MATH. SCAND. 55 (1984), 285–309

THE SUPPORT OF FUNCTIONS AND DISTRIBUTIONS WITH A SPECTRAL GAP

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Various aspects of the following general uniqueness problem in Fourier analysis have been considered by several authors:

PROBLEM A. Suppose that f is a function or tempered distribution supported on a set A in d-dimensional Euclidean space $\mathbf{R}^d$, the d-dimensional torus $\mathbf{T}^d$ or the d-dimensional integer lattice $\mathbf{Z}^d$, such that its Fourier transform $\hat{f}$ is supported on B. Then, given some *a priori* conditions on f and/or $\hat{f}$, which conditions on the sets A and B imply that $f=0$, and conversely, when does there exist a non-zero function or distribution such that $\operatorname{supp}(f)\subset A$, $\operatorname{supp}(\hat{f})\subset B$?

The appropriate definition of the support may vary from case to case. Normally we use the closed supports of distribution theory but sometimes we just consider $\{x; f(x)\neq 0\}$.

The case when $A=\mathbf{T}\setminus I$, where I is an interval and $a=|A|<2\pi$, has been treated carefully in Levinsons book [13]. By duality such a uniqueness problem is equivalent to a closure problem on a finite interval. The general closure problem for complex exponentials was solved by Beurling and Malliavin in their famous paper [7], where they give necessary and sufficient conditions for the span of the exponentials $\{e^{i\lambda_n x}\}$ to be dense in $L^2(0,a-\varepsilon)$ for all $\varepsilon>0$. For a more recent survey of questions of closure and uniqueness see Redheffer [14].

If $f\in L^1(\mathbf{T})$ and $A=\mathbf{T}\setminus E$, where E is a set of positive measure, and there is an integer n_0 such that $\hat{f}(n)=0$ for $n\leqq n_0$, then $f=0$ (second theorem of F. and M. Riesz). Conversely it is known (see e.g. Katznelson [12, Lemma 3.13]) that given a closed set E of measure 0, there is always a function f holomorphic and $\neq 0$ in the open and continuous in the closed unit disc such that the boundary values of f vanish on E. For absolutely convergent Taylor series the sets of uniqueness were studied by Carleson [10].

In the case of the real line it is well-known that if $f\in L^1(\mathbf{R})$ and $\hat{f}(x)=0$,

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Received November 11, 1981; in revised form May 15, 1983.

$x \leqq x_0$, then f cannot vanish on a set of positive measure unless it is identically equal to 0.

Except for this result, to the author's knowledge, very little is written about uniqueness for functions on R or R^d.

One simple result of this type is the following theorem, where a question by H. S. Shapiro was affirmatively answered.

THEOREM 1. *Let $f \in L^p(\mathsf{R}^d)$, $1 \leqq p \leqq \infty$, and $\hat{f}$ be its Fourier transform. Suppose that the sets $\{t \in \mathsf{R}^d; f(t) \neq 0\}$ and $\{x \in \hat{\mathsf{R}}^d; \hat{f}(x) \neq 0\}$ both have finite Lebesgue measure. Then $f=0$.*

This was proved by the author in [3]. Later Amrein and Berthier [2] gave another proof of Theorem 1 using Hilbert space methods. They also prove the following existence theorem, which is in a sense complementary to Theorem 1.

THEOREM 2. (Amrein and Berthier [2].) *Assume that $\complement A \subset \mathsf{R}^d$ and $\complement B \subset \hat{\mathsf{R}}^d$ both have finite Lebesgue measure. Then the set of functions satisfying $f = f\chi_A$, $\hat{f} = \hat{f}\chi_B$, form an infinitely dimensional subspace of $L^2(\mathsf{R}^d)$.*

Here χ_E denotes as usual the characteristic function of the set E.

However the following question that arises naturally in this connection remains to our knowledge unsolved:

PROBLEM B. Does there exist a not identically vanishing function in $L^1(\mathsf{R}^d)$ supported on a set of finite Lebesgue measure such that its Fourier transform vanishes on a set of infinite measure?

However it may very well hold that the Fourier transform of a not identically vanishing $L^1(\mathsf{R}^d)$ (or even $L^\infty(\mathsf{R}^d)$) function supported on a set of finite measure vanishes on a cube in R^d. This follows from Theorem 6 below.

The main aim of this paper is to investigate the following question:

PROBLEM C. Given a set $A \subset \mathsf{R}$; when does there exist a distribution in $\mathscr{S}'(\mathsf{R})$ or a function in $L^p(\mathsf{R})$, not identically vanishing and supported on A such that its Fourier transform vanishes on an interval of length 2a (or some nonempty open interval)?

In the case of functions this may be viewed as the special case $d\Delta = \chi_A dx$ of a problem posed by H. Dym [11].

PROBLEM D. Let $d\Delta$ be a measure of finite total variation on the real line and

$$\tilde{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\, d\Delta(t) .$$

Determine

$$a_0(\Delta) = \inf\{a>0\,;\ \tilde{f}(t)=0 \text{ for } |t|\leqq a \ \Rightarrow\ f=0 \text{ a.e. } (d\Delta)\} .$$

Problem C with f replaced by a measure of finite total variation has been treated by Beurling [5] and de Branges [8]. Beurling gives in [5, Theorem IV and Corollary 4.1] a nice sufficient condition for uniqueness, but it is not necessary. We shall however see that his condition in a sense is best possible — in the respect that it gives only a condition on the complement of supp (f) (Theorem 6). De Branges [8, Theorem 66], gives a necessary and sufficient condition for uniqueness in the case of measures of finite total variation. However his condition is sometimes hard to verify.

We will give a new proof of Beurling's theorem based on the Fourier-Carleman transform and harmonic majorants. Furthermore we will give an improved version of his theorem (Theorem 7), in the case the support is a sequence of regularily distributed intervals and also give an existence theorem (Theorem 6), which shows that Theorem 7 is best possible. It moreover shows the existence of an L^∞ function $\neq 0$ supported on a set of finite measure, and with a given spectral gap.

2. A uniqueness theorem depending on one-sided quasianalyticity.

The following result is a consequence of Theorem XXV in Levinson [13]:

PROPOSITION 1. *Suppose that $f \in L^1(\mathbf{R})$ and that its Fourier-transform $\hat{f}$ vanishes on an interval. Let*

$$F(u) = \int_u^\infty |f(t)|\, dt$$

and suppose that

$$\int_1^\infty \frac{\log|F(u)|}{1+u^2}\, du = -\infty .$$

Then $f=0$.

Beurling [5, Corollary 2.1] proved this result under the less restrictive assumption that $\hat{f}$ vanishes on a set of positive measure.

The following uniqueness result is a direct consequence of Proposition 1

THEOREM 3. *Suppose $f \in L^1(\mathbf{R}) \cap L^p(\mathbf{R})$ for some $p>1$, put $E=\{t \in \mathbf{R}\,; f(t)\neq 0\}$*

and suppose $\hat{f}$ vanishes on an interval (or more generally on a set of positive measure). Let

$$\begin{cases} \Psi_1(u) = \int_u^{\infty} \chi_E(t)\,dt \\ \Psi_2(-u) = \int_{-\infty}^{u} \chi_E(t)\,dt\,. \end{cases}$$

If

$$\int_1^{\infty} \frac{\log \Psi_j(u)}{1+u^2}\,du = -\infty$$

for $j=1$ or 2, then $f=0$.

PROOF. We prove the result when the decay condition holds on the positive real axis ($j=1$). Let q be the dual exponent of p. We have

$$F(u) = \int_u^{\infty} |f(t)|\,dt \leqq \|f\|_p \left(\int_u^{\infty} \chi_E(t)\,dt \right)^{1/q}.$$

Proposition 1 (alternatively Beurling's improvement of Proposition 1) immediately gives the result.

COROLLARY. *Suppose that*

(i) $f \in L^p(\mathrm{R})$ *for some* $p>1$,

(ii) $\hat{f}$ *vanishes on some interval,*

(iii) $\{t \in \mathrm{R}\ ;\ f(t) \neq 0,\ t \geqq 0\} \subseteq \bigcup_{k=1}^{\infty} [x_k - d_k, x_k + d_k]$ *with* $0 < c_1 \leqq x_k - x_{k-1} \leqq c_2$,

(iv) $\sum_{n=1}^{\infty} \frac{\log s_n}{n^2} = -\infty$, *where* $s_n = \sum_{k=n}^{\infty} d_k$.

Then $f=0$.

The condition (iv) is essentially best possible as follows from Theorem 6.

Note that it is essential in Theorem 3 and its corollary that $f \in L^p$ for some $p>1$. For $p=1$ there are counterexamples (see Theorem 9).

3. The Fourier-Carleman transform.

Let f be a function such that $e^{-\varepsilon|t|}f(t) \in L^1(\mathrm{R})$ for all $\varepsilon>0$. We then define its Fourier-Carleman transform as follows (cf. Bremermann [9], $z=x+iy$):

$$\mathscr{F}(f,z) = \begin{cases} \int_{-\infty}^{0} f(t)e^{-izt}\,dt, & y>0 \\ -\int_{0}^{\infty} f(t)e^{-izt}\,dt, & y<0 . \end{cases}$$

For a function $\hat{f}$ such that $|\hat{f}(\omega)|e^{-\varepsilon|\omega|} \in L^1(\mathsf{R})$ for all $\varepsilon>0$, the inverse Fourier-Carleman transform is similarily defined as

$$\mathscr{F}^{-1}(\hat{f},z) = \begin{cases} -\frac{1}{2\pi}\int_{0}^{\infty} \hat{f}(\omega)e^{iz\omega}, & y>0 \\ \frac{1}{2\pi}\int_{-\infty}^{0} \hat{f}(\omega)e^{iz\omega}\,d\omega, & y<0 . \end{cases}$$

The following is an immediate consequence of Plancherel's theorem.

LEMMA 1. *If $f \in L^p(\mathsf{R})$, $1 \leqq p \leqq 2$, the Cauchy transform of f is equal to the inverse Fourier-Carleman transform of its Fourier transform $\hat{f}$, that is,*

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{z-t}\,dt = \mathscr{F}^{-1}(\hat{f},z) \tag{3.1}$$

for $y \neq 0$.

(3.1) immediately shows that $\mathscr{F}^{-1}(\hat{f},z)$ has an analytic continuation across the intervals, which compose the open set $\mathsf{R}\setminus \operatorname{supp}(f)$, where $\operatorname{supp}(f) = \overline{\{t \in \mathsf{R};\, f(t) \neq 0\}}$.

Moreover, if $\hat{f} \in L^p(\hat{\mathsf{R}})$ has a gap in its support, $\operatorname{supp}(\hat{f}) \cap [-A,A] = \emptyset$, a simple estimate using Hölder's inequality shows that (q is the dual exponent of p)

$$|\mathscr{F}^{-1}(\hat{f},z)| \leqq \frac{1}{2\pi}\frac{1}{q^{1/q}}\frac{1}{|y|^{1/q}}e^{-A|y|}\|\hat{f}\|_p . \tag{3.2}$$

The Fourier-Carleman transform (and the inverse Fourier-Carleman transform) may also be defined for a tempered distribution T (Bremermann [9]) and estimates of the type (3.2) hold, if $\operatorname{supp}(\hat{T}) \cap (-A,A) = \emptyset$. In fact, if T is a tempered distribution, there are integers m and k such that

$$|\mathscr{F}^{-1}(\hat{T},z)| \leqq C_T \frac{|z|^m}{|y|^k} e^{-A|y|} , \tag{3.3}$$

and $\mathscr{F}^{-1}(\hat{T},z)$ has an analytic continuation across $\mathsf{R}\setminus\operatorname{supp}(T)$.

Also results analogous to the Paley-Wiener theorem hold.

To state these results we first need the following:

DEFINITION. A function F, holomorphic in the upper half plane is said to be of *bounded type* if it may be written as

$$F = G/H ,$$

where G and H are bounded holomorphic functions in the upper half plane.

By a theorem, which in the case of the unit disc is due to R. Nevanlinna, each function F of bounded type in the upper halfplane may be represented as (see e.g. de Branges [8, p. 22])

$$F(z) = B(z)\exp\{-ihz+G(z)\} , \tag{3.4}$$

where $B(z)$ is a Blaschke product (in the upper halfplane), h is a real number and $G(z)$ is a function holomorphic in the upper halfplane such that

$$\operatorname{Re} G(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{d\mu(t)}{(t-x)^2+y^2}$$

for some real measure μ satisfying

$$\int_{-\infty}^{\infty}\frac{d|\mu|}{1+t^2} < \infty .$$

That the representation (3.4) holds is actually a necessary and sufficient condition for F to be of bounded type.

When F is a function of bounded type, the *mean type* of F is defined as the number h in the representation (3.4).

Of course a similar theory exists for the lower halfplane. If F is of bounded type in the lower halfplane then

$$F(z) = B(z)\exp\{ihz+G(z)\}, \qquad \operatorname{Im} z < 0 ,$$

where B is a Blaschke product and $\operatorname{Re} G$ is the Poisson integral of a measure. The number h is called the mean type of F in the lower halfplane. (Note the sign change relative to (3.4).)

LEMMA 2. *Suppose that F is a function holomorphic in $\mathbf{C}\setminus E$, where E is a closed subset of $\mathbf{R}$. Let Π_+ and Π_- denote the open upper and lower halfplanes respectively. Assume that $F|_{\Pi_+}$ and $F|_{\Pi_-}$ both have continuous extensions to the closed halfplanes and that the boundary values*

$$F_1(x) = \lim_{y\to+0} F(x+iy) \quad \textit{and} \quad F_2(x) = \lim_{y\to-0} F(x+iy)$$

both belong to $L^p(\mathbf{R})$, $1 \leqq p \leqq 2$. Furthermore assume that F is of bounded type in both the upper and lower halfplanes and has mean type $-a$ in both halfplanes. Then $f = F_1 - F_2$ is a function in $L^p(\mathbf{R})$ such that $\hat{f}|_{(-a,a)} = 0$ a.e.

Proof. The function $H(z)=e^{-iaz}F(z)$ is of bounded type in the upper halfplane and of non-positive mean type. Clearly

$$\lim_{y\to\pm 0} H(\cdot+iy)\in L^p(\mathbf{R}) .$$

By de Branges [8, Theorem 12], extended to the L^p-case, $1\leqq p\leqq 2$, it follows that

$$H(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{H(t)}{t-z}\,dt, \qquad \operatorname{Im} z>0 .$$

By Lemma 1

$$H(z) = \int_0^\infty e^{iz\xi}\,\hat{H}(\xi)\,d\xi, \qquad \operatorname{Im} z>0 .$$

Hence

$$\begin{aligned} F(z) &= e^{iaz}\int_0^\infty e^{iz\xi}\hat{H}(\xi)\,d\xi \\ &= \int_a^\infty e^{iz\xi}\hat{H}(\xi-a)\,d\xi = \int_a^\infty e^{iz\xi}h_1(\xi)\,d\xi . \end{aligned}$$

In a similar way

$$F(z) = -\int_{-\infty}^{-a} e^{-iz\xi}h_2(\xi)\,d\xi, \qquad \text{for } \operatorname{Im} z<0 ,$$

and consequently the Fourier transform of $F_1(x)-F_2(x)$ is 0 a.e. on $(-a,a)$.

4. Uniqueness results.

4.1. *Beurlings uniqueness theorem.*

To illustrate our methods, we will give a new proof of a version of a theorem of Beurling [5].

Theorem 4. *Suppose that T is a tempered distribution on $\mathbf{R}$ and that $\complement \operatorname{supp}(T)$ contains the disjoint union of closed intervals*

$$\bigcup_{n=1}^{\infty}\,[l_n-a_n,\ l_n+a_n] ,$$

where

$$0<l_1<l_2<\ldots<l_n<\ldots\nearrow\infty ,$$

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{l_n}\right)^2 = \infty \tag{4.1}$$

and that $\hat{T}=0$ on an open interval.

Then $T=0$.

PROOF. We will first reduce the theorem to the case, when there is a constant b such that

(i) $a_n \geqq b > 0$ for all n
(ii) $T \in \mathscr{S}(\mathbf{R})$.

$\mathscr{S}(\mathbf{R})$ denotes the Schwartz class of rapidly decreasing, infinitely differentiable functions.

Sum over those n for which $a_n \leqq b$:

$$\sum_{a_n \leqq b} \left(\frac{a_n}{l_n}\right)^2 \leqq \sum_{\substack{a_n \leqq b \\ n \geqq 2}} \frac{a_n b}{l_n l_{n-1}} + \frac{b^2}{l_1^2} \leqq \sum_{n=2}^{\infty} \frac{b(l_n - l_{n-1})}{l_n l_{n-1}} + \frac{b^2}{l_1^2}$$
$$= \sum_{n=2}^{\infty} b\left(\frac{1}{l_{n-1}} - \frac{1}{l_n}\right) + \frac{b^2}{l_1^2} = \frac{b}{l_1} + \frac{b^2}{l_1^2} < \infty .$$

Therefore there is a subfamily of intervals, such that (i) and (4.1) hold. Then convolve first $\hat{T}$ and then T with a $C^\infty(\mathbf{R})$ function of sufficiently small compact support to obtain the reduced situation.

Now we turn to the proof of Theorem 4 assuming (i) and (ii). Let

$$\begin{cases} F(z) = (2\pi/\|\hat{T}\|_1)\mathscr{F}^{-1}(\hat{T}, z) \\ u(z) = -\log|F(z)| . \end{cases}$$

$u(z)$ is superharmonic in $\Omega = \mathbf{C} \setminus \operatorname{supp}(T)$ and $u(z) \geqq A|y|$. Now take its average over circles centered at x, $l_n - a_n/2 \leqq x \leqq l_n + a_n/2$, of radius $a_n/2$. The superharmonicity gives that

$$u(x) \geqq \frac{A}{\pi} a_n, \quad l_n - \frac{a_n}{2} \leqq x \leqq l_n + \frac{a_n}{2} ,$$

and clearly

$$u(i) \geqq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x)dx}{1+x^2} \geqq \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{l_n - a_n/2}^{l_n + a_n/2} \frac{u(x)}{1+x^2} dx$$
$$\geqq AC_1 \sum_{n=1}^{\infty} \int_{l_n - a_n/2}^{l_n + a_n/2} \frac{a_n dx}{1+x^2} = C_2 A \sum_{n=1}^{\infty} \left(\frac{a_n}{l_n}\right)^2 = \infty ,$$

where C_1 and C_2 are numerical constants.

The rôle of the point $z=i$ is not particular and it follows that $F \equiv 0$, so Theorem 4 is proved.

4.2. *A uniqueness theorem for Fourier transforms of measures.*

In this section we will need the notion of a "local harmonic measure" $\beta_E(x)$ introduced in [4]. It is defined as follows:

Let E be a closed subset of $\mathbf{R}$, let α be a real number, $0<\alpha<1$, and let K_x be the square in the complex plane with midpoint $x \in \mathbf{R}$, sidelength $\alpha|x|$ and the sides parallell to the coordinate axes. Let $w^x(z)$ be the harmonic measure of ∂K_x in $K_x \setminus E$, i.e. let $w^x(z)$ solve the Dirichlet problem

$$w^x(z) = \begin{cases} 0 & \text{on } E \\ 1 & \text{on } \partial K_x \end{cases}$$

$$\Delta w^x(z) = 0 \quad \text{in } \Omega = K_x \setminus E .$$

(All points of E are assumed to be regular for Dirichlet's problem.) Then $\beta_E(x)$ is defined as $w^x(x)$.

THEOREM 5. *Let $\mu \in M(\mathbf{R})$, the set of measures of finite total variation, and suppose there is an open interval I such that* $\operatorname{supp}(\mu) \cap I = \emptyset$. *Suppose that all points of $E = \operatorname{supp}(\hat{\mu})$ are regular for Dirichlet's problem and that E is so thin that*

$$\int_{|x| \geq 1} \frac{\beta_E(x)}{|x|}\, dx = \infty . \tag{4.2}$$

Then $\mu = 0$.

Note that the condition (4.2) is independent of α. This is a consequence of Theorem 4 in [4].

PROOF. Without loss of generality assume that $I = (-A, A)$, $A>0$. Define

$$F(z) = \mathscr{F}^{-1}(\mu, z) = \begin{cases} -\dfrac{1}{2\pi}\displaystyle\int_A^{\infty} e^{iz\omega}\, d\mu(\omega), & y>0 \\[2ex] \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{-A} e^{iz\omega}\, d\mu(\omega), & y<0 . \end{cases}$$

It follows that $F(z)$ is holomorphic in $\Omega = \mathbf{C} \setminus E$ and that

$$|F(z)| \leqq \frac{\|\mu\|}{2\pi} e^{-A|y|} .$$

Let

$$u(z) = -\log\left(\frac{|F(z)|}{\|\mu\|} 2\pi\right) \geqq A|y| .$$

The situation is now analogous to the one in the proof of Theorem 4 in [4] and we conclude that $u(z) \equiv \infty$ for $y \neq 0$, hence $F=0$ and $\mu=0$.

Note that if the condition

$$\int_{|x| \geqq 1} \frac{\beta_E(x)}{|x|} dx < \infty$$

holds, Theorem 4 of [4] proves the existence of a positive harmonic function u, zero on E such that $u(z) \geqq |y|$. To give a converse of Theorem 5 we would like to prove the existence of a nonzero function F, holomorphic in in Ω and satisfying $|F(z)| \leqq e^{-A|y|}$. However the function u constructed is not necessarily $= -\log|F|$ for some function F holomorphic in Ω. The converse problem will instead be dealt with by the methods of section 5.

5. A general existence theorem for functions with small support and a spectral gap.

We will construct functions with a spectral gap supported on closed sets, which are disjoint unions of closed intervals of positive distance to each other. A support set may be written

$$E = \bigcup_{n=-\infty}^{\infty} [x_n - d_n, x_n + d_n] \tag{5.1}$$

and its complement is

$$\complement E = \bigcup_{n=-\infty}^{\infty} (l_n - a_n, l_n + a_n)\,, \tag{5.2}$$

where

$$\begin{cases} l_n + a_n = x_n - d_n \\ l_n - a_n = x_{n-1} + d_{n-1}\,. \end{cases} \tag{5.3}$$

The main result in this section is

THEOREM 6. *Let*

$$E = \bigcup_{n=-\infty}^{\infty} [x_n - d_n, x_n + d_n], \qquad d_n > 0\,,$$

be a disjoint union of closed intervals and let the sequences $\{l_n\}_{n=-\infty}^{\infty}$ *and* $\{a_n\}_{n=-\infty}^{\infty}$ *be defined by* (5.3). *Assume that there are constants* C_1, C_2 *and* C_3 *such that*

(i) *if* $l_n l_m > 0$ *and* $C_1^{-1} \leqq \left|\frac{l_n}{l_m}\right| \leqq C_1$, *then* $C_2^{-1} \leqq \left|\frac{a_n}{a_m}\right| \leqq C_2$,

(ii) $C_1^{-1} \leqq \left|\frac{l_{n+1}}{l_n}\right| \leqq C_1$,

(iii) $a_n \geqq C_3 \max(1, d_n)$,

(iv) $\sum_{n=-\infty}^{\infty} \left(\frac{a_n}{l_n}\right)^2 \left[\log^+\left(\frac{a_n}{d_n}\right) + 1\right] < \infty$.

Then given any real numbers $A > 0$ *and* p, $1 \leqq p < \infty$, *there is a nonzero function* $f \in L^1(\mathbf{R}) \cap L^p(\mathbf{R}) \cap C^\infty(\mathbf{R})$ *supported on* E *that* $\hat{f}|_{(-A,A)} = 0$.

PROOF. Pick $p \geqq 2$ and $A > 0$. By Lemma 2 it is enough to construct a function F, holomorphic in $\mathbf{C} \setminus E$ such that F is of mean type $\leqq -A$ both in the upper and lower halfplanes and $F(x \pm i0) \in L^1(\mathbf{R}) \cap L^p(\mathbf{R}) \cap C^\infty(\mathbf{R})$.

Let N be an integer satisfying $\pi N > A$. $F(z)$ is constructed as

$$F(z) = e^{-NG(z)} M(z),$$

where

$$\operatorname{Re} G(z) = u(z) = \lim_{T \to \infty} \int_{-T}^{T} \log\left|1 - \frac{z}{t}\right| d\mu(t), \tag{5.4}$$

$d\mu$ is a positive measure supported on E such that

$$\mu(t) = \int_0^t d\mu(u)$$

is close to t for all t, $-\infty < t < \infty$, in a sense made precise later and

$$\int_{x_n - d_n}^{x_n + d_n} d\mu(t) \quad \text{is an integer for all } n. \tag{5.5}$$

The property (5.5) makes it possible to define the function $e^{-N(u(z) + i\tilde{u}(z))}$ as a holomorphic function in $\mathbf{C} \setminus E$, since the conjugate function $\tilde{u}(z)$ is well-defined mod 2π in $\mathbf{C} \setminus E$. M is a Beurling-Malliavin multiplier (Beurling and Malliavin [6]), i.e. an entire function of exponential type $< \pi N - A$ "multiplying down" $|e^{-NG(x)}| = e^{-Nu(x)}$ on the real axis. We require that $M(x)$ decays so fast that

$$\begin{cases} |M(x)| e^{-pNu(x)} \in L^1(\mathbf{R}) \\ |M(x)| e^{-Nu(x)} \in L^1(\mathbf{R}) \\ |M(x)| \leqq 1, \quad x \in \mathbf{R}. \end{cases} \tag{5.6}$$

These conditions imply that $M(x) e^{-Nu(x)} \in L^1(\mathbf{R}) \cap L^p(\mathbf{R})$.

$\mu(t)$ is defined as $\mu(t) = t + \varphi(t)$, where $\varphi(t)$ is a C^∞ function such that

(a) $\varphi'(t) = -1$ on $(l_n - a_n, l_n + a_n)$
(b) $\varphi(x_n + d_n) - \varphi(x_n - d_n) = -2d_n + \text{integer}$
(c) $|\varphi(x_n - d_n) + a_n| \leqq 1$
(d) $|\varphi(x_n + d_n) - a_{n+1}| \leqq 1$
(e) $\varphi(0) = 0$
(f) $\|\varphi'\|_\infty \leqq C \dfrac{a_n}{d_n}$ on $[x_n - d_n, x_n + d_n]$
(g) $\|\varphi\|_\infty \leqq Ca_n$ on $[l_n - a_n, x_n + a_n]$.

The estimate (g) and (iv) immediately give

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|}{1+t^2} dt \leqq C \sum_{n=-\infty}^{\infty} \left(\frac{a_n}{l_n}\right)^2 < \infty .$$

Let us also observe that (iv) implies that $a_n/l_n \to 0$ as $n \to \pm\infty$ and consequently also

$$\frac{\varphi(t)}{t} \to 0 \quad \text{as } t \to \pm\infty . \tag{5.7}$$

We now claim that the limit in (5.4) exists. For $\operatorname{Im} z > 0$ a partial integration gives

$$\int_{-T}^{T} \log\left|1 - \frac{z}{t}\right| d\mu(t) = \operatorname{Re}\left[\mu(t)\cdot\log\left(1 - \frac{z}{t}\right)\right]_{-T}^{T} + \operatorname{Re} z \int_{-T}^{T} \frac{\mu(t)}{t(z-t)} dt .$$

From (5.7) it follows that

$$\lim_{T\to\infty} [\mu(t)\cdot\log(1 - z/t)]_{-T}^{T} = 0 .$$

Hence

$$z \int_{-T}^{T} \frac{\mu(t)}{t(z-t)} dt = z \int_{-T}^{T} \frac{t + \varphi(t)}{t(z-t)} dt$$

$$\to -i\pi z + z \int_{-\infty}^{\infty} \frac{\varphi(t)}{t(z-t)} dt$$

as $T \to \infty$ and we have proved our claim.

We conclude that

$$e^{-NG(z)} = \exp\left\{ \pm \pi N i z + Nz \int_{-\infty}^{\infty} \frac{-\varphi(t)}{t(z-t)} dt \right\}$$
$$= \exp\{\pm \pi N i z\} F_1(z) ,$$

where the + sign (− sign) is chosen in the upper (lower) halfplane.

To apply Lemma 2 it is necessary to show that it is possible to multiply down the boundary values $|e^{-NG(x)}|=e^{-Nu(x)}$ with an entire function $M(z)$ of exponential type $\varepsilon<\pi N-A$ so that (5.6) holds.

Moreover we have to show that $M(z)e^{-NG(z)}$ is of mean type $\leqq -A$ in both halfplanes.

Let $\psi(t)=\varphi(t)/t$ and $\tilde{\psi}$ be its Hilbert transform. Note that $\varphi(t)=O(t)$ as $t \to 0$. By redefining φ on an interval $[x_n-d_n, x_n+d_n]$ we may accomplish that $\tilde{\psi}(0)=0$. We have

$$|F_1(x)| = \exp\{-N\pi x\tilde{\psi}(x)\} .$$

Let

$$\sigma(x) = N\pi(|\tilde{\psi}(x)|+|\tilde{\psi}(-x)|), \qquad x>0 ,$$

and

$$\sigma(x) = -\sigma(-x), \qquad x<0 .$$

Then $|F_1(x)|\leqq\exp\{x\sigma(x)\}$ and by Lemma I and III of Beurling and Malliavin [6] it follows that it is possible to multiply down

$$|F_1(x)| = e^{-Nu(x)}$$

by a function M of small exponential type so that (5.6) holds provided that

$$\sigma(z) = P_z\sigma = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\sigma(t)}{(x-t)^2+y^2}\,dt$$

has finite Dirichlet integral,

$$\mathscr{D}(\sigma) = \int_0^{\infty}\int_0^{\infty} |\operatorname{grad}\sigma|^2\,dxdy < \infty ,$$

and

$$\int_0^{\infty} \frac{\sigma(t)}{t}\,dt < \infty .$$

The Dirichlet integral can be expressed in $\sigma(x)$ by means of the Douglas functional (cf. e.g. Ahlfors [1, Theorem 2–5] for the corresponding formula in the case of the unit circle, which is easily transferred to the upper halfplane)

$$\mathscr{I}(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\sigma(x)-\sigma(y)}{x-y}\right)^2 dxdy .$$

Since

$$\mathscr{D}(|\tilde{\psi}(x)|) = \mathscr{I}(|\tilde{\psi}(x)|) \leqq \mathscr{I}(\tilde{\psi}(x)) = \mathscr{D}(\tilde{\psi}) = \mathscr{D}(\psi) = \mathscr{I}(\psi) ,$$

it is enough to verify that $\mathscr{I}(\psi)<\infty$.

We have

$$\mathcal{I}(\psi) \leqq \frac{1}{2\pi} \iint_{\substack{|u| \leqq 1 \\ |t| \leqq 1}} \left(\frac{\psi(u)-\psi(t)}{u-t}\right)^2 dudt + \frac{1}{2\pi} \iint_{|t| \geqq 1} + \frac{1}{2\pi} \iint_{|u| \geqq 1} .$$

The integral over the unit square is finite. By symmetry it is sufficient to estimate

$$\int_1^\infty dt \int_{-\infty}^\infty \left(\frac{\psi(t)-\psi(u)}{t-u}\right)^2 du .$$

Furthermore, by interchanging the order of integration

$$\int_1^\infty dt \int_{|u-t| \geqq \alpha t} \left(\frac{\psi(t)-\psi(u)}{t-u}\right)^2 du$$

$$\leqq C \int \frac{\psi^2(u)}{1+|u|} du \leqq C \sum_{n=-\infty}^{\infty} \left(\frac{a_n}{l_n}\right)^3 < \infty .$$

Hence it only remains to estimate

$$\int_1^\infty dt \int_{|u-t| \leqq \alpha t} \left(\frac{\psi(u)-\psi(t)}{u-t}\right)^2 du .$$

Assume first that $t \in I_n = [l_n - a_n, l_n + a_n]$. Clearly

$$\int_{I_n} \left(\frac{\psi(u)-\psi(t)}{u-t}\right)^2 du \leqq C \frac{a_n}{l_n^2} .$$

An explicit calculation gives that the integrals over the adjacent intervals I_{n+1}, I_{n-1}, and J_n, J_{n-1} with

$$J_n = [x_n - d_n, x_n + d_n]$$

gives a contribution which may be estimated by

$$C \frac{a_n}{l_n^2} \log^+ \frac{a_n}{d_n} .$$

Furthermore the remaining values of u are contained in the set $\{u; \gamma a_n \leqq |u-t| \leqq \alpha t\}$, where γ is independent of n and

$$\int_{\gamma a_n \leqq |u-t| \leqq \alpha t} \left(\frac{\psi(u)-\psi(t)}{u-t}\right)^2 du \leqq C \left(\frac{a_n}{l_n}\right)^2 \int_{|u| \geqq \gamma a_n} \frac{1}{u^2} du \leqq C' \frac{a_n}{l_n^2} .$$

Similar estimates hold also for $t \in J_n$. Summing up we get

$$\int_1^\infty dt \int_{|u-t|\leqq \alpha t} \left(\frac{\psi(u)-\psi(t)}{u-t}\right)^2 du$$
$$\leqq \sum_{n=1}^{\infty} \int_{I_n \cup J_n} dt \int_{|u-t|\leqq \alpha t} \left(\frac{\psi(u)-\psi(t)}{u-t}\right)^2 du$$
$$\leqq C \sum_{n=-\infty}^{\infty} \left(\frac{a_n}{l_n}\right)^2 \left[\log^+ \frac{a_n}{d_n} + 1\right] < \infty .$$

We turn to the proof that

$$\exp\left\{\int_{-\infty}^{\infty} \frac{z\psi(t)}{z-t} dt\right\}$$

is of bounded type in both the upper and lower halfplanes. This follows from

LEMMA 3. *Let $F_1(z)=\exp\{-zf(z)\}$, where f is analytic in the upper halfplane Π_+ and has a finite Dirichlet integral*

$$\mathscr{D}(f) = \iint_{\Pi_+} |f'(z)|^2 \, dxdy < \infty .$$

Then F_1 is of bounded type in the upper halfplane.

PROOF. By Lemma 12 in de Branges [8] it follows that given any real number $\varepsilon>0$ there is an analytic function g with $\mathscr{D}(g)<\infty$ and such that

$$\begin{aligned} \operatorname{Re} -zf(z) &\leqq \operatorname{Re} -zg(z) \\ \operatorname{Re} izg'(z) &\geqq -\varepsilon . \end{aligned}$$

By the proof in de Branges [8, p. 260], of the Beurling-Malliavin theorem $\exp\{-zg(z)\}$ is of bounded type in the upper halfplane. By Herglotz' theorem there is a real number $\varkappa \geqq 0$ and a positive measure μ such that

$$\operatorname{Re}(-zg(z))-\operatorname{Re}(-zf(z)) = \varkappa y + \int_{-\infty}^{\infty} \frac{y\, d\mu(t)}{(x-t)^2+y^2} . \tag{5.8}$$

Hence $F_1(z)$ is of bounded type and the proof of Lemma 3 is complete.

We return to the proof of Theorem 6. It follows that $F(z)=M(z)e^{-NG(z)}$ is of mean type $\leqq -N\pi+\varepsilon$ in both halfplanes.

It only remains to verify that

$$\int_0^\infty (\sigma(t)/t)\, dt < \infty$$

or equivalently that

(5.9) $$\int_{-\infty}^{\infty} |\tilde{\psi}(t)|/(1+|t|)\,dt < \infty .$$

By (5.8)

$$\operatorname{Re}(-zf(z)) = cy + \int_{-\infty}^{\infty} \frac{y\,d\nu(t)}{(x-t)^2+y^2}$$

for some measure ν such that

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+t^2} < \infty .$$

Since $\tilde{\psi}$ is continuous (5.9) follows.

REMARK. Under additional regularity assumptions on the sequences $\{a_n\}_{n=-\infty}^{\infty}$, $\{d_n\}_{n=-\infty}^{\infty}$ it is possible to construct the function f i Theorem 6 such that also $f \in L^1(\mathbf{R}) \cap L^{\infty}(\mathbf{R})$. One such condition is that $a_n[\log^+(a_n/d_n)+1]$ has a majorant $\{q_n\}_{n=-\infty}^{\infty}$, which is increasing for $n>0$ and decreasing for $n<0$ and satisfies

$$\sum_{n=-\infty}^{\infty} \frac{a_n q_n}{l_n^2} < \infty .$$

In this case the Beurling-Malliavin theorem may be replaced by Theorem XXVI in Levinson [13].

6. A general uniqueness theorem for distributions supported on regularily distributed intervals.

We shall frequently use the expression $A_n \sim B_n$, which means:

there is a universal constant C such that $C^{-1} \leqq |A_n/B_n| \leqq C$.

The following result is an improvement of Beurling's theorem (Theorem 4) for intervals with the same regular distribution as those in the previous section:

THEOREM 7. *Suppose that f is a tempered distribution on the real line such that*

(i) $$\operatorname{supp}(f) \cap [0,\infty) \subset \bigcup_{n=1}^{\infty} [x_n-d_n, x_n+d_n] ,$$

where $[x_n-d_n, x_n+d_n]$, $n=1,2,\ldots$, are disjoint and nondegenerate. Let the sequences $\{l_n\}_{n=2}$ and $\{a_n\}_{n=2}^{\infty}$ be defined by

$$\begin{cases} l_n + a_n = x_n - d_n \\ l_n - a_n = x_{n-1} + d_{n-1} \end{cases}$$

and assume that

(ii) $l_n \sim l_m \Rightarrow a_n \sim a_m$ *and* $\max(1, \log(1/d_n)) \sim \max(1, \log(1/d_m))$,
(iii) $a_n \to \infty$ *as* $n \to \infty$ *and* $a_n \geqq Cd_n$ *for some constant* $C > 0$,
(iv) $\sum_{n=1}^{\infty} (a_n/l_n)^2[\log^+ a_n/d_n + 1] = \infty$,
(v) $\hat{f}|_{(-a,a)} = 0$ *for some* $a > 0$.

Then $f = 0$.

PROOF. By (3.3) the inverse Fourier-Carleman transform of $\hat{f}$, $F(z)$ satisfies the estimate

$$|F(z)| \leqq C\frac{1}{|y|^k} e^{-a|y|}|z|^M .$$

Hence $u(z) = -\log(|F(z)|/C)$ is superharmonic for $z \in \Omega = \mathbf{C} \setminus E$ and satisfies the inequality

$$u(z) \geqq a|y| + k\log|y| - M\log|z| .$$

The proof will be based on the same principles as in Theorem 5. Thus we want to prove

$$\int_{-\infty}^{\infty} \frac{u(x+i)}{1+x^2}\,dx = \infty . \tag{6.1}$$

Let $u_1(z) = u(z) + M\log|z|$. Then

(I) $u_1(z) \geqq a|y| + k\log|y|$.
(II) $u_1(z)$ is superharmonic in $\Omega_0 = \Omega \setminus \{0\}$.
(III) $\displaystyle\int_{-\infty}^{\infty} \frac{u_1(x+i)}{1+x^2}\,dx = \infty$ iff $\displaystyle\int_{-\infty}^{\infty} \frac{u(x+i)}{1+x^2}\,dx = \infty$.

The difference from the situation in Theorem 5 is the occurrence of the term $k\log|y|$ in (I).

On the circles $C_n = \{z \in \mathbf{C};\ |z - x_n| = (1+\delta)d_n\}$, where δ is chosen so that C_n are disjoint, we have

$$u_1(z) \geqq k \cdot \log d_n - C' .$$

Let the union of the corresponding closed discs be denoted by E and let

$$K_{\alpha,\xi} = \{(x,y);\ |x-\xi| \leqq \alpha\xi, |y| \leqq \alpha\xi\}, \qquad \xi > 0 ,$$

$\Omega_{\alpha,\xi} = K_{\alpha,\xi} \setminus E$, $R_{\alpha,\xi} = \partial K_{\alpha,\xi}$, $R^{(1)}_{\alpha,\xi} = R_{\alpha,\xi} \cap \{|y| = \alpha\xi\}$ and $E_{\alpha,\xi} = E \cap K_{\alpha,\xi}$. The real number α is chosen so that the assumption in (ii) is true for $(l_m, 0)$, $(l_n, 0) \in K_{\alpha,\xi}$.

We claim that for some constants c_i, $i = 1, \ldots, 5$,

$$(6.2)\qquad u_1(x+i) \geqq c_1 l_n \frac{\log^+ \frac{a_n}{d_n} - c_3}{\frac{l_n}{a_n} + c_2 \log^+ \frac{a_n}{d_n}} - c_4 a_n - c_5, \qquad l_n - \frac{a_n}{2} \leqq x \leqq l_n - \frac{a_n}{2}.$$

This will be proved by the method of harmonic minorants on Ω_{α, l_n}. The information available is that $u_1(z)$ is superharmonic and

$$u_1(z) \geqq \begin{cases} a|y| + k \log |y|, & z \in R_{\alpha, l_n} \\ k \log d_j - C, & z \in C_j,\ x_j \in K_{\alpha, l_n}. \end{cases}$$

As a comparison function we will use the function $v(z/a_n)$, where $v(z)$ is the logarithmic potential of a measure. Let the normalized lengths in the scale a_n be denoted by a prime, i.e. let $x'_k = x_k/a_n$, $d'_k = d_k/a_n$, $a'_k = a_k/a_n$, and $l'_k = l_k/a_n$. The function $v(z)$ is defined by

$$v(z) = \lim_{T \to \infty} \int_{-T}^{T} \log |1 - z/t|\, d\mu(t),$$

where

$$\begin{cases} d\mu|_{[(1-\alpha)l'_n,\,(1+\alpha)l'_n]} = \sum_{(1-\alpha)l'_n \leqq x'_k \leqq (1+\alpha)l'_n} \frac{1}{2}(a'_{k-1} + a'_k)\delta_{x'_k} \\ d\mu|_{\complement[(1-\alpha)l'_n,\,(1+\alpha)l'_n]} = dt. \end{cases}$$

By a calculation it is easy to establish that

$$(6.3)\qquad v(z) \leqq \begin{cases} \pi|y| + c_6, & z \in R_{\alpha, l'_n}, \\ \frac{a'_{j-1} + a'_j}{2} \log d'_j + c_7, & |z - x'_j| = (1+\delta) d'_j,\ x'_j \in R_{\alpha, l'_n}, \end{cases}$$

$$v\left(x + i\frac{1}{a_n}\right) \geqq c_8, \qquad l'_n - \frac{a'_n}{2} \leqq x \leqq l'_n + \frac{a'_n}{2}.$$

Let w_1, w_2 solve the Dirichlet problems

$$w_1(z) = \begin{cases} k \cdot \log |y|, & |y| \leqq 1, & z \in R_{\alpha, l_n} \\ 0 & |y| > 1, & z \in R_{\alpha, l_n} \\ 0 & & z \in E_{\alpha, l_n} \end{cases}$$

$$w_2(z) = \begin{cases} k \cdot \log |y|, & |y| \leqq 1, & z \in R_{\alpha, l_n} \\ 0 & |y| > 1, & z \in R_{\alpha, l_n}. \end{cases}$$

It is clear that $w_1(z) \geqq w_2(z)$ and $w_2(x+i) \geqq C$ for $(1-\alpha/2)l_n \leqq x \leqq (1+\alpha/2)l_n$. We conclude that

$$u_2(z) = u_1(z) - w_1(z) - c_9 k v(z/a_n) + c_{10}$$

satisfies

$$u_2(z) \geqq \begin{cases} (a - c_{11}/a_n)|y|, & z \in R_{\alpha, l_n} \\ 0 & z \in C_j,\ x_j \in R_{\alpha, l_n} . \end{cases}$$

Therefore $u_2(z)$ may be estimated using harmonic measures. We define

$$\omega_1(z) = \omega(z, \Omega_{\alpha, l_n}, R^{(1)}_{\alpha, l_n}) ,$$
$$\omega_2(z) = \omega(z, \Omega_{\alpha, l_n}, R_{\alpha, l_n}) ,$$

where $\omega(z, \Omega, E)$ as usual denotes the harmonic measure of E in the domain Ω evaluated at z. For large n

$$u_2(z) \geqq \frac{\alpha a}{2} l_n \omega_1(z) ,$$

but the harmonic measure ω_2 is easier to estimate. By a result analogous to Lemma 7 in [4] we have $\omega_1(l_n) \geqq \frac{1}{2}\omega_2(l_n)$. This is however not quite sufficient for our purposes. Let $\omega_3(z)$ solve the Dirichlet problem

$$\omega_3(z) = \begin{cases} 1 & |y| = \alpha l_n,\ z \in K_{\alpha, l_n} \\ -1 & x = (1-\alpha) l_n,\ x = (1+\alpha) l_n,\ z \in K_{\alpha, l_n} . \end{cases}$$

As in the proof of Lemma 7 in [4] it follows that

$$\omega_1(z) \geqq \tfrac{1}{2}[\omega_2(z) + \omega_3(z)] . \tag{6.4}$$

It is easily established that

$$\omega_3(x+i) \geqq -\text{const}\,(x/l_n)^2 . \tag{6.5}$$

It remains to estimate $\omega_2(z)$. Using again $v(z/a_n)$ as a comparison function (6.3) yields

$$\omega_2(x+i) \geqq c_1' \frac{\log^+ \dfrac{a_n}{d_n} - c_3}{\dfrac{l_n}{a_n} + c_2 \log^+ \dfrac{a_n}{d_n}} , \tag{6.6}$$

and combining our estimates (6.2) follows. We turn to the proof of (6.1). If $\sum_{n \geqq 1} (a_n/l_n)^2 = \infty$, Theorem 7 follows from Beurling's theorem (Theorem 4). Therefore we may without loss of generality assume that the sum is finite.

The situation can now be divided up into two cases:

CASE 1. $l_n/a_n \leqq \frac{1}{2}\log(a_n/d_n)$ for infinitely many n. The homogeneity properties of a_n and $\log(1/d_n)$ imply that $(l_k/a_k) < C \log(a_k/d_k)$ for those k such that $|l_k - l_n| \leqq \alpha l_n$. From (6.4), (6.5), and (6.6) it follows that there is a constant $\gamma > 0$ such that

$$(6.7) \qquad \omega_1(x+i) \geqq C > 0,$$

for

$$x \in \bigcup_{\{k;\, |l_k - l_n| \leqq \gamma l_n\}} [l_k - a_k/2, l_k + a_k/2] .$$

Since $a_k \geqq c d_k$, we realize that (6.7) holds on a subset of measure $\geqq \varkappa l_n$ of the interval $G_n = [l_n - \gamma \cdot l_n,\ l_n + \gamma \cdot l_n]$, where $\varkappa$ is a fixed constant. Chose the sequence $\{n_k\}$ for which $l_{n_k}/a_{n_k} \sim \log(a_{n_k}/d_{n_k})$ so rapidly increasing so that the intervals G_{n_k} are disjoint. Since $u_1(x+i) \geqq 0$ it follows from (6.7) that

$$\int_{-\infty}^{\infty} \frac{u_1(x+i)}{1+x^2} \geqq \sum_{k=1}^{\infty} \int_{l_{n_k} - \gamma l_{n_k}}^{l_{n_k} + \gamma l_{n_k}} C \frac{l_{n_k}}{1+x^2} dx + O(1) = \infty .$$

CASE 2. $l_n/a_n > \frac{1}{2} \log(a_n/d_n)$ for $n \geqq n_0$.

By (6.2) it follows that

$$u_1(x+i) \geqq C a_n \log a_n/d_n - C' a_n' \qquad x \in [l_n - a_n/2,\ l_n + a_n/2]$$

for $n \geqq n_0$. We get

$$\int_{-\infty}^{\infty} \frac{u_1(x+i)}{1+x^2} \geqq C \sum_{n=n_0}^{\infty} \int_{l_n - \frac{1}{2}a_n}^{l_n + \frac{1}{2}a_n} a_n \log^+ \frac{a_n}{d_n} \frac{1}{1+x^2} dx + O(1)$$

$$\geqq C' \sum_{n=n_0}^{\infty} \left(\frac{a_n}{l_n}\right)^2 \log^+ \frac{a_n}{d_n} + O(1) = \infty .$$

Hence $F = 0$ and the proof of Theorem 7 is finished.

7. Functions supported on small intervals around the integers.

The following result is a completion of Theorem 3 and its corollary in the L^1 case.

THEOREM 8. *Suppose that $f \in L^1(\mathbf{R})$ and $\hat{f}|_I = 0$ for some interval I with $|I| > 2\pi$ and*

$$\operatorname{supp}(f) \cap \{\xi > 0\} \subseteqq \bigcup_{n=1}^{\infty} [n - d_n, n + d_n], \qquad d_n > 0 ,$$

where $d_n \to 0$ as $n \to \infty$. If moreover

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{-\log d_n}{n^2} = \infty,$

(ii) $\log d_n \sim \log d_m$ *if* $n \sim m$,

then $f = 0$.

It follows from Theorem 9 below that given any sequence $\{d_n\}_{n\in Z}$, $d_n>0$, there is an L^1-function $f\neq 0$ such that $\operatorname{supp}(f)\subset\bigcup_{n=-\infty}^{\infty}[n-d_n,\ n+d_n]$ and $\hat{f}|_{[-\pi,\pi]}=0$. This shows that the condition $|I|>2\pi$ is essential in Theorem 8. Moreover the condition

$$\sum_{n=1}^{\infty}\frac{-\log d_n}{n^2}=\infty$$

is essential by Theorem 6 above.

Proof of Theorem 8. The basic idea of the proof is the same as that of Theorem 7. By making a translation if necessary we may assume $\hat{f}|_{(-A,A)}=0$ with $A>\pi$. Now form

$$F(z) = \mathscr{F}^{-1}(\hat{f},z)\,.$$

Since $\hat{f}|_{(-A,A)}=0$ and $\hat{f}\in L^{\infty}$ it follows that

$$|F(z)| \leqq \frac{\|\hat{f}\|_{\infty}}{2\pi|y|}e^{-A|y|}\,.$$

and F is holomorphic in $\Omega=\mathbf{C}\setminus(\bigcup_{n=-\infty}^{\infty}[n-d_n,n+d_n])$. The function

$$u(z) = -\log\left(\frac{|F(z)|}{\|\hat{f}\|_{\infty}}2\pi\right) \geqq A|y|+\log|y|$$

and is superharmonic in Ω. It follows that

$$u(z) \geqq \log d_n \quad \text{on } |z-n| = 2d_n\,.$$

We now use the function $\log|\sin\pi z|$ as a harmonic minorant of $u(z)$ on the domain

$$K_n = \{(x,y)\,;\ |x-n|\leqq\alpha n,\ |y|\leqq\alpha n\}\setminus\left(\bigcup_{k=[(1-\alpha)n]-1}^{[(1+\alpha)n]+1}\{z\,;\ |z-k|\leqq 2d_k\}\right)$$

and obtain

$$u(x+i) \geqq C(A-\pi)n\cdot\frac{\min\limits_{(1-\alpha)n\leqq m\leqq(1+\alpha)n}\log\dfrac{1}{d_m}}{C'n+\min\limits_{(1-\alpha)n\leqq m\leqq(1+\alpha)n}\log\dfrac{1}{d_m}}+O(1)$$

for $n-\frac{1}{2}\leqq x\leqq n+\frac{1}{2}$, $n\geqq n_0$, where n_0 is chosen so large that $d_n\leqq\frac{1}{4}$ for $n\geqq\alpha n_0$.

Since

$$\sum_{n=1}^{\infty}\frac{-\log d_n}{n^2}=\infty\,.$$

and $\log d_n \sim \log d_m$ if $n \sim m$, it follows as in the proof of Theorem 7 that

$$\int_{-\infty}^{\infty} \frac{u(x+i)}{1+x^2} dx = \infty$$

and we conclude that $F=0$.

8. Constructions of L^1-functions with small supports.

In this section we will construct L^1-functions supported on very small sets, which are unions of intervals, the midpoints of which in some sense are close to an arithmetic progression. This material is related to some constructions of L. de Branges [8] of measures supported on a sequence of points with a gap in their spectrum.

THEOREM 9. *Suppose there exists an entire function $S(z)$, which is real for real z and has only real simple zeros $\{x_n\}_{n=-\infty}^{\infty}$, such that $S(z)$ is of bounded type and of mean type a in the upper half-plane, and such that*

$$\sum_n \frac{1}{|S'(x_n)|} \frac{1}{(|x_n|+1)^m} < \infty \tag{8.1}$$

for some integer $m>0$. Then, for any choice of the sequence $\{d_n\}$, $d_n>0$, there is a function $f \in L^1(\mathbf{R})$ supported on the set $\bigcup_{n=-\infty}^{\infty} [x_n - d_n, x_n + d_n]$ such that $\hat{f}|_{[-a,a]} = 0$.

COROLLARY. *For any sequence $\{d_n\}_{n \in \mathbf{Z}}$, $d_n>0$ there is a non zero function $f \in L^1$, supported on $\bigcup_{n=-\infty}^{\infty} [n-d_n, n+d_n]$ such that $\hat{f}|_{[-\pi,\pi]}=0$.*

This is the result showing that the condition $|I|>2\pi$ is essential in Theorem 8.

PROOF OF COROLLARY. As the function $S(z)$ in Theorem 9 we choose $\sin \pi z$. We only have to verify that $S(z)$ is of bounded type and mean type π. This follows immediately from Theorems 10 and 11 in the Branges [8]. (The other conditions are obviously satisfied.)

Theorem 9 has the disadvantage that the existence of a function S with the required properties may be hard to verify. The following theorem has a hypothesis, which is more easily checked but is less precise than Theorem 9 as for the length of the interval, where $\hat{f}$ is $=0$.

THEOREM 10. *Let $\{x_n\}_{n=-\infty}^{\infty}$ be an increasing sequence of real numbers and let $n(t)$ be the counting function, i.e.*

$$n(t) = \begin{cases} \#\{k\ ;\ 0 \leqq x_k \leqq t,\ t \geqq 0\} \\ -\#\{k\ ;\ t < x_k < 0,\ t < 0\}\,. \end{cases}$$

Assume that there is a constant $\gamma > 0$ *such that*

(i) $|x_{k+1} - x_k| \geqq \gamma > 0$ *for all* k,

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{|n(t) - at|}{1+t^2}\,dt < \infty\,.$

Then for every $\varepsilon > 0$ *and any sequence* $\{d_n\}$, $d_n > 0$, *there is a function* $f \in L^1$ *supported on* $\bigcup_{n=-\infty}^{\infty} [x_n - d_n,\ x_n + d_n]$ *such that* $\hat{f}|_{[-\pi a+\varepsilon,\, \pi a - \varepsilon]} = 0$.

Proof of Theorem 9. Without loss of generality we may assume that the intervals $[x_n - d_n, x_n + d_n]$ are disjoint. We pick an arbitrary zero of S, x_1 say, and consider the function $(z - x_1)^m S(z)$. Then we claim that the following partial fraction decomposition holds

$$\frac{1}{(z-x_1)^m S(z)} = \sum_{n \neq 1} \frac{1}{(x_n - x_1)^m S'(x_n)}\,\frac{1}{z - x_n} + \sum_{v=1}^{m+1} \frac{A_v}{(z-x_1)^v}\,, \tag{8.2}$$

where the constants A_v are chosen to correspond to the Laurent expansion of $1/[(z-x_1)^m S(z)]$ at $z = x_1$.

Let H be the difference between the left hand side and the right hand side of (8.2). H is then an entire function and of mean type 0 in both the upper and the lower halfplane. By a theorem of Krein, (see e.g. de Branges [8, Problem 37]) H is an entire function of exponential type 0. Moreover H is bounded on the imaginary axis. By Phragmén-Lindelöf, H is bounded in the complex plane and hence constant. That the constant is 0 follows by letting z tend to ∞ along the imaginary axis.

Since $S(z)$ is of exponential type (again by Krein's theorem) it has a Hadamard factorization

$$S(z) = Ae^{\alpha z} z^m \prod_{\substack{n=-\infty \\ x_n \neq 0}}^{\infty} (-z/x_n) e^{z/x_n}\,.$$

We now construct a function (cf. (8.2))

$$\begin{aligned} T(z) = A\exp\Bigg\{\alpha z + \frac{m+1}{2d_1}\int_{x_1-d_1}^{x_1+d_1} \log(1 - z/t)\,dt \\ + \sum_{\substack{n=-\infty \\ n \neq 1}}^{\infty} \frac{1}{2d_n}\int_{x_n-d_n}^{x_n+d_n} \log(1 - z/t)\,dt + \frac{z}{x_n}\Bigg\}\,. \end{aligned} \tag{8.3}$$

This function is holomorphic in $\mathbf{C} \setminus \bigcup_{n=-\infty}^{\infty} [x_n - d_n,\ x_n + d_n] = \mathbf{C} \setminus E$ independent of the chosen branches of the logarithm. To be strictly formal we should proceed as in section 7. (The function T is slightly modified if $x_1 = 0$.) It is easily established that

$$|1/T(z)| \leqq C|1/S(z)|,$$

when $z \in A = \{z\ ;\ |z-x_n| \geqq 2d_n \text{ for all } n\}$.

By chosing contours, which avoid E suitably in the upper and lower halfplane we prove using (8.2) and (8.3) that

$$\frac{1}{T(z)} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{z-t}\,dt,$$

where

$$f(t) = \frac{1}{T(x+i0)} - \frac{1}{T(x-i0)} \in L^1.$$

By Lemma 1

$$\text{(8.4)} \qquad \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{z-t}\,dt = \begin{cases} -\dfrac{1}{2\pi}\displaystyle\int_0^{\infty} e^{iz\xi}\hat{f}(\xi)\,d\xi & \operatorname{Im} z > 0 \\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{0} e^{iz\xi}\hat{f}(\xi)\,d\xi & \operatorname{Im} z < 0. \end{cases}$$

Moreover since $1/T(x+i)$ is bounded and $T(z)$ is of mean type a, we may use a Phragmén-Lindelöf type result to conclude that

$$\left|\frac{1}{T(w+i)}\right| \leqq Ae^{-a\operatorname{Im} w}, \quad \operatorname{Im} w > 0.$$

By (8.4)

$$\frac{1}{T(w+i)} = \frac{1}{2\pi}\int_0^{\infty} e^{iw\xi}e^{-\xi}\hat{f}(\xi)\,d\xi.$$

As in the proof of Lemma 2 it follows that $e^{-\xi}\hat{f}(\xi) = 0$ for $0 \leqq \xi \leqq a$ and hence $\hat{f}(\xi)|_{[0,a]} = 0$. The proof that $\hat{f} = 0$ for $-a \leqq \xi \leqq 0$ is completely analogous.

Proof of Theorem 10. Under the hypothesis of this theorem, it follows from Theorem 67 in de Branges [8] that for every $\varepsilon > 0$ there is a function $S(z)$, real for real z, which has only real simple zeros at the points x_n and such that

$$\sum_n \frac{1}{|S'(x_n)|} < \infty.$$

The partial fraction decomposition

$$\frac{1}{S(z)} = \sum_n \frac{1}{S'(x_n)(z - x_n)}$$

is then valid. The proof now proceeds in complete analogy with that of Theorem 9.

ACKNOWLEDGMENT. I wish to thank professors J. E. Björk, L. Carleson, B. Kjellberg, and H. S. Shapiro for their kind interest and valuable suggestions during my work on this paper.

I also want to thank docent Matts Essén, who carefully read a preliminary version of the article and made several useful comments on it.

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