MATH. SCAND. 56 (1985), 5-14

VALUE REGIONS FOR CONTINUED FRACTIONS $K(a_n/1)$ WHOSE ELEMENTS LIE IN PARABOLIC REGIONS

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1. Introduction.

A continued fraction $K(a_n/1)$ can be obtained in terms of non singular linear fractional transformations

$$s_n(w) = \frac{a_n}{1+w}, \qquad a_n \neq 0,\ n \geqq 1 .$$

Define, inductively,

$$S_1(w) = s_1(w),\ S_n(w) = S_{n-1}(s_n(w)), \qquad n \geqq 2 .$$

Then the nth *approximant* f_n of the continued fraction $K(a_n/1)$ is given by

$$f_n = S_n(0), \qquad n \geqq 1 .$$

A nonempty subset E of $\mathbf{C}$ is called an *element region* for $K(a_n/1)$ if

$$0 \neq a_n \in E, \qquad n \geqq 1 .$$

A nonempty subset V of $\hat{\mathbf{C}} = \mathbf{C} \cup [\infty]$ is said to be a *value region corresponding* to the element region E if

$$E \subset V \quad \text{and} \quad \frac{E}{1+V} \subset V .$$

Here $E/(1+V)$ is understood to be the set

$$E/(1+V) = [u = a/(1+v) : a \in E,\ v \in V] .$$

It is easily seen that the approximants f_n of a continued fraction $K(a_n/1)$ whose elements satisfy $a_n \in E$, $n \geqq 1$ are in every value region which corresponds to the element region E.

Research supported in part by the United States National Science Foundation under Grant No. MCS 8202230.

Received May 30, 1983.

Let V_i, $i \in I$, be a family of value regions corresponding to a fixed element region E. Then $\bigcap [V_\iota : i \in I]$ is also a value region for E. In view of this result there is always a *best value region* $V(E)$ for each element region E. The region $V(E)$ is exactly the set of all approximants of continued fractions $K(a_n/1)$ with elements in E. The closure $c(V(E))$ of $V(E)$ in $\hat{\mathbf{C}}$ contains in addition the values of all limit points of sequences $\{f_n\}$ of continued fractions $K(a_n/1)$ with elements in E.

Value regions and element regions were first discussed by Scott and Wall [6] in 1941. For a detailed account of value regions and their use in convergence theory see Chapter 4 of [2].

We shall here be concerned with parabolic element regions, with focus at the origin,

$$E(p,\alpha) = \left[z = re^{i\theta} : r \leqq \frac{2k}{1-\cos(\theta - 2\alpha)}\right],$$

where $-\pi/2 < \alpha < \pi/2$, $0 < p \leqq 1/2$ and

$$k = k(p,\alpha) = p(1-p)\cos^2\alpha \,. \tag{1.1}$$

Another way of writing $E(p,\alpha)$, which we shall use later, is

$$E(p,\alpha) = [z = k\zeta^2 e^{i2\alpha} : |\mathrm{Im}\,\zeta| \leqq 1] \,. \tag{1.2}$$

It will also be convenient to introduce notation for certain half planes. We set

$$V(\beta, q) = [v : \mathrm{Re}\,(ve^{-i\beta}) \geqq -q\cos\beta] \,.$$

Value regions corresponding to element regions $E(p,0)$ were first studied by Leighton and Thron [5] in 1942. They obtained the region $H(p)$, bounded by the hyperbola

$$\left[v = Re^{i\varphi} : R = \frac{2p(1-p)}{1-2p-\cos\varphi}\right]$$

and containing the origin, as the best value region for $E(p,0)$. Value regions for $E(p,\alpha)$, $-\pi/2 < \alpha < \pi/2$, were recently [3] investigated by the present authors. We refer to that article for a discussion of the uses of value regions among which the derivation of truncation error bounds may be one of the most important.

Here we shall improve the results of [3]. As in that article we shall first derive value regions which are angular openings, but with vertices on the negative real axis (rather than on the ray $-re^{i\alpha}$ as was the case in [3]). We shall then show that the intersection of all of these angular openings, for a fixed $E(p,\alpha)$, is indeed the best value region for that element region. A parametric description of the boundary of the value region, which is not, in general, a hyperbola, will also be given.

2. Mappings of half planes.

We begin with a strengthened form of a lemma which was originally proved in [7].

LEMMA 2.1. *Let* $-\pi/2<\theta<\pi/2$, $0<\varphi<2\pi$ *and* $0\leqq t\leqq 1$ *be independent variables in their intervals and let* F *be given by*

$$F = F(\varphi,\theta,t) = \frac{2t\cos\theta}{1-\cos\varphi}e^{i(\theta+\varphi)}\,.$$

Then the range of F *is exactly the half plane* $\operatorname{Re}F\geqq -1$.

PROOF. Clearly the range is star shaped with respect to the origin and symmetric with respect to the real axis. For a fixed $\eta=\theta+\varphi$, $-\pi/2<\eta<\pi/2$ we have

$$\frac{\cos\theta}{1-\cos\varphi} = \frac{\cos\theta}{1-\cos(\eta-\theta)}\,.$$

This expression can be made as large as we please by choosing θ to be sufficiently close to η. Thus all values in the half plane $\operatorname{Re}F>0$ are assumed. For $\theta+\varphi=\pi/2$ we have

$$\frac{\cos\theta}{1-\cos\varphi} = \frac{\sin\varphi}{1-\cos\varphi} = \cot\frac{\varphi}{2}\,.$$

This expression also can be made arbitrary large by taking φ close to 0 (and $\varphi>0$). Hence the whole imaginary axis belongs to the range of F.

We always have

$$\operatorname{Re}\left(\frac{2\cos\theta}{1-\cos\varphi}e^{i(\theta+\varphi)}\right) = \frac{2\cos\theta\cos(\theta+\varphi)}{1-\cos\varphi}$$

$$= \frac{\cos(2\theta+\varphi)+\cos\varphi}{1-\cos\varphi} = -1+\frac{1+\cos(2\theta+\varphi)}{1-\cos\varphi} \geqq -1\,.$$

Let $\eta=\theta+\varphi$, $\pi/2<\eta<3\pi/2$, then

$$\cos(2\theta+\varphi) = \cos(\eta+\theta) = -1$$

for $\eta+\theta=\pi$, that is for $\theta=\pi-\eta$. This value is always in the interval $-\pi/2<\theta<\pi/2$ so that all values on the line $\operatorname{Re}F=-1$ are also assumed.

The lemma will now be used in the proof of the following result.

THEOREM 2.1. *For* $-\pi/2<\alpha<\pi/2$, $0<p\leqq 1/2$, $k=p(1-p)\cos^2\alpha$ *and*

$$|\sin\delta| \leqq (1-2p)\cos\alpha$$

let

$$D = D(\delta) = D(p,\alpha,\delta) = \left(1-\sqrt{1-\frac{4k}{\cos(\alpha+\delta)\cos(\alpha-\delta)}}\right)\Big/2 . \tag{2.1}$$

Then

$$\frac{E(p,\alpha)}{1+V(\alpha+\delta, D(\delta))} = V(\alpha-\delta, D(-\delta)) .$$

PROOF. One has $1+V(\beta,q)=V(\beta,q-1)$. Now assume that $q<1$, then

$$v \in \frac{1}{1+V(\beta,q)}$$

iff $v=e^{-i\beta}re^{i\theta}\cos\theta$, $r\leqq 1/((1-q)\cos\beta)$, $-\pi/2<\theta<\pi/2$. Further, any $a\in E(p,\alpha)$ can be written as $a=te^{2i\alpha}$, $t\geqq 0$, or

$$a = e^{2i\alpha}\frac{2kse^{i\varphi}}{1-\cos\varphi},\quad 0<\varphi<2\pi,\ 0<s\leqq 1 .$$

(The points of the ray $a=te^{2i\alpha}$, $t\geqq 0$, are insignificant, and shall be omitted in the subsequent argument.) Thus

$$w \in \frac{E(p,\alpha)}{1+V(\alpha+\delta,q)}$$

is of the form

$$w = \left(\frac{e^{2i\alpha}2kse^{i\varphi}}{1-\cos\varphi}\right)\left(\frac{e^{-i(\alpha+\delta)}r'e^{i\theta}\cos\theta}{\cos(\alpha+\delta)(1-q)}\right)$$
$$= e^{i(\alpha-\delta)}\frac{k}{\cos(\alpha+\delta)(1-q)}\frac{2r's\cos\theta}{1-\cos\varphi}e^{i(\theta+\varphi)},$$

where $0<r'\leqq 1$, $0<s\leqq 1$, $-\pi/2<\theta<\pi/2$, $0<\varphi<2\pi$. From Lemma 2.1 it then follows that w covers the half plane $V(\alpha-\delta,q')$, where

$$q' = \frac{k}{(1-q)\cos(\alpha+\delta)\cos(\alpha-\delta)} .$$

An analogous argument yields

$$\frac{E(p,\alpha)}{1+V(\alpha-\delta,q')} = V(\alpha+\delta,q''),$$

where

$$q'' = \frac{k}{(1-q')\cos(\alpha+\delta)\cos(\alpha-\delta)}.$$

We would like to have $q''=q$. This will be the case if

$$q-qq' = k/(\cos(\alpha+\delta)\cos(\alpha-\delta))$$

and

$$q'-qq' = k/(\cos(\alpha+\delta)\cos(\alpha-\delta)).$$

Hence we must have $q=q'$ and

$$q^2-q+k/(\cos(\alpha+\delta)\cos(\alpha-\delta)) = 0.$$

Thus

$$q = \left(1\pm\sqrt{1-\frac{4k}{\cos(\alpha+\delta)\cos(\alpha-\delta)}}\right)\Big/2.$$

We note that, since

$$\cos(\alpha+\delta)\cos(\alpha-\delta) = \cos^2\alpha-\sin^2\delta,$$

one can write

$$1-\frac{4k}{\cos(\alpha+\delta)\cos(\alpha-\delta)} = \frac{\cos^2\alpha(1-2p)^2-\sin^2\delta}{\cos^2\alpha-\sin^2\delta}.$$

From this one obtains the restriction on δ given in the statement of the theorem. Since we want small regions we choose the minus sign in front of the square root. This completes the proof of Theorem 2.1.

It does *not* follow that $V=V(\alpha+\delta,D(\delta))$ or $V'=V(\alpha-\delta,D(-\delta))$ are value regions for $E(p,\alpha)$. Since, however,

$$\frac{E}{1+V} = V' \quad \text{and} \quad \frac{E}{1+V'} = V$$

it is true that

$$\frac{E}{1+V\cap V'} \subset V\cap V'.$$

We cannot expect equality here and in fact it can be shown that it does not hold. Since $0 \in V \cap V'$ and hence $E \subset V \cap V'$, it follows that $V \cap V'$ is a value region for $E(p, \alpha)$.

The boundary of $V(\alpha+\delta, D(\delta))$ is the straight line which passes through $-D(\delta)\cos(\alpha+\delta)e^{i(\alpha+\delta)}$ and has slope $\tan(\pi/2+\alpha+\delta)$. From this one obtains the equation of the boundary to be

$$(x+D(\delta))\cos(\alpha+\delta)+y\sin(\alpha+\delta) = 0\,. \tag{2.2}$$

The equation of the boundary of $V(\alpha-\delta, D(-\delta))$ is

$$(x+D(-\delta))\cos(\alpha-\delta)+y\sin(\alpha-\delta) = 0\,.$$

Since $D(\delta)=D(-\delta)$ the vertex of the angular opening

$$A(p,\alpha,\delta) = V(\alpha+\delta, D(\delta)) \cap V(\alpha-\delta, D(-\delta))$$

is the point $-D(\delta)$ on the negative real axis and $A(p, \alpha, \delta)$ can be written as

$$A(p,\alpha,\delta) = [v : |\arg(v+D)-\alpha| \leqq \pi/2-|\delta|]\,. \tag{2.3}$$

From now on we shall indicate that D is a function of δ only if we want to emphasize this fact. Normally we shall just write D. The following result has now been proved.

THEOREM 2.2. *The angular opening $A(p,\alpha,\delta)$, given by* (2.3), *where D is defined in* (2.1) *and $k=p(1-p)\cos^2\alpha$, $-\pi/2<\alpha<\pi/2$, $0<p\leqq 1/2$, is a value region corresponding to the element region $E(p,\alpha)$ for every δ for which $|\sin\delta| \leqq (1-2p)\cos\alpha$.*

We conclude this section with a brief glance at some special cases. For $\alpha=0$ one has $|\sin\delta| \leqq (1-2p)$,

$$A(p,0,\delta) = [v : |\arg(v+D)| \leqq \pi/2-|\delta|]$$

and

$$D = \left(1-\frac{\sqrt{(1-2p)^2-\sin^2\delta}}{\cos\delta}\right)\Big/2\,.$$

For $\delta=0$ we obtain easily $A(p,\alpha,0)=V(\alpha,p)$. For $p=1/2$ the range of δ reduces, independent of α, to the single value $\delta=0$. The value region $A(1/2,\alpha,0)$ is then $V(\alpha,1/2)$ as has been known since 1942 [4].

The case $p=0$ requires special treatment. Our results are not valid in this case, but as $p \to 0$ the range for δ approaches $|\delta|<\pi/2-|\alpha|$. Also $D \to 0$ so that the best choice for δ is $\pi/2-|\alpha|$. If one defines $E(0,\alpha)$ to be the ray $\arg z=2\alpha$, then, as was observed by Henrici and Pfluger [1] in connection with S-fractions in 1966, a value region for $E(0,\alpha)$ is indeed

$$A(0,\alpha,\pi/2-|\alpha|) = \begin{cases}[v:\ 0\leqq \arg v\leqq 2\alpha] & \text{if } \alpha\geqq 0\,,\\ [v:\ 2\alpha\leqq \arg v\leqq 0] & \text{if } \alpha<0\,.\end{cases}$$

The half plane $V(\alpha,1/2)$ is known [4] to be the best value region for $E(1/2,\alpha)$, which is itself a best conditional convergence region [4]. For $p<1/2$ the half plane $V(\alpha,p)=A(p,\alpha,0)$ is a value region but not a best value region since choices of δ other than $\delta=0$ are possible and since the best value region is the intersection of all value regions for $E(p,\alpha)$.

3. Intersection of value regions and bestness.

We recall that D is given by (2.1) so that

$$2D-1 = -\sqrt{1-\frac{4k}{\cos(\alpha+\delta)\cos(\alpha-\delta)}}$$

and

$$k = p(1-p)\cos^2\alpha = D(1-D)\cos(\alpha+\delta)\cos(\alpha-\delta)\,. \tag{3.1}$$

Implicit differentiation with respect to δ yields

$$\begin{aligned}0 &= D'(1-D)\cos(\alpha+\delta)\cos(\alpha-\delta)-DD'\cos(\alpha+\delta)\cos(\alpha-\delta)\\ &\quad +D(1-D)(-\sin(\alpha+\delta)\cos(\alpha-\delta)+\cos(\alpha+\delta)\sin(\alpha-\delta))\end{aligned}$$

and hence (we shall from now on assume $D<1/2$).

$$D' = \frac{dD}{d\delta} = \frac{D(1-D)\sin 2\delta}{(1-2D)\cos(\alpha+\delta)\cos(\alpha-\delta)}\,. \tag{3.2}$$

To get the envelope of the family of lines (2.2) and thus the boundary of the intersection of all $A(p,\alpha,\delta)$, $|\sin\delta|<(1-2p)\cos\alpha$, we differentiate (2.2) with respect to δ. Thus one obtains

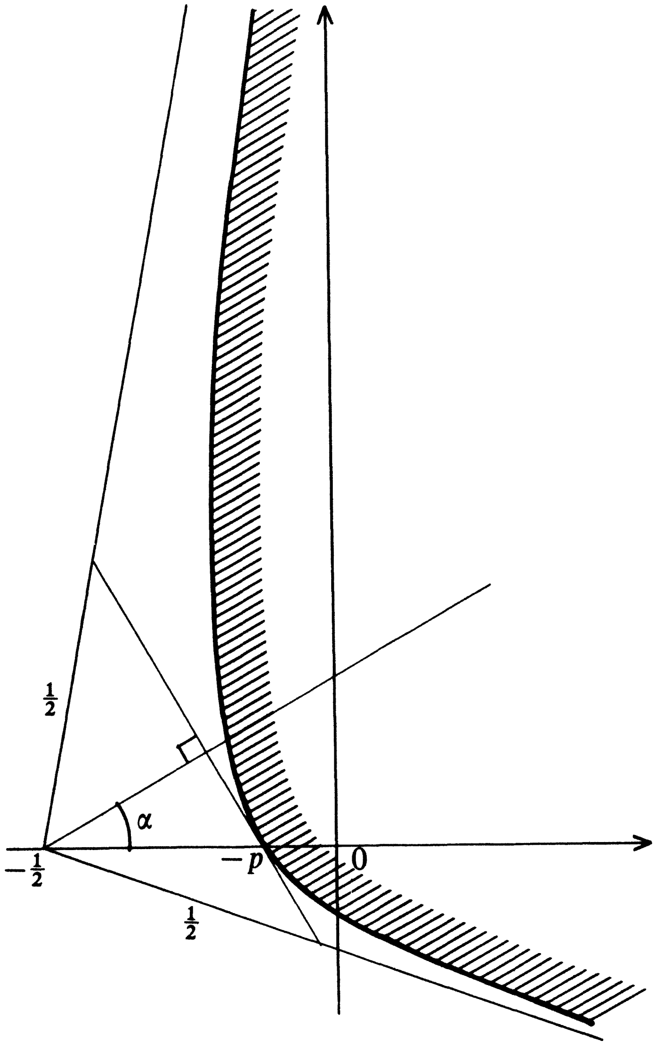
$$-(x+D)\sin(\alpha+\delta)+y\cos(\alpha+\delta)+D'\cos(\alpha+\delta) = 0\,.$$

Combining the two equations one arrives at

$$\begin{aligned}x+D &= D'\sin(\alpha+\delta)\cos(\alpha+\delta)\,,\\ y &= -D'\cos^2(\alpha+\delta)\,.\end{aligned}$$

For $z=x+iy$ a point on the envelope we then have

$$z+D = \frac{D(1-D)}{(1-2D)}\,\frac{\sin 2\delta}{\cos(\alpha-\delta)}\,(-i)e^{i(\alpha+\delta)}$$



and therefore

$$z = -D\left(1+\frac{(1-D)i\sin 2\delta}{(1-2D)\cos(\alpha-\delta)}e^{i(\alpha+\delta)}\right)$$

and

$$1+z = (1-D)\left(1+\frac{-Di\sin 2\delta}{(1-2D)\cos(\alpha-\delta)}e^{i(\alpha+\delta)}\right).$$

Write

$$a_j = -D(1-D)\cos(\alpha+\delta)\cos(\alpha-\delta)(1-i\tan\theta_j)^2 e^{2i\alpha}, \qquad j=1,2 .$$

Then in view of (1.2) and (3.1) the point a_j is on the boundary of $E(p,\alpha)$. We further require that

$$\tan\theta_1 = ((1-D)\tan(\alpha+\delta)+(-D)\tan(\alpha-\delta))/(1-2D), \tag{3.3}$$
$$\tan\theta_2 = ((-D)\tan(\alpha+\delta)+(1-D)\tan(\alpha-\delta))/(1-2D).$$

Finally, introduce

$$w = -D\left(1-\frac{(1-D)i\sin 2\delta}{(1-2D)\cos(\alpha+\delta)}e^{i(\alpha-\delta)}\right),$$

so that

$$1+w = (1-D)\left(1-\frac{(-D)i\sin 2\delta}{(1-2D)\cos(\alpha+\delta)}e^{i(\alpha-\delta)}\right).$$

Then one can write

$$\begin{aligned}1+z &= (1-D)e^{i(\alpha+\delta)}\left(\cos(\alpha+\delta)-i\sin(\alpha+\delta)-\frac{Di\sin 2\delta}{(1-2D)\cos(\alpha-\delta)}\right)\\ &= (1-D)e^{i(\alpha+\delta)}\cos(\alpha+\delta)\left(1-i\frac{(1-D)\tan(\alpha+\delta)+(-D)\tan(\alpha-\delta)}{1-2D}\right)\\ &= (1-D)e^{i(\alpha+\delta)}\cos(\alpha+\delta)(1-i\tan\theta_1).\end{aligned}$$

In a similar way one shows that

$$w = -De^{i(\alpha-\delta)}\cos(\alpha-\delta)(1-i\tan\theta_1).$$

Hence $w(1+z)=a_1$ or $w=a_1/(1+z)$. An analogous argument leads to $z=a_2/(1+w)$. Thus

$$z = \frac{a_2}{1}\,\frac{}{+}\,\frac{a_1}{1+z}.$$

This equation has also a second solution u. We must have $uz=-a_2$. Since $(-1-w)z=-a_2$ it follows that $-1-w=u$. Clearly $-1-w\notin A(p,\alpha,\delta)$ and thus the periodic continued fraction

$$\frac{a_2}{1}\,\frac{}{+}\,\frac{a_1}{1}\,\frac{}{+}\,\frac{a_2}{1}\,\frac{}{+}\cdots$$

converges to z. The following theorem has now been proved.

THEOREM 3.1. *Let* $-\pi/2<\alpha<\pi/2$, $0<p<1/2$. *Then*

$$B(p,\alpha) = \bigcap [A(p,\alpha,\delta) : |\sin\delta| \leqq (1-2p)\cos\alpha] ,$$

where $A(p,a,\delta)$ is given in (2.3), *is the closure of the best value region for the element region $E(p,\alpha)$.*

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