikov system, with $k$ invariant $\alpha$. We shall show that we can get information about the structure of $H^*(E_\alpha; Z_p)$ as a module over $\mathscr{A}(p)$ from the triple product structure of $\mathscr{A}(p)$.

Consider the following exact triangle:

$$\begin{array}{ccc} HZ_p & \xrightarrow{\alpha} & HZ_p \\ {\scriptstyle p}\nwarrow & & \swarrow{\scriptstyle i} \\ & E_\alpha & \end{array}$$

Passing to cohomology, we have

$$\begin{array}{ccccc} \mathscr{A}(p) = HZ_p{}^*HZ_p & \xleftarrow{\alpha^*} & HZ_p{}^*HZ_p = \mathscr{A}(p) \\ {\scriptstyle p^*}\searrow & & \nearrow{\scriptstyle i^*} \\ & HZ_p{}^*E_\alpha & \end{array}$$

Let $\beta \in \mathscr{A}(p)$ be such that $\beta\alpha = 0$. Then $\alpha^*(\beta) = \beta\alpha = 0$, so, by exactness, $\beta = i^*(x)$ for some $x \in HZ_p{}^*E_\alpha$. Let $\gamma \in \mathscr{A}$ be such that $\gamma\beta = 0$, and consider $\gamma \cdot x$. We have

$$i^*(\gamma \cdot x) = \gamma i^*(x) = \gamma\beta = 0 ,$$

so, again by exactness, $\gamma \cdot x = p^*(y)$ for some $y \in \mathscr{A}(p)$.

THEOREM 3.1. *The indeterminacy in the definition of $y$ is $\mathscr{A}\alpha + \gamma\mathscr{A}$, and $y = \langle \gamma, \beta, \alpha \rangle$ modulo this indeterminacy.*

PROOF. Consider the following diagram:

$$\begin{array}{ccccccc} HZ_p & \xrightarrow{\alpha} & HZ_p & \xrightarrow{\beta} & HZ_p & \xrightarrow{\gamma} & HZ_p \\ & & & {\scriptstyle i}\searrow & \uparrow{\scriptstyle x} & & \uparrow{\scriptstyle y} \\ & & & & E_\alpha & \xrightarrow[p]{} & HZ_p . \end{array}$$

Regard $x$ and $y$ as maps $x\colon E_\alpha \to HZ_p$ and $y\colon HZ_p \to HZ_p$. Then since $\beta = i^*(x)$ and $\gamma \cdot x = p^*(y)$, this diagram is homotopy commutative. The theorem then follows from Theorem 2.1.

As a corollary, we have the following result of Kristensen and Pedersen. (They prove this for the case $p = 2$ in [8, Theorem 3.1], but their proof works equally well for odd $p$.)

COROLLARY 3.2. *Let $\alpha \in \mathscr{A}(p)$, and consider the stable two-stage Postnikov system*

$$\begin{array}{ccc} HZ_p & \xrightarrow{i} & E \\ & & \downarrow{\scriptstyle p} \\ & & HZ_p \xrightarrow{\alpha} HZ_p . \end{array}$$

*Let $x \in H^*(E; Z_p)$ be such that $i^*(x) \neq 0$, and let $\gamma \in \mathscr{A}(p)$ be such that $i^*(\gamma \cdot x) = 0$. Then*

$$\gamma \cdot x = p^*\langle \gamma, i^*(x), \alpha \rangle .$$

PROOF. Let $\beta = i^*(x)$. Then $\beta\alpha = \alpha^*(\beta) = \alpha^* i^*(x) = 0$, and $\gamma\beta = \gamma i^*(x) = i^*(\gamma \cdot x) = 0$. Then the result follows from the theorem.

THEOREM 3.3. *$\langle \gamma, \beta, \mathrm{Sq}^1 \rangle = 0$ for any $\beta, \gamma \in \mathscr{A}(2)$ such that $\beta\,\mathrm{Sq}^1 = 0$ and $\gamma\beta = 0$.*

PROOF. Consider the exact triangle

$$\begin{array}{ccc} HZ_2 & \xrightarrow{\mathrm{Sq}^1} & HZ_2 \\ & {}_{p}\nwarrow \quad \swarrow_{i} & \\ & E & \end{array}$$

Passing to homotopy, we have the exact sequence

$$\to \pi_i(HZ_2) \xrightarrow{i_*} \pi_i(E) \xrightarrow{p_*} \pi_i(HZ_2) \to \pi_{i-1}(HZ_2) \to$$

Since $\pi_i(HZ_2) = 0$ if $i \neq 0$ and $\pi_0(HZ_2) = Z_2$, it follows that $\pi_i(E) = 0$ if $i \neq 0$ and $\pi_0(E) = Z_2 \oplus Z_2$ or $Z_4$; that is $E = H(Z_2 \oplus Z_2) = HZ_2 \wedge HZ_2$ or $E = HZ_4$. But if $E = HZ_2 \wedge HZ_2$, then this exact triangle is trivial, and therefore $\mathrm{Sq}^1 = 0$. Since $\mathrm{Sq}^1 \neq 0$, it follows that $E = HZ_4$.

We therefore have

$$\begin{array}{ccc} HZ_2 & \xrightarrow{\mathrm{Sq}^1} & HZ_2 \\ & {}_{p}\nwarrow \quad \swarrow_{i} & \\ & HZ_4 & \end{array}$$

Passing to cohomology with $Z_2$-coefficients, we have

$$\begin{array}{ccc} H^*(HZ_2) & \xleftarrow{\mathrm{Sq}^1} & H^*(HZ_2) \\ & {}^{p^*}\searrow \quad \nearrow^{i^*} & \\ & H^*(HZ_4) & \end{array}$$

Let $x \in H^*(HZ_4)$ be such that $i^*(x) = \beta$. Then by Theorem 3.1,

$$\gamma \cdot x = p^*\langle \gamma, \beta, \mathrm{Sq}^1 \rangle .$$

But $H^k(HZ_4; Z_2) = \lim_{n \to \infty} H^{n+k}(Z_4, n; Z_2)$, and by Serre's results on $H^*(Z_4, n; Z_2)$ [9], we know that $\gamma \cdot x = \gamma\beta = 0$. Thus $\langle \gamma, \beta, \mathrm{Sq}^1 \rangle = 0$.

As another application of Theorem 3.1, we shall show that the existence of a four-stage Postnikov system implies the vanishing of a certain triple product.

THEOREM 3.4. *Consider the following diagram of spectra and maps of spectra:*

$$\begin{array}{ccccc} HZ_p & \xrightarrow{i_3} & E_3 & & \\ & & \downarrow{p_3} & & \\ HZ_p & \xrightarrow{i_2} & E_2 & \xrightarrow{\tau_3} & HZ_p \\ & & \downarrow{p_2} & & \\ HZ_p & \xrightarrow{i_1} & E_1 & \xrightarrow{\tau_2} & HZ_p \\ & & \downarrow{p_1} & & \\ & & E_0 & \xrightarrow{\tau_1} & HZ_p \end{array}$$

*Here* $E_0 = H\mathbf{Z}_p$ *and*

$$\begin{array}{ccc} E_{j-1} & \xrightarrow{\tau_j} & H\mathbf{Z}_p \\ {\scriptstyle p_j}\nwarrow & & \swarrow{\scriptstyle i_j} \\ & E_j & \end{array}$$

*is an exact triangle for* $j=1,2,3$. *Let* $\alpha=\tau_1$, $\beta=\tau_2 i_1$, *and* $\gamma=\tau_3 i_2$. *Then* $\alpha,\beta,\gamma \in \mathscr{A}(p)$, $\langle\gamma,\beta,\alpha\rangle$ *is defined, and* $\langle\gamma,\beta,\alpha\rangle=0$ *modulo its indeterminacy.*

PROOF. The fact that the three triangles are exact implies that $\beta\alpha=0$ and $\gamma\beta=0$. Taking cohomology with $\mathbf{Z}_p$ coefficients we have the following

$$\begin{array}{ccccc} \mathscr{A}(p) & \xleftarrow{i_3^*} & H^*(E_3) & & \\ & & \uparrow{\scriptstyle p_3^*} & & \\ \mathscr{A}(p) & \xleftarrow{i_2^*} & H^*(E_2) & \xleftarrow{\tau_3^*} & \mathscr{A}(p) \\ & & \uparrow{\scriptstyle p_2^*} & & \\ \mathscr{A}(p) & \xleftarrow{i_1^*} & H^*(E_1) & \xleftarrow{\tau_2^*} & \mathscr{A}(p) \\ & & \uparrow{\scriptstyle p_1^*} & & \\ & & \mathscr{A}(p) & \xleftarrow{\tau_1^*} & \mathscr{A}(p) \end{array}$$

Now $\tau_1^*(1)=\alpha$, $i_1^*\tau_2^*(1)=\beta$, and $i_2^*\tau_3^*(1)=\gamma$. Let

$$u_1 = \tau_2^*(1) \in H^*(E_1) .$$

Then $i_1^*(u_1)=i_1^*\tau_2^*(1)=\beta$. Since

$$i_1^*(\gamma\cdot u_1)=\gamma\cdot i_1^*(u_1) = \gamma\cdot\beta = 0 ,$$

we have $\gamma\cdot u_1 = p_1^*(y)$ for some $y \in \mathscr{A}(p)$. By Theorem 3.1, $y \in \langle\gamma,\beta,\alpha\rangle$.

Let $u_2=\tau_3^*(1) \in H^*(E_2)$. Then $i_2^*(u_2)=\gamma$. Thus

$$\begin{aligned} \gamma\cdot u_1 &= \gamma\cdot\tau_2^*(1) \\ &= \tau_2^*(\gamma\cdot 1) \\ &= \tau_2^*(\gamma) \\ &= \tau_2^* i_2^*(u_2) \\ &= 0 . \end{aligned}$$

Thus we may choose $y$ to be $0 \in \mathscr{A}(p)$; that is, $0 \in \langle\gamma,\beta,\alpha\rangle$. This proves the theorem.

E. Thomas has computed the stable Postnikov tower for $BO(n)$ [2, pp. 132–150]. The first few terms look like this:

$$
\begin{array}{ccccc}
K(\mathbf{Z},n+7) & \xrightarrow{i_3} & E_3 & & \\
 & & \downarrow{\scriptstyle p_3} & & \\
K(\mathbf{Z},n+3) & \xrightarrow{i_2} & E_2 & \xrightarrow{j_3} & K(\mathbf{Z},n+8) \\
 & & \downarrow{\scriptstyle p_2} & & \\
K(\mathbf{Z}_2,n+1) & \xrightarrow{i_1} & E_1 & \xrightarrow{j_2} & K(\mathbf{Z},n+4) \\
 & & \downarrow{\scriptstyle p_1} & & \\
 & & K(\mathbf{Z}_2,n) & \xrightarrow{j_1} & K(\mathbf{Z}_2,n+2)
\end{array}
$$

Thomas shows that $j_1 = \mathrm{Sq}^2$, $j_2 i_1 = \delta_2 \mathrm{Sq}^2$, and $j_3 i_2 = \delta_2 \mathrm{Sq}^4 \pm \delta_3 P_3^1$, where $\delta_2$ and $\delta_3$ are the appropriate Bocksteins. When we pass to cohomology with $\mathbf{Z}_2$ coefficients, we have $j_1^* = \mathrm{Sq}^2$, $(j_2 i_1)^* = \mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^3$, and $(j_3 i_2)^* = \mathrm{Sq}^1\mathrm{Sq}^4 \pm 0 = \mathrm{Sq}^5$. Thus we have the following results, due to Kristensen [5]:

COROLLARY 3.5. $\langle \mathrm{Sq}^5, \mathrm{Sq}^3, \mathrm{Sq}^2 \rangle = 0$.

COROLLARY 3.6. $\langle \gamma, \beta, \mathrm{Sq}^2 \rangle = 0$ *for any* $\beta, \gamma \in \mathscr{A}$ *such that* $\mathrm{Sq}^2 = 0$ *and* $\gamma\beta = 0$.

PROOF. Toda [13] has shown that the following sequence is exact:

$$
\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\cdot \mathrm{Sq}^2} & \mathscr{A} \\
{\scriptstyle \cdot \mathrm{Sq}^3}\uparrow & & \downarrow{\scriptstyle \cdot \mathrm{Sq}^2} \\
\mathscr{A}/\mathscr{A}\mathrm{Sq}^1 & \xleftarrow[\cdot \mathrm{Sq}^5]{} & \mathscr{A}/\mathscr{A}\mathrm{Sq}^1 .
\end{array}
$$

Since $\beta\mathrm{Sq}^2 = 0$, there is an $a \in \mathscr{A}$ such that $\beta = a\mathrm{Sq}^3$. Then $0 = \gamma\beta = \gamma a \mathrm{Sq}^3$. Thus there is a $b \in \mathscr{A}$ such that $\gamma a = b\mathrm{Sq}^5$. Then by Corollary 2.8, we have

$$
\begin{aligned}
\langle \gamma, \beta, \mathrm{Sq}^2 \rangle &= \langle \gamma, a\mathrm{Sq}^3, \mathrm{Sq}^2 \rangle \\
&= \langle \gamma a, \mathrm{Sq}^3, \mathrm{Sq}^2 \rangle \\
&= \langle b\mathrm{Sq}^5, \mathrm{Sq}^3, \mathrm{Sq}^2 \rangle \\
&= b\langle \mathrm{Sq}^5, \mathrm{Sq}^3, \mathrm{Sq}^2 \rangle \\
&= b \cdot 0 = 0 .
\end{aligned}
$$

Note that the maximum indeterminacy is that of the first triple product, $\mathscr{A}\mathrm{Sq}^2 + \gamma\mathscr{A}$. Thus $\langle \gamma, \beta, \mathrm{Sq}^2 \rangle = 0$ modulo its indeterminacy.

## 4. Triple Products and the Canonical Antiautomorphism.

In this section we shall study the effect of the canonical antiautomorphism of the Steenrod algebra on triple products. We shall use the Pontrjagin duality functor of E. Brown and M. Comenetz. First, let $G$ be an abelian group. Recall that its Pontrjagin dual is the group

$$c(G) = \mathrm{Hom}\,(G, \mathbf{R}/\mathbf{Z}) .$$

This duality functor has the following properties:

1. If $p$ is a prime, $c(\mathsf{Z}_p) \approx \mathsf{Z}_p$.
2. If $G$ is finite, $c(G) \approx G$ noncanonically.
3. If $G$ is finite, $cc(G) \approx G$ canonically.
4. $c(\mathsf{Z}) \approx \mathsf{R}/\mathsf{Z}$.

E. Brown and M. Comenetz [3] have shown how to define the Pontrjagin dual of a spectrum. It is defined as follows: Let $E$ be a spectrum. If $c(E)$ is the dual of $E$, then we want

$$c(E)^*(X) \approx c(E_*(X))$$

for all spectra $X$. So we define $c(E)$ to be a spectrum that represents the cohomology theory $c(E_*(\cdot))$; by the stable version of the Brown Representation Theorem, such a spectrum exists. (See, for example, [1, Theorem 3.12, p. 156].) Then $c(E)$ is called the Pontrjagin dual of $E$. Note that $c(E)$ is not canonically defined. Brown and Comenetz then show how to make $c$ into a functor (non-canonically); that is, given a map $f\colon E \to F$, they define a map $c(f)\colon c(F) \to c(E)$. They then show that $c$ has the following properties:

THEOREM 4.1. (1) *$c$ is an additive, antiexact, contravariant, degree-preserving functor. (A contravariant functor $c\colon \mathscr{S} \to \mathscr{S}$, where $\mathscr{S}$ is the stable category, is called antiexact if, whenever $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$ is an exact triangle, then so is $c(X) \xleftarrow{c(f)} c(Y) \xleftarrow{c(g)} c(Z) \xleftarrow{-c(h)} c(X)$.)*

(2) *Call $E$ an $f$-spectrum if $\pi_q(E)$ is finite for all $q$. If $E$ is an $f$-spectrum, then $cc(E)$ is naturally equivalent to $E$.*

(3) *If $E_1$ and $E_2$ are $f$-spectra, then $c$ induces a homomorphism*

$$c : [E_1, E_2]_* \to [c(E_2), c(E_1)]_*$$

*which is an isomorphism*

(4) *If $E$ is an $f$-spectrum, then*

$$\pi_q(c(E)) \approx \pi_{-q}(E) \quad \textit{for all } q\ .$$

*In particular, if $G$ is a finite abelian group, then $c(HG) \approx HG$.*

(5) *Choose $c(H\mathsf{Z}_p)$ to be $H\mathsf{Z}_p$. Then if $f\colon H\mathsf{Z}_p \to H\mathsf{Z}_p$ is a map, the map $c(f)\colon H\mathsf{Z}_p \to H\mathsf{Z}_p$ is defined canonically. Furthermore, the isomorphism*

$$c : [H\mathsf{Z}_p, H\mathsf{Z}_p]_* \to [H\mathsf{Z}_p, H\mathsf{Z}_p]_*$$

*is the canonical antiautomorphism of the Steenrod algebra*

$$\chi\colon \mathscr{A}(p) \to \mathscr{A}(p)\ .$$

Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ be maps of spectra such that $gf \simeq *$ and $hg \simeq *$, and consider the following homotopy-commutative diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ & & & \searrow{\scriptstyle i} & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle a} \\ & & & & E_f & \xrightarrow{j} & A \end{array}$$

By Theorem 2.1, $[a] = \langle h, g, f \rangle$ modulo indeterminacy. Now apply the functor $c$ to this diagram:

$$\begin{array}{ccccccc} cA & \xleftarrow{cf} & cB & \xleftarrow{cg} & cC & \xleftarrow{ch} & cD \\ & & & \nwarrow{\scriptstyle ci} & \downarrow{\scriptstyle cb} & & \downarrow{\scriptstyle ca} \\ & & & & cE_f & \xleftarrow{cj} & cA . \end{array}$$

Since $c$ is antiexact, the triangle

$$cA \xrightarrow{-cj} cE_f \xrightarrow{ci} cB \xrightarrow{cf} cA$$

is an exact triangle. We therefore have the following homotopy commutative diagram:

$$\begin{array}{ccccccc} cA & \xrightarrow{-cj} & cE_f & & & & \\ {\scriptstyle -ca}\uparrow & & {\scriptstyle cb}\uparrow & \searrow{\scriptstyle ci} & & & \\ cD & \xrightarrow[ch]{} & cC & \xrightarrow[cg]{} & cB & \xrightarrow[cf]{} & cA \end{array}$$

By Theorem 2.2, $[-ca] = -\langle cf, cg, ch \rangle$. Thus we have

$$c\langle h, g, f \rangle = c[a] = [ca] = \langle cf, cg, ch \rangle .$$

THEOREM 4.2. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *be such that* $gf \simeq *$ *and* $hg \simeq *$. *Then* $c\langle h, g, f \rangle = \langle cf, cg, ch \rangle$.

COROLLARY 4.3. *Let* $\alpha$, $\beta$, *and* $\gamma \in \mathscr{A}(p)$ *be such that* $\beta\alpha = 0$ *and* $\gamma\beta = 0$. *Then*

$$\chi\langle \gamma, \beta, \alpha \rangle = \langle \chi(\alpha), \chi(\beta), \chi(\gamma) \rangle .$$

PROOF. Let $A = B = C = D = HZ_p$, and use part 5 of Theorem 4.1.

COROLLARY 4.4. $\langle Sq^1, \beta, \alpha \rangle = 0$ *for any* $\alpha, \beta \in \mathscr{A}(2)$ *such that* $\beta\alpha = 0$ *and* $Sq^1 \beta = 0$.

PROOF. Since $\chi(Sq^1) = Sq^1$, we have by Theorems 4.2 and 3.3,

$$\begin{aligned} \chi\langle Sq^1, \beta, \alpha \rangle &= \langle \chi(\alpha), \chi(\beta), \chi(Sq^1) \rangle \\ &= \langle \chi(a), \chi(\beta), Sq^1 \rangle \\ &= 0 . \end{aligned}$$

Therefore $\langle \mathrm{Sq}^1, \beta, \alpha \rangle = 0$.

COROLLARY 4.5. $\langle \mathrm{Sq}^2, \beta, \alpha \rangle = 0$ *for any* $\alpha, \beta \in \mathscr{A}(2)$ *such that* $\beta\alpha = 0$ *and* $\mathrm{Sq}^2\beta = 0$.

PROOF. Since $\chi(\mathrm{Sq}^2) = \mathrm{Sq}^2$, this corollary follows from Theorem 4.2 and Corollary 3.6.

## BIBLIOGRAPHY


1. J. F. Adams, *Stable homotopy and generalized homology*, University of Chicago Press, Chicago, Ill., 1974.
2. Raoul Bott, *Lectures on K(X)*, mimeographed lecture notes, Department of Mathematics, Harvard University (no date).
3. Edgar H. Brown, Jr. and Michael Comenetz, *Pontrjagin duality for generalized homology and cohomology theories*, Amer. J. Math. 98 (1976), 1–27.
4. Leif Kristensen, *On secondary cohomology operations* II, (Conf. on Algebraic Topology, Univ. of Illinois at Chicago Circle, Chicago, Ill., 1968), pp. 117–133. Univ. of Illinois at Chicago Circle, Chicago, Ill., 1969.
5. Leif Kristensen, *Massey products in Steenrod's algebra*, (Proc. Advanced Study Inst. on Algebraic Topology, Aarhus, 1970), Vol. II, pp. 240–55. Mat. Inst., Aarhus Univ., Aarhus, 1970.
6. Leif Kristensen and Ib Madsen, *On evaluation of higher order cohomology operations*, Math. Scand. 20 (1967), 114–130.
7. Leif Kristensen and Ib Madsen, *On the structure of the operation algebra for certain cohomology theories*, (Conf. on Algebraic Topology, Univ. of Illinois at Chicago Circle, Chicago, Ill., 1968), pp. 134–160. Univ. of Illinois at Chicago Circle, Ill., Chicago, 1969.
8. Leif Kristensen and Erik Kjær Pedersen, *The $\mathscr{A}$-module structure for the cohomology of two-stage spaces*, Math. Scand. 30 (1972), 95–106.
9. Jean-Pierre Serre, *Cohomologic modulo 2 des complexes d'Eilenberg-MacLane*, Comment. Math. Helv. 27 (1953), 198–232.
10. E. Spanier, *Secondary operations on mappings and cohomology*, Ann. of Math. (2) 75 (1962), 260–282.
11. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations* (Ann. of Math. Studies 50), Princeton University Press, Princeton, N. J., 1962.
12. H. Toda, *Generalized Whitehead products and homotopy groups of spheres*, J. Inst. Polytech. Osaka City Univ. 3 (1952), 43–82.
13. H. Toda, *On exact sequences in Steenrod algebra* mod 2, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 31 (1958), 33–64



DEPARTMENT OF MATHEMATICS
OBERLIN COLLEGE
OBERLIN, OHIO 44074
U.S.A.

Current address:
DEPARTMENT OF MATHEMATICS AND
COMPUTER SCIENCE
UNIVERSITY OF RICHMOND
RICHMOND, VIRGINIA 23173
U.S.A.