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A NEW CHARACTERIZATION OF THE INTERPOLATION SPACES BETWEEN L^p AND L^q

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Abstract.

It is shown that a normed space X is an interpolation space with respect to the couple (L^p, L^q) if and only if it is an interpolation space with respect to each of the couples (L^1, L^q) and (L^p, L^∞). The case of spaces with weights is also discussed. Examples are given showing that analogous results do not hold in a more general setting.

1. Introduction.

The interpolation spaces with respect to the couple (L^p, L^q) have been characterized by Gunnar Sparr [11]. His characterizations is in terms of the K-functional for (L^p, L^q) and generalizes earlier results of A. P. Calderon [3], Lorentz and Shimogaki [7], and Sedaev and Semenov [10], [9]. In this paper we present an alternative characterization of these interpolation spaces which may be more convenient for certain applications. The characterization and its proof are closely related to the paper [4] where an alternative proof is given of Sparr's theorem. The basic strategy in [4] is to split up operators and functions in a way which enables the Sparr characterization to be deduced from analogous simpler results for the couples (L^1, L^q) and (L^p, L^∞). A similar strategy will play an essential rôle here also.

We shall use terminology and notation as in [1] and [4]. Thus for example $\mathscr{L}_c(A)$ denotes the class of all operators mapping the Banach space A into itself with norms not exceeding c and, for any compatible couple of Banach spaces (A_0, A_1), $\mathscr{L}_{c_0}(A_0) \cap \mathscr{L}_{c_1}(A_1)$ denotes the class of operators on $A_0 + A_1$ whose restrictions to A_0 and A_1 are in $\mathscr{L}_{c_0}(A_0)$ and $\mathscr{L}_{c_1}(A_1)$, respectively. We shall also say that an intermediate space A with respect to (A_0, A_1) is a c-*interpolation space* for some constant $c > 0$ if $\mathscr{L}_1(A_0) \cap \mathscr{L}_1(A_1) \subset \mathscr{L}_c(A)$ (i.e. the restrictions to A of all operators in $\mathscr{L}_1(A_0) \cap \mathscr{L}_1(A_1)$ are in $\mathscr{L}_c(A)$.) The non increasing rearrangement of a measurable function f is denoted by f^*.

Our principal result is as follows:

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THEOREM A. *Let* $1 \leqq p \leqq q \leqq \infty$. *Let* (Ω, Σ, μ) *be an arbitrary measure space. Let X be a normed c-interpolation space with respect to* $(L^1(\mu), L^q(\mu))$ *and also with respect to* $(L^p(\mu), L^\infty(\mu))$. *Then X is a* βc*-interpolation space with respect to* $(L^p(\mu), L^q(\mu))$ *where the constant* β *depends only on p and q.*

Using the Riesz–Thorin Theorem, we obtain our new characterization as the following immediate corollary of Theorem A:

COROLLARY. *X is an interpolation space with respect to* $(L^p(\mu), L^q(\mu))$ *if and only if it is an interpolation space with respect to both* $(L^1(\mu), L^q(\mu))$ *and* $(L^p(\mu), L^\infty(\mu))$.

The proof of Theorem A is presented in Section 2. Several of the steps are reminiscent of arguments in [4] and indeed the reader who is familiar with the details of [4] will perceive that careful modifications and appropriate applications of Banach limits and other devices to the proofs in Section 4 of [4] would yield a large part of what we require here. However we have chosen to make our presentation self contained and in fact it also contains a proof of Sparr's theorem which is considerably shorter and simpler than those given in [11] and [4].

In Section 3 we discuss generalizations of our main result to the case of spaces with weights. In the light of these it would be natural to conjecture that for any Banach spaces B_1, B_p, B_q, B_∞ such that B_p is an interpolation space with respect to (B_1, B_q) and B_q is an interpolation space with respect to (B_p, B_∞), the interpolation spaces with respect to (B_p, B_q) are precisely those spaces which are interpolation spaces with respect to each of the couples (B_1, B_q) and (B_p, B_∞). However in the latter part of Section 3 we describe counterexamples which show that such a conjecture is false, even if we restrict B_1, B_p, B_q, B_∞ to be rearrangement invariant spaces.

REMARKS. (1) It should be mentioned that Sparr's original proof [11] applies also to couples of weighted spaces (L^p_v, L^q_w), (see Section 3 below) and also to the case where p and q may assume values less than 1, provided that the underlying measure space is suitably restricted. Our approach does not seem to be applicable in this latter context. (See however [5, Section 4] for a related result.)

(2) Our approach here yields Lorentz and Shimogaki's description of the interpolation spaces with respect to (L^p, L^∞) without their requirement ([7, p. 207 (1.2)]) that the norms of the spaces be semicontinuous. (Cf. part (i) of the lemma in Section 2.)

(3) In fact the theorems of Sparr and Lorentz and Shimogaki are both special

cases of [6, Theorem 2], but the proof of that theorem gives a much poorer estimate of the constant involved.

2. The proof of Theorem A.

Let X satisfy the hypotheses of the theorem. We begin by observing that X is an intermediate space with respect to $(L^p, L^q) = (L^p(\mu), L^q(\mu))$, that is

$$L^p \cap L^q \subset X \subset L^p + L^q,$$

where the embeddings are continuous. Indeed, for $f \in L^p \cap L^q$, $f = f_1 + f_2$, where

$$f_1 = f\chi_{\{|f| > f^*(1)\}} \quad \text{and} \quad f_2 = f\chi_{\{|f| \leqq f^*(1)\}}.$$

Clearly $f_1 \in L^1 \cap L^q \subset X$ and $f_2 \in L^p \cap L^\infty \subset X$ so that $f \in X$. Furthermore

$$\|f\|_X \leqq \text{const.}\, \|f\|_{L^p \cap L^q}.$$

For the second inclusion note that any $f \in X$ is an element of $(L^p + L^\infty) \cap (L^1 + L^q)$. Thus, in view of the estimates of Holmstedt, Kree and Peetre for K-functionals (see e.g. [4, p. 216]),

$$\int_0^1 f^*(s)^p\, ds \quad \text{and} \quad \int_1^\infty f^*(s)^q\, ds$$

are both finite, $f \in L^p + L^q$ and in fact the embedding $X \subset L^p + L^q$ is continuous.

REMARK. In the case $p = q$ the above argument shows that $X = L^p = L^q$.

Now let T be an operator in $\mathscr{L}_1(L^p) \cap \mathscr{L}_1(L^q)$ and let $f \in X$. It follows from Holmstedt's estimates for $K(t, f; L^p, L^q)$ that

$$\left(\int_0^{t^\alpha} [(Tf)^*(s)]^p\, ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty [(Tf)^*(s)]^q\, ds\right)^{1/q}$$
$$\leqq \beta_0\left[\left(\int_0^{t^\alpha} f^*(s)^p\, ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty f^*(s)^q\, ds\right)^{1/q}\right]$$

for all $t > 0$, where $\alpha = (1/p - 1/q)^{-1}$ and β_0 is a constant depending only on p and q. (It is not difficult to show for example that $\beta_0 \leqq \max[(1 + 2^{1/p}(q/p - 1)^{-1/q}), (2^{1/q} + (1 - p/q)^{-1/q})]$.) Letting $g = \beta_0^{-1} Tf$ we may rewrite the above inequality in the form

$$\text{(1)} \quad \left(\int_0^{t^\alpha} g^*(s)^p\, ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty g^*(s)^q\, ds\right)^{1/q}$$
$$\leqq \left(\int_0^{t^\alpha} f^*(s)^p\, ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty f^*(s)^q\, ds\right)^{1/q}.$$

We shall show that if (1) is satisfied for all $t>0$, then for any $\varepsilon>0$ there exist two operators $U \in \mathscr{L}_{1+\varepsilon}(L^p) \cap \mathscr{L}_{1+\varepsilon}(L^\infty)$ and $V \in \mathscr{L}_{1+\varepsilon}(L^1) \cap \mathscr{L}_{1+\varepsilon}(L^q)$ such that $g=Uf+Vf$. By hypothesis U and V are both in the class $\mathscr{L}_{c(1+\varepsilon)}(X)$ so it will follow that

$$\|g\|_X \leqq 2c(1+\varepsilon)\|f\|_X, \quad \text{i.e. } \|Tf\|_X \leqq 2\beta_0 c(1+\varepsilon)\|f\|_X \quad \text{for all } \varepsilon>0\,.$$

This will prove Theorem A with $\beta=2\beta_0$.

Remark. The proof of the existence of U and V as above such that $g=(U+V)f$ will in fact be valid for arbitrary f and g in L^p+L^q satisfying (1) for all $t>0$. Since by the Riesz–Thorin Theorem $U+V \in \mathscr{L}_{2+2\varepsilon}(L^p) \cap \mathscr{L}_{2+2\varepsilon}(L^q)$, then this will also furnish a proof of Sparr's theorem, namely that X is an interpolation space with respect to (L^p, L^q) if and only if for all $f \in X$, $g \in L^p+L^q$, $K(t,g;\, L^p,L^q) \leqq K(t,f;\, L^p,L^q)$ for all $t>0$ implies that $g \in X$ with appropriate norm estimate.

Before constructing the operators U and V we make two simplifications: Firstly we may suppose without loss of generality that f and g are each functions of the form $\sum_{-\infty}^{\infty} r^n \chi_{E_n}$ for some constant $r>1$ and disjoint sets E_n each of finite measure. To see this, for any choice of $\varepsilon>0$ let

$$\bar{f} = \sum_{n=-\infty}^{\infty} (1+\varepsilon)^n \chi_{\{x \mid (1+\varepsilon)^n \leqq |f(x)| < (1+\varepsilon)^{n+1}\}}$$

$$\tilde{g} = \sum_{n=-\infty}^{\infty} (1+\varepsilon)^n \chi_{\{x \mid (1+\varepsilon)^{n-1} < |g(x)| \leqq (1+\varepsilon)^n\}}$$

Except possibly in the case $q=\infty$ for which Theorem A is a triviality, the sets of constancy of $\bar{f}$ and $\tilde{g}$ all have finite measure. Clearly (1) still holds with f and g replaced by $\bar{f}$ and $\tilde{g}$ and if we can deduce that $\tilde{g}=\tilde{U}\bar{f}+\tilde{V}\tilde{g}$ for suitable operators $\tilde{U}, \tilde{V}$ in $\mathscr{L}_{1+\varepsilon}(L^p) \cap \mathscr{L}_{1+\varepsilon}(L^\infty)$ and $\mathscr{L}_{1+\varepsilon}(L^1) \cap \mathscr{L}_{1+\varepsilon}(L^q)$, respectively, then the required operators U and V satisfying $g=Uf+Vf$ can be obtained by composing $\tilde{U}$ and $\tilde{V}$ with operators of pointwise multiplication by $\bar{f}/f$ and $g/\tilde{g}$ in the obvious way. This multiplies our estimates of the norms of U and V by an extra factor of $(1+\varepsilon)^2$ but, since ε can be taken arbitrarily small, the final conclusion is unaffected.

Secondly we may suppose without loss of generality that (Ω, Σ, μ) is $\mathbf{R}_+ = (0,\infty)$ equipped with Lebesgue measure, and that f and g are non-negative non-increasing functions, since we can extend this case to the more general one with the help of operators S_0 and S_1 such that $S_0 f = f^*$, $S_1 g^* = g$, S_0 maps $L^r(\Omega, d\mu)$ to $L^r(\mathbf{R}_+, dx)$ with norm one and S_1 maps $L^r(\mathbf{R}_+, dx)$ to $L^r(\Omega, d\mu)$ with norm one for $r=1,\infty$. The construction of such operators is described in [3] for the case when Ω is σ-finite, which is the only case we need consider here since, for $q<\infty$, f and g have σ-finite supports. (Non σ-finite measure spaces are considered in [4].) Moreover, if as in our case, f and g assume countably many different values each on sets of finite measure, then it is easy to write explicit formulae for S_0 and S_1 as sums of averaging operators.

In view of the above we can suppose from here on that $L^r = L^r(\mathrm{R}_+, dx)$ for $r = 1, p, q, \infty$ and that

$$f = \sum_{n=-\infty}^{\infty} (1+\varepsilon)^{-n} \chi_{a_n, a_{n+1})} \quad \text{and} \quad g = g = \sum_{n=-\infty}^{\infty} (1+\varepsilon)^{-n} \chi_{[b_n, b_{n+1})}$$

where $(a_n)_{n=-\infty}^{\infty}$ and $(b_n)_{n=-\infty}^{\infty}$ are non decreasing sequences in $[0,\infty)$. Each sequence has at most two cluster points, at 0 and at the right endpoint of the interval supporting f or g, respectively, if that interval is bounded.

Let

$$\alpha(t) = \int_0^t (f(s)^p - g(s)^p)\, ds \quad \text{and} \quad \beta(t) = \int_t^\infty (f(s)^q - g(s)^q)\, ds .$$

These are piecewise linear functions with vertices at points $t=a_n$, $t=b_n$ of the above sequences. Thus the sets

$$A = \{t>0 \mid \alpha(t) \geqq 0\} \quad \text{and} \quad B = \{t>0 \mid \beta(t) \geqq 0\}$$

are each unions of (possibly infinite) sequences of disjoint closed subintervals of R_+, $A = \bigcup_i A_i$, $B = \bigcup_j B_j$. By (1) $\max(\alpha(t), \beta(t)) \geqq 0$ for all $t>0$, so $A \cup B = \mathrm{R}_+$.

Since $\alpha(t)=0$ at the left endpoint c_i of the interval A_i it follows that

$$\int_{c_i}^t (\chi_{A_i} g)^p\, ds \leqq \int_{c_i}^t (\chi_{A_i} f)^p\, ds \quad \text{for all } t \geqq c_i . \tag{2}$$

Similarly, since $\beta(t)=0$ at the right endpoint of the interval B_j we have that

$$\int_t^\infty (\chi_{B_j} g)^q\, ds \leqq \int_t^\infty (\chi_{B_j} f)^q\, ds \quad \text{for all } t \geqq 0 . \tag{3}$$

We now have to apply the following lemma, whose proof will be deferred till later.

LEMMA. *Let f and g be non-negative non-increasing functions on R_+ belonging to $L^p + L^q$.*

(i) *If g assumes at most countably many values and*

$$\int_0^t g(s)^p\, ds \leqq \int_0^t f(s)^p\, ds \quad \textit{for all } t>0 , \tag{4}$$

then there exists an operator $U \in \mathscr{L}_1(L^p) \cap \mathscr{L}_1(L^\infty)$ such that $Uf = g$.

(ii) *If f assumes at most countably many values and*

$$\int_t^\infty g(s)^q\,ds \leqq \int_t^\infty f(s)^q\,ds \quad \text{for all } t>0\,, \tag{5}$$

then there exists an operator $V \in \mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$ *such that* $Vf=g$.

Applying this lemma to suitable translates of the functions $\chi_{A_i}g$, $\chi_{A_i}f$ and of $\chi_{B_j}g$, $\chi_{B_j}f$ we deduce from (2) and (3) that there exist operators $U_i \in \mathscr{L}_1(L^p) \cap \mathscr{L}_1(L^\infty)$ and $V_j \in \mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$ such that $U_i(f\chi_{A_i}) = g\chi_{A_i}$ and $V_j(f\chi_{B_j}) = g\chi_{B_j}$. By disjointness of the sets A_i, the operator U defined by

$$Uh = \sum_i \chi_{A_i} U_i(h\chi_{A_i}) \quad \text{for} \quad h \in L^p + L^\infty$$

is in $\mathscr{L}_1(L^p) \cap \mathscr{L}_1(L^\infty)$ and similarly the operator V defined by

$$Vh = \chi_{\mathbf{R}_+ \setminus A} \sum_j \chi_{B_j} V_j(h\chi_{B_j}) \quad \text{for } h \in L^1 + L^q$$

is in $\mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$. It is easy to see that $Uf + Vf = g$ as required.

Before embarking on the proof of the lemma which will complete the proof of Theorem A we remark that if f and g are simple, part (i) is exactly Lemma 4 of [7, p. 212] and part (ii) is Theorem 2 of [4, p. 226]. As already hinted at above we could use somewhat elaborate limiting processes to deduce our lemma from these two results. However we shall proceed directly. Our arguments are simpler than those of [7] and [4] in the simple function case and naturally adapt to the general case also.

We also indicate that there will be considerable similarity between the proofs of (i) and (ii) and that the "right pictures" to draw to accompany the proofs are the graphs of the functions $P(x,\varphi)$, $Q(x,\varphi)$ defined for all $x>0$ by

$$P(x,\varphi) = \int_0^x \varphi(s)^p\,ds, \qquad Q(x,\varphi) = \int_x^\infty \varphi(s)^q\,ds\,, \tag{6}$$

where φ is f, g or some other non-negative non-increasing function in $L^p + L^q$.

Proof of (i). Our first step is to show that for any non-negative non-increasing φ in $L^p + L^q$ (or indeed in $L^p + L^\infty$) and any linear function $l(x) = ax + b$ with $a \geqq 0$, $b \geqq 0$ there exists an operator $S \in \mathscr{L}_1(L^p) \cap \mathscr{L}_1(L^\infty)$ such that $S\varphi$ is non-negative and non-increasing and

$$P(x, S\varphi) = \min\,(l(x), P(x,\varphi)) \quad \text{for all } x>0\,. \tag{7}$$

(See Figure 1.)

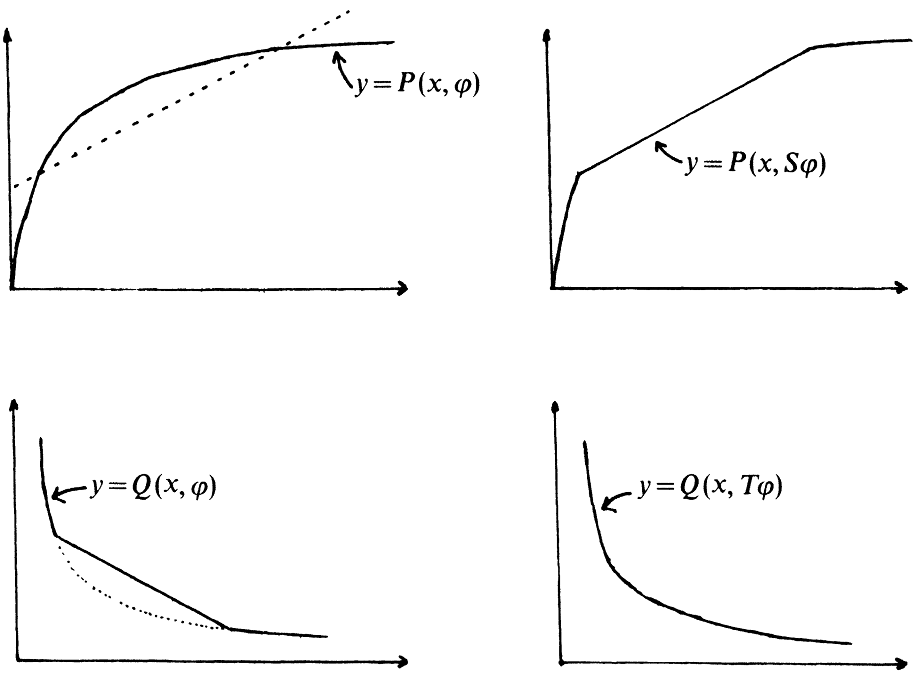

Figure 1. The action of the operators S and T.
S "cuts corners" from the graph of $P(x, \varphi)$, whereas T "sticks corners" onto the graph of $Q(x, \varphi)$.

Subsequently we shall obtain U as a suitable composition of operators of this type.

To construct S, we let $J = \{x \geqq 0 \mid l(x) \leqq P(x, \varphi)\}$. If J is empty or a single point then of course S is the identity operator. Otherwise J is either a bounded or semi-infinite interval, in view of the concavity of $P(x, \varphi)$. If $J = [c, \infty)$ is semi-infinite, then $\int_c^x \varphi(s)^p\, ds \geqq a(x-c)$ for all $x \geqq c$ which implies that $\varphi(x)^p \geqq a$ for all $x \geqq c$. Thus we may define S by

$$Sh = \chi_{[0,c)} h + \chi_{[c,\infty)} a^{1/p} h/\varphi$$

for all $h \in L^p + L^\infty$ and obtain (7).

Alternatively, if J is bounded then we define S by

$$Sh = h\chi_{\mathbf{R}_+ \setminus J} + |J|^{-1/p} \left(\int_J h\psi\, ds \right) \chi_J$$

for all $h \in L^p + L^\infty$ where

$$\psi = \varphi^{p-1} \Big/ \left(\int_J |\varphi|^p\, ds \right)^{1/p'} .$$

Since $\int_J \psi^{p'}\, ds = 1$ it follows immediately that $S \in \mathscr{L}_1(L^p) \cap \mathscr{L}_1(L^\infty)$.

Furthermore, since $S\varphi$ is constant on J and satisfies

$$\int_J (S\varphi)^p\,ds = \int_J \varphi^p\,ds\,,$$

we also obtain (7).

Now let f and g satisfy the hypotheses in part (i) of the lemma. Let $(I_n)_{n=1,2\ldots}$ be a finite or infinite sequence containing all the intervals of constancy of g. We shall consider the case of an infinite sequence, the finite sequence case being an easier variant of the same argument. The restriction of $P(x,g)$ to each of the intervals I_n thus coincides on I_n with a linear function $l_n(x)=a_nx+b_n$, where $a_n=(g|_{I_n})^p\geqq 0$ and also $b_n\geqq 0$ since, by concavity, $P(x,g)\leqq l_n(x)$ for all $x\geqq 0$. Since

$$P(x,g) = \inf_n l_n(x) \leqq P(x,f)$$

we have that

$$P(x,g) = \lim_{n\to\infty} P_n(x)\,,$$

where

$$P_1(x) = \min\,(l_1(x),P(x,f)) \qquad \text{and} \qquad P_n(x) = \min\,(l_n(x),P_{n-1}(x))\,.$$

We may thus obtain a sequence of operators $(S_n)_{n=1}^\infty$ by the above construction, such that $S_n\in\mathscr{L}_1(L^p)\cap\mathscr{L}_1(L^\infty)$ and for the functions f_n defined by $f_1=S_1f$, $f_n=S_nf_{n-1}$,

$$P(x,f_n) = P_n(x) = \min\,(l_n(x),P(x,f_{n-1}))\,.$$

Now for all x not in the closure of I_n, $l_n(x)>P(x,g)$ and, since $P(x,g)$ and $P_{n-1}(x)$ coincide on each I_m for $m\leqq n-1$, the set

$$J_n = \{x\geqq 0 \mid l_n(x)\leqq P_{n-1}(x)\}$$

is disjoint from the interior of each I_m. We deduce that $(S_nh)\chi_{I_m}=h\chi_{I_m}$ for all $m\leqq n-1$, and all $h\in L^p+L^\infty$.

Let $U_n=S_nS_{n-1}\ldots S_1$ and define U by

$$Uh = \sum_{n=1}^\infty (U_nh)\chi_{I_n} \qquad \text{for all } h\in L^p+L^\infty\,.$$

Since Uh and U_nh coincide on $\bigcup_{m=1}^{n-1} I_m$ for all n and $\bigcup_{m=1}^\infty I_m=\mathbf{R}_+$ it follows immediately that $U\in\mathscr{L}_1(L^p)\cap\mathscr{L}_1(L^\infty)$. Finally Uf restricted to I_n coincides with $U_nf=f_n$ and $P(x,f_n)$ and $P(x,g)$ coincide for $x\in I_n$. By differentiation, $Uf=g$ on I_n for each n and thus $Uf=g$.

PROOF OF (ii). The function $Q(x,\varphi)$ as defined in (6) plays a rôle analogous to that of $P(x,\varphi)$ in the proof of (i). Our first step is to show that, given any non-increasing linear function $l(x)=ax+b$ and any non-negative non-increasing function $\varphi \in L^1+L^q$ with the property that, for some non-negative non-increasing convex function $Q(x)$,

$$(8) \qquad Q(x,\varphi) = \max(l(x), Q(x)) \quad \text{for all } x \geqq 0,$$

then there exists an operator $T \in \mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$ such that $T\varphi$ is non-negative and non-increasing and

$$(9) \qquad Q(x, T\varphi) = Q(x) \quad \text{for all } x > 0 .$$

(See Figure 1.) Subsequently we shall obtain V as a composition of operators of this type.

To construct T we let

$$J = \{x \geqq 0 \mid l(x) \geqq Q(x)\} = \{x \geqq 0 \mid l(x) = Q(x,\varphi)\} .$$

As before the cases where J is empty or a single point can be treated by taking T to be the identity operator. Otherwise J is an interval. If J is semi-infinite then necessarily $a=0$, since $Q(x) \geqq 0$ for all $x>0$. But then also $b=0$ since $\lim_{x\to\infty} Q(x,\varphi)=0$. Consequently $Q(x)=Q(x,\varphi)$ and again T is the identity operator. Thus we can assume that J is a bounded interval $J=[\alpha,\beta]$.

We introduce the function ξ defined by

$$\xi(x) = \left(-\frac{d}{dx} Q(x)\right)^{1/q} .$$

Since $Q(x)$ is absolutely continuous on each compact subinterval of R_+ it is clear that ξ is defined for almost every x in R_+ and $Q(x,\xi)=Q(x)$. The given properties of $Q(x)$ thus imply that ξ coincides almost everywhere with a non-negative non-increasing function on R_+. By differentiation, $\varphi(x)=\xi(x)$ for a.e. $x \notin J$ whereas on J we have $\varphi(x)=(-a)^{1/q}$ a.e. and

$$\int_J \xi(s)^q \, ds = Q(\alpha,\xi)-Q(\beta,\xi) = Q(\alpha,\varphi)-Q(\beta,\varphi) = \int_J \varphi(s)^q \, ds .$$

The operator T is defined by

$$Th = h\chi_{\mathsf{R}_+ \setminus J} + |J|^{-1} \int_J h(s)\, ds (-a)^{-1/q} \xi \chi_J \quad \text{for all } h \in L^1+L^q .$$

It is a routine matter to verify that $T \in \mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$ and that $T\varphi=\xi$ so that T has all the required properties.

Now let f and g satisfy the hypotheses in part (ii) of the lemma. Let $(I_n)_{n=1,2,\ldots}$ be a finite or infinite sequence containing all the intervals of constancy of f. As before we shall consider the case of an infinite sequence, leaving the reader to complete the details of the easier finite case.

On each interval I_n the function $Q(x, f)$ coincides with a linear function $l_n(z) = a_n x + b_n$ where $a_n = -(f \mid I_n)^p \leqq 0$. We define a sequence of non-negative non-increasing convex functions $(Q_n(x))_{n=1}^{\infty}$ by

$$Q_1(x) = \max(l_1(x), Q(x, g))$$

and

$$Q_n(x) = \max(l_n(x), Q_{n-1}(x))$$

for each $n \geqq 2$. The functions f_n defined for a.e. $x \in \mathsf{R}_+$ by

$$f_n(x) = \left(-\frac{d}{dx} Q_n(x)\right)^{1/q}$$

are non-negative and non-increasing and satisfy $Q(x, f_n) = Q_n(x)$ for each n. We may thus obtain a sequence of operators $(T_n)_{n=1}^{\infty}$ by the above construction, such that $T_n \in \mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$ and $T_n f_n = f_{n-1}$ for $n \geqq 2$ and $T_1 f_1 = g$.

By convexity $l_n(x) < Q(x, f)$ for all x not in the closure of I_n. Also, since $Q(x, g) \leqq Q(x, f)$ for all x, the functions $Q(x, f)$ and

$$Q_{n-1}(x) = \max\left(\max_{1 \leqq m \leqq n-1} l_m(x), Q(x, g)\right)$$

coincide on each I_m for $m \leqq n-1$. Consequently the set

$$J_n = \{x \geqq 0 \mid l_n(x) \geqq Q_{n-1}(x)\}$$

is disjoint from the interior of each I_m for $m \leqq n-1$. It follows from the construction of T_n that $T_n(h\chi_{I_m}) = h\chi_{I_m}$ for all $m \leqq n-1$ and all $h \in L^1 + L^q$.

Let $V_n = T_1 T_2 \ldots T_n$ and define V by

$$Vh = \sum_{n=1}^{\infty} V_n(h\chi_{I_n})$$

for all $h \in L^1 + L^q$. Since

$$V(h\chi_{\bigcup_{m=1}^{n-1} I_m}) = V_n(h\chi_{\bigcup_{m=1}^{n-1} I_m}) \quad \text{for all } n$$

and $\bigcup_{m=1}^{\infty} I_m = \mathsf{R}_+$, a simple density argument shows that $V \in \mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$.

Finally we must show that $Vf = g$. Since $Q(x, f)$ and $Q_{n-1}(x) = Q(x, f_{n-1})$ coincide on each I_m, $m \leqq n-1$ so do f and f_{n-1} (except possibly on a subset of measure zero). Thus

$$Vf = \sum_{m \leqq n-1} V_m(f_{n-1}\chi_{I_m}) + V(f\chi_{\bigcup_{m \geqq n} I_m})$$

$$= V_{n-1}(f_{n-1}\chi_{\bigcup_{m\leqq n-1} I_m}) + V(f\chi_{\bigcup_{m\geqq n} I_m})$$
$$= V_{n-1}(f_{n-1}) - V_{n-1}(f_{n-1}\chi_{\bigcup_{m\geqq n} I_m}) + V(f\chi_{\bigcup_{m\geqq n} I_m}) .$$

The first term equals the function g for each n and the third term tends to zero in $L^1 + L^q$ as n tends to infinity. Thus the proof that $Vf = g$ can be reduced to showing that the sequence $(\varepsilon_n)_{n=1}^{\infty}$ defined by

$$\varepsilon_n = \|f_{n-1}\chi_{\bigcup_{m\geqq n} I_m}\|_{L^1+L^q}$$

has a subsequence which tends to zero.

At this stage we have to be more specific about how we choose the order in which the intervals of constancy of f appear in the sequence $(I_n)_{n=1}^{\infty}$. If one of these intervals has left endpoint zero then the order of the sequence is irrelevant for our purposes and the reader will be able to show that $\lim_{n\to\infty}\varepsilon_n = 0$ by a simpler version of the argument to be presented here. Accordingly we can assume that there exists a subsequence which we will denote by $(A_n)_{n=1}^{\infty}$ of the intervals of constancy of f such that, if α_n and β_n denote the left and right endpoints respectively of A_n, then $\beta_{n+1} \leqq \alpha_n$ and

$$\lim_{n\to\infty} \alpha_n = 0 . \tag{10}$$

For each n we can choose a finite (possibly empty) sequence of other intervals of constancy of f, $B_{n,1}, B_{n,2} \ldots B_{n,k_n}$ all lying to the right of A_n and all distinct from all the intervals A_m and all intervals $B_{m,j}$ for $m < n$, such that

$$\int_{[\alpha_n, \infty)\setminus E_n} f(x)^q\, dx < 1/n \tag{11}$$

where

$$E_n = \bigcup_{m=1}^{n} \left(A_m \cup \bigcup_{j=1}^{k_m} B_{m,j} \right) .$$

To form the sequence $(I_n)_{n=1}^{\infty}$ we take the intervals A_n and $B_{n,j}$ in the following order: $B_{1,1}, B_{1,2}, \ldots B_{1,k_1}, A_1, B_{2,1}, \ldots B_{2,k_2}, A_2, B_{3,1}, \ldots$. If f is zero on a (necessarily semi-infinite) interval let $B_{1,1}$ be that interval. The conditions (10) and (11) guarantee that all other intervals of constancy (where f is strictly positive) appear in the sequence. Let $m_n = k_1 + k_2 + \ldots + k_n + n$ so that $I_{m_n} = A_n$ and $E_n = \bigcup_{j=1}^{m_n} I_j$. Then

$$\varepsilon_{m_n+1} = \|f_{m_n}\chi_{\bigcup_{m\geqq m_n+1} I_m}\|_{L^1+L^q}$$
$$= \|f_{m_n}\chi_{\mathbf{R}_+\setminus E_n}\|_{L^1+L^q}$$
$$\leqq \|f_{m_n}\chi_{(0,\alpha_n]}\|_{L^1} + \|f_{m_n}\chi_{[\alpha_n,\infty)\setminus E_n}\|_{L^q} .$$

For each x, $0<x\leqq\alpha_n$, $Q(x,f_{m_n})$ either coincides with $Q(x,g)$ or with the linear function $l_{m_n}(x)$ which coincides with $Q(x,f_{m_n})$ and $Q(x,f)$ on $A_n=I_{m_n}$. Thus on $(0,\alpha_m]$ we have

$$f_{m_n}(x) \leqq g(x)+f|_{A_n} \leqq g(x)+f(x) .$$

Consequently

$$\|f_{m_n}\chi_{[0,\alpha_n)}\|_{L^1} \leqq \int_0^{\alpha_n} f(x)+g(x)\,dx$$

and this term clearly tends to zero as n tends to infinity since both f and g are in L^1+L^q. To treat the second term $\|f_{m_n}\chi_{[\alpha_n,\infty)\setminus E_n}\|_{L^q}$ we note that $[\alpha_n,\infty)\setminus E_n$ is the union of finitely many disjoint intervals $F_1, F_2, \ldots$ and each endpoint of each of these intervals is also an endpoint of some interval I_m with $m\leqq m_n$ on which $Q(x,f)$ and $Q(x,f_{m_n})$ coincide. Consequently, for each such interval F_j with endpoints γ_j, δ_j,

$$\begin{aligned}\int_{F_j} f_{m_n}(x)^q\,dx &= Q(\gamma_j, f_{m_n})-Q(\delta_j, f_{m_n})\\ &= Q(\gamma_j, f)-Q(\delta_j, f)\\ &= \int_{F_j} f(x)^q\,dx .\end{aligned}$$

(This also holds when $\delta_j=\infty$ since then both integrals equal $Q(\gamma_j, f_{m_n}) = Q(\gamma_j, f)$.) It follows that

$$\int_{[\alpha_n,\infty)\setminus E_n} f_{m_n}(x)^q\,dx = \int_{[\alpha_n,\infty)\setminus E_n} f(x)^q\,dx < 1/n ,$$

so that the second term above converges to zero and $\lim_{n\to\infty}\varepsilon_{m_n+1}=0$. This completes the proof of part (ii) of the lemma and with it the proof of Theorem A.

3. Some generalizations and counter examples.

In this section we first briefly discuss analogues of the results of the previous section for spaces with weights and then give examples showing that such results do not hold in a more general setting.

Sparr originally formulated and proved his theorem [11] for a couple of weighted L^p spaces (L^p_v, L^q_w), thus also including the case $p=q$ which had been investigated earlier by Sedaev [9] and Sedaev and Semenov [10]. (Here v and w denote positive measurable functions on Ω and the spaces L^p_v and L^q_w are normed by

$$\|f\|_{L^p_v} = \left(\int (|f|v)^p \, d\mu\right)^{1/p} \quad \text{and} \quad \|f\|_{L^q_w} = \left(\int (|f|w)^q \, d\mu\right)^{1/q} .)$$

As remarked in [4, p. 234], if $p \neq q$, Sparr's theorem for weighted spaces follows from the case of unweighted spaces with the help of a "Stein–Weiss" transformation (cf. [12]). Similarly, in the present context, one can readily deduce that X is an interpolation space with respect to (L^p_v, L^q_w) if and only if it is an interpolation space with respect to both $(L^1_{\gamma u}, L^q_w)$ and (L^p_v, L^∞_u) where γ and u are weight functions defined by

$$u = (w^q v^{-p})^{1/(q-p)} \quad \text{and} \quad \gamma = v^p u^{-p} = w^q u^{-q} .$$

(Cf. [4, p. 234.])

If $p = q$ the above arguments cannot be applied. However one can obtain an analogue of Theorem A for the couple (L^p_v, L^p_w). Let us first recall that for each $t > 0$

$$K(t, f; L^p_v, L^p_w) \sim \left(\int_{E_t} (|f|v)^p \, d\mu\right)^{1/p} + t\left(\int_{\Omega \setminus E_t} (|f|w)^p \, d\mu\right)^{1/p}$$

where

$$E_t = \{\omega \in \Omega \mid v(\omega) \leqq tw(\omega)\} .$$

As indicated in [4, pp. 234, 235] it is possible to give a proof of Sparr's theorem for this couple, essentially by using the two functionals

$$\tilde{P}(x, f) = \int_{E_x} (|f|v)^p \, d\mu, \quad \tilde{Q}(x, f) = \left(\int_{\Omega \setminus E_x} (|f|w)^p \, d\mu\right)^{1/p}$$

in rôles analogues to those played by $P(x, f)$ and $Q(x, f)$ in Section 2 above. In particular we use the following analogue of the lemma of Section 2.

LEMMA 3.1. *For any $\varepsilon > 0$, if $f, g \in L^p_v + L^p_w$ and either*

(i) $\tilde{P}(x, g) \leqq \tilde{P}(x, f)$ *for all* $x \geqq 0$

or

(ii) $\tilde{Q}(x, g) \leqq \tilde{Q}(x, f)$ *for all* $x \geqq 0$

then there exists an operator $T \in \mathscr{L}_{1+\varepsilon}(L^p_v) \cap \mathscr{L}_{1+\varepsilon}(L^p_w)$ such that $Tf = g$.

Note that w does not appear explicitly in the condition (i) and in fact we can show that the operator T whose existence follows from (i) is in the class $\mathscr{L}_{1+\varepsilon}(L^p_v) \cap \mathscr{L}_{1+\varepsilon}(L^p_{w_\infty})$ for any weight function w_∞ having the property that for each $x > 0$ the set

$$E_x^\infty = \{\omega \in \Omega \mid v(\omega) \leqq x w_\infty(\omega)\}$$

coincides with E_y for some $y = y(x) > 0$. Similarly condition (ii) implies the existence of T such that $Tf = g$ and $T \in \mathscr{L}_{1+\varepsilon}(L_{v_1}^p) \cap \mathscr{L}_{1+\varepsilon}(L_w^p)$ where v_1 can be v or any other weight function having the property that for each $x > 0$ the set

$$E_x^1 = \{\omega \in \Omega \mid v_1(\omega) \leqq x w(\omega)\}$$

coincides with E_y for some $y = y(x) > 0$.

Now suppose that v_1 and w_∞ both have the properties described above and also that any space X which is intermediate with respect to both $(L_{v_1}^p, L_w^p)$ and $(L_v^p, L_{w_\infty}^p)$ is necessarily also intermediate with respect to (L_v^p, L_w^p), (for example we can require

$$\min(\max(v_1, w), \max(v, w_\infty)) \leqq \max(v, w)$$

and

$$\max(\min(v_1, w), \min(v, w_\infty)) \geqq \min(v, w)\ .)$$

Then, analogously to Theorem A, we can show that if X is an interpolation space with respect to both the couples $(L_{v_1}^p, L_w^p)$ and $(L_v^p, L_{w_\infty}^p)$ then it is an interpolation space with respect to (L_v^p, L_w^p).

In order to obtain a necessary and sufficient condition in the style of the corollary of Theorem A we need to know, in addition to all the above hypotheses, that L_v^p is an interpolation space with respect to $(L_{v_1}^p, L_w^p)$ and L_w^p is an interpolation space with respect to $(L_v^p, L_{w_\infty}^p)$. Here one can invoke a theorem of Peetre ([1, pp. 116–119]) and require that $v = wh_1(v_1/w)$ and $w = vh_\infty(w_\infty/v)$, where h_1 and h_∞ are quasi concave functions. (If h_1 and h_∞ are strictly increasing then this will also automatically imply the condition of coincidence of each of the sets E_x^1 and E_x^∞ with sets E_y for suitable values of y.)

EXAMPLE. Given weight functions v and w we choose numbers α, β, $0 < \alpha < \beta < 1$ and let

$$v_1 = [v^{1/\alpha}/w^{1/\beta}]^{\alpha\beta/(\alpha-\beta)}$$

and

$$w_\infty = v^{1/\alpha} v_1^{1-1/\alpha} = w^{1/\beta} v_1^{1-1/\beta}\ .$$

It follows that $v = v_1^{1-\alpha} w_\infty^\alpha$ and $w = v_1^{1-\beta} w_\infty^\beta$ and v_1 and w_∞ have all the required properties. Consequently X is an interpolation space with respect to (L_v^p, L_w^p) if and only if it is an interpolation space with respect to $(L_{v_1}^p, L_w^p)$ and also with respect to $(L_v^p, L_{w_\infty}^p)$.

The above examples suggest that Theorem A and its corollary might be merely a special case of a much more general result. However in the remainder of this section we show that the most natural formulation of such a generalisation which one might conjecture is false. More specifically we give examples of spaces B_1, B_p, P_q and B_∞ (where the notation here has been chosen to stress the comparison with Theorem A and its corollary) such that B_p is an interpolation space with respect to (B_1, B_q) and B_q is an interpolation space with respect to (B_p, B_∞), and there exists a space A which is an interpolation space with respect to both (B_1, B_q) and (B_p, B_∞) but not with respect to (B_p, B_q).

We shall use Lorentz spaces ($L^{p,q}$ spaces) as defined in [1, p. 8], and interpolation spaces $(A_0, A_1)_{\theta,r}$ and $[A_0, A_1]^\theta$ obtained by the real method ([1, Chapter 3]) and the "second" Calderón complex method ([1, Chapter 4], [2]) respectively.

Let $r_1, r_\infty, s_1, s_\infty$ be numbers in $(1, \infty)$ such that $r_1 \neq r_\infty$, $s_1 \neq s_\infty$. Let

$$B_1 = L^{r_1,\infty} \oplus L^{s_1,1} \quad \text{and} \quad B_\infty = L^{r_\infty,\infty} \oplus L^{s_\infty,1}$$

where the underlying measure space is $\mathbf{R}_+$ with Lebesgue measure.

Let

$$B_p = (B_1, B_\infty)_{1/4,\infty} = (L^{r_1,\infty}, L^{r_\infty,\infty})_{1/4,\infty} \oplus (L^{s_1,1}, L^{s_\infty,1})_{1/4,\infty} = L^{r_*,\infty} \oplus L^{s_*,\infty}$$

where $1/r_* = 3/4r_1 + 1/4r_\infty$, $1/s_* = 3/4s_1 + 1/4s_\infty$. ([1, p. 113]). Similarly we take

$$B_q = (B_1, B_\infty)_{3/4,\infty} = L^{r_{**},\infty} \oplus L^{s_{**},\infty},$$

where $1/r = 1/3r_1 + 2/3r_{**} = (1/r_1 + 1/r_\infty)/2$ and similarly $1/s = 1/3s_1 + 2/3s_{**}$ and $B_q = (B_p, B_\infty)_{2/3,\infty}$. Now let

$$\begin{aligned} A &= [B_1, B_q]^{2/3} \\ &= [L^{r_1,\infty}, L^{r_{**},\infty}]^{2/3} \oplus [L^{s_1,1}, L^{s_{**},\infty}]^{2/3} \\ &= L^{r,\infty} \oplus L^{s,3}, \end{aligned}$$

where $1/r = 1/3r_1 + 2/3r_{**} = (1/r_1 + 1/r_\infty)/2$ and similarly $1/s = 1/3s_1 + 2/3s_{**} = (1/s_1 + 1/s_\infty)/2$. (See [2, pp. 124–125].) A similar calculation shows that $A = [B_p, B_\infty]^{1/3}$. Thus A is an interpolation space with respect to both (B_1, B_q) and (B_p, B_∞) but not with respect to (B_p, B_q) since, for example if we take $r_1 = s_1$ and $r_\infty = s_\infty$, the operator T defined by $T(f, g) = (g, f)$ is bounded on B_p and B_q but not on A.

We now modify the above construction so as to obtain another counterexample in which all of the spaces B_1, B_p, B_q, B_∞ and A are rearrangement invariant spaces of measurable functions on $\mathbf{R}_+$. Our approach is related to some ideas of V. I. Ovčinnikov ([8]). We simply replace direct sums by intersections and take

$$B_1 = L^{r_1,\infty} \cap L^{s_1,1}, \qquad B_\infty = L^{r_\infty,\infty} \cap L^{s_\infty,1}$$

$$B_p = L^{r_*,\infty} \cap L^{s_*,\infty}, \qquad B_q = L^{r_{**},\infty} \cap L^{s_{**},\infty}$$

and

$$A = L^{r,\infty} \cap L^{s,3} .$$

Here, as before, $r_1, r_\infty, s_1, s_\infty$ are in $(1,\infty)$ and $r_*, r_{**}, r, s_*, s_{**}$ and s are defined by the same formulae as above. However we now require that $s_1 = r_1'$ and $s_\infty = r_\infty'$, where, as usual, u' denotes the conjugate index of u, $(1/u + 1/u' = 1)$. It follows that $s_* = r_*'$, $s_{**} = r_{**}'$ and $s = r'$. We also choose both r_1 and r_∞ greater than 2 which ensures that $r_1 > s_1$, $r_\infty > s_\infty$, $r_* > s_*$, $r_{**} > s_{**}$ and $r > s$.

We require the following lemma whose proof will be given later.

LEMMA 3.2. *Let $E_j = L^{\alpha_j,\beta_j} \cap L^{\gamma_j,\delta_j}$ with $\alpha_j > \gamma_j$ for $j = 0, 1$, where all Lorentz spaces are taken on the same arbitrary measure space (X, Σ, μ). Then*

$$(1) \qquad [E_0, E_1]^\theta = [L^{\alpha_0,\beta_0}, L^{\alpha_1,\beta_1}]^\theta \cap [L^{\gamma_0,\delta_0}, L^{\gamma_1,\delta_1}]^\theta$$

$$(2) \qquad (E_0, E_1)_{\theta,\varrho} = (L^{\alpha_0,\beta_0}, L^{\alpha_1,\beta_1})_{\theta,\varrho} \cap (L^{\gamma_0,\delta_0}, L^{\gamma_1,\delta_1})_{\theta,\varrho}$$

for all $\theta \in (0,1)$ and $\varrho \in [1,\infty]$.

Using this lemma we can show in almost exactly the same fashion as in the previous counterexample that

$$B_p = (B_1, B_\infty)_{1/4,\infty} = (B_1, B_q)_{1/3,\infty}$$

$$B_q = (B_1, B_\infty)_{3/4,\infty} = (B_p, B_\infty)_{2/3,\infty}$$

and

$$A = [B_1, B_q]^{2/3} = [B_p, B_\infty]^{1/3} .$$

Thus, as before, A is an interpolation space with respect to both (B_1, B_q) and (B_p, B_∞) and it remains to find an example of an operator T which is bounded on B_p and on B_q but not on A. We shall define T by $Tf(x) = \int_0^{x^{-1}} f(t)\,dt$.

If $f \in L^{u,\infty}$ for $u \in (1,\infty)$, then

$$|Tf(x)| \leqq \int_0^{x^{-1}} f^*(t)\,dt$$

$$\leqq \|f\|_{L^{u,\infty}} \int_0^{x^{-1}} t^{-1/u}\,dt$$

$$\leqq \|f\|_{L^{u,\infty}} (1 - 1/u)^{-1} x^{-1/u'} .$$

Thus T maps $L^{u,\infty}$ into $L^{u',\infty}$ and consequently is a bounded operator on

$L^{u,\infty} \cap L^{u',\infty}$ for all u and in particular on B_p and on B_q. Since $s = r' < r$ the function $f(x) = x^{-1/r}\chi_{(0,1]}(x)$ is in $A = L^{r,\infty} \cap L^{s,3}$. However $Tf(x) = s \min(1, x^{-1/s})$ is not in $L^{s,3}$ and hence T is not a bounded operator on A.

PROOF OF LEMMA 3.2. We shall prove only (1), the proof of (2) being almost identical. First we observe that the inclusion

$$[E_0, E_1]^\theta \subset [L^{\alpha_0,\beta_0}, L^{\alpha_1,\beta_1}]^\theta \cap [L^{\gamma_0,\delta_0}, L^{\gamma_1,\delta_1}]^\theta$$

is an immediate consequence of the inclusions $E_j \subset L^{\alpha_j,\beta_j}$, $E_j \subset L^{\gamma_j,\delta_j}$, for $j = 0, 1$, all inclusions being continuous embeddings.

Now, turning to the proof of the reverse inclusion, we suppose that f is an element of the space

$$[L^{\alpha_0,\beta_0}, L^{\alpha_1,\beta_1}]^\theta \cap [L^{\gamma_0,\delta_0}, L^{\gamma_1,\delta_1}]^\theta .$$

This space is of course $L^{\alpha,\beta} \cap L^{\gamma,\delta}$, where $1/\alpha = (1-\theta)/\alpha_0 + \theta/\alpha_1$ and β, γ and δ are defined by analogous formulae. In particular it follows that $\alpha > \gamma$.

At this point it is convenient to assume that the measure space X is non atomic and infinite. (The extension to the case of an arbitrary measure space is routine.) Let

$$W_0 = \{x \in X \mid |f(x)| > f^*(1)\} \quad \text{and}$$
$$W_n = \{x \in X \mid f^*(n) \geqq |f(x)| > f^*(n+1)\} \quad \text{for } n = 1, 2, \dots .$$

If $f^*(t)$ is strictly decreasing and strictly positive for all $t > 0$, then $\mu(W_n) = 1$ for each n. (Otherwise we can replace f by a larger function $\tilde{f}$ in $L^{\alpha,\beta} \cap L^{\gamma,\delta}$ with these properties and show that $\tilde{f}$ is in $[E_0, E_1]^\theta$ implying the same for f.) We have

$$|f| \leqq f_0 + f_1 \quad \text{where } f_0 = |f|\chi_{W_0}$$

and

$$f_1 = \sum_{n=1}^{\infty} \left(\int_{W_{n-1}} |f| \, d\mu \right) \chi_{W_n} ,$$

and clearly both f_0 and f_1 are in $L^{\alpha,\beta} \cap L^{\gamma,\delta}$.

Let μ_0 be the measure on X defined by $\mu_0(E) = \mu(E \cap W_0)$. Let μ_1 be the (atomic) measure on X such that each W_n is an atom and $\mu_1(W_n) = 1$ for $n = 1, 2, \dots$ and $\mu_1(X \setminus \bigcup_{n=1}^{\infty} W_n) = 0$. Let I_0 be the natural embedding of the space of functions supported on W_0 into the space of functions on X. Let I_1 be the natural embedding of the space of functions which assume constant values on each of the sets W_n, $n = 1, 2 \dots$ into the space of functions on X. It is a straightforward matter to show that I_0 maps $L^{\alpha_j,\beta_j}(\mu_0)$ boundedly into E_j and

I_1 maps $L^{\gamma_j,\delta_j}(\mu_1)$ boundedly into E_j for $j=0,1$, since $\alpha_j>\gamma_j$. Consequently I_0 maps $L^{\alpha,\beta}(\mu_0)$ boundedly into $[E_0,E_1]^\theta$ and I_1 maps $L^{\gamma,\delta}(\mu_1)$ boundedly into $[E_0,E_1]^\theta$. So

$$|f| \leqq f_0+f_1 = I_0 f_0 + I_1 f_1 \in [E_0,E_1]^\theta .$$

This completes the proof of the lemma.

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