MATH. SCAND. 56 (1985), 15-28

CLASSIFICATION OF EQUIDIMENSIONAL CONTACT UNIMODULAR MAP GERMS

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Let A denote the set of all $\mathcal{K}$-finite germs $(\mathbf{C}^n,0) \to (\mathbf{C}^n,0)$ of holomorphic mappings. The objective of the present paper is to give a complete classification of all $\mathcal{K}$-unimodular germs in A under $\mathcal{K}$-equivalence. We denote by m the kernel rank of a map germ in A. The case $m \leqq 2$ was discussed in a previous paper [1]. In section 1 we show that the case $m \geqq 4$ gives rise only to germs of modality $\geqq 2$, so that we can restrict our attention to the next case $m=3$.

Then the 2-jet of a germ (with $n=m=3$) can be thought of as a net of conics and it is an easy matter to produce normal forms for these [2, 5].

The idea now is to add higher order terms to these normal forms and then to use determinacy, discarding any case in which at least two moduli appear.

This is done in section 2 using the technique of *complete transversals* which we have introduced in [1] for the weighted-homogeneous jets and which we extend here (Proposition 1.5) to the general case.

The sheer classification consists of 17 series of singularities (with one, two or three natural indices), 5 exceptional map germs and countably many families with one parameter λ. For each of these families we describe in section 3 a *basic invariant* i.e. an algebraic map φ such that $f_\lambda \sim f_\mu$ iff $\varphi(\lambda)=\varphi(\mu)$. A nice consequence of this classification is the following result, based on the work of du Plessis on genericity of ($\mathcal{A}$-smooth) finite determinacy [4].

Finite determinacy of equidimensional map germs $f\colon (K^n,0) \to (K^0,0)$ $K=\mathbf{R},\mathbf{C}$ holds in general if and only if $n \leqq 13$. Many thanks are due to Prof. A. du Plessis who communicated to us some results in the first section and to the referee for his useful suggestions.

Received May 24, 1982; in revised form August 9, 1983.

1. Reduction to the case $n=3$ and other preliminaries.

It is enough to consider only map germs $f \in A$ having zero 1-jet: then $j^2 f$ is a linear system of quadric hypersurfaces depending on $n^2(n+1)/2$ parameters. The corresponding contact group $K_0^2 = \mathrm{Cl}\,(n)^2$ has dimension $2n^2$ and hence the number of moduli for any orbit is at least

$$N(n) = n^2(n+1)/2 - 2n^2 = n^2(n-3)/2 .$$

It follows that $N(n) \geqq 8$ for $n > 3$ and hence the unimodular map germs are to be found only when $n \leqq 3$. The case $n \leqq 2$ is settled in [1] and hence we shall suppose from now on that $n=3$.

Let dim (f) be the dimension of the linear system of conics associated with a given germ $f \in A$.

LEMMA 1.1. *If the map germ $f \in A$ is unimodular, then*

$$\dim(f) = 3$$

PROOF. The set $P = \{f \in J^2(3,3);\ \dim(f) \leqq 2\}$ is a constructible K_0^2-invariant subset of $J^2(3,3)$.

Using the classification of pencils of conics [2] we see that P consists of a finite number of K_0^2 orbits, all of which are contained in the closure of the orbit through

$$g = (x^2 - y^2, x^2 - z^2, 0) .$$

It is easy to see that a complete transversal of g in $J^3(3,3)$ is given by the vector space

$$\{(0, 0, \lambda_1 x^3 + \lambda_2 y^3 + \lambda_3 z^3 + \lambda_4 xyz);\ \lambda \in \mathbf{C}^4\}$$

and that $\mathrm{cod}_0\,(g) \geqq 3$. Using Lemma 1.4.i [1] we get that any germ $f \in A$ with $j^2 f \in P$ has modality $\geqq 3$.

Hence, if the map germ f is unimodular, then $j^2 f$ is a net of conics. Now the classification of nets of conics is known from [2] (see also [5] where the real case is mainly considered) and runs as follows.

Type	Normal form	Codimension
1	$(x^2+\lambda yz, y^2+\lambda zx, z^2+\lambda xy)$	1
2	(xy, x^2+yz, y^2+xz)	1
3	$(xy, x^2+yz, (x+z)^2)$	1
4	(x^2+yz, xy, xz)	2
5	(x^2+yz, xy, z^2)	2
6	(x^2+yz, y^2, z^2)	2
7	(xy, yz, xz)	3
8	$(x^2, yz+z^2, xy)$	3
9	(xz, x^2+z^2, yz)	3
10	(x^2, y^2, z^2)	3
11	(xy, xz, z^2)	4
12	(y^2, z^2, xy)	4
13	(x^2+yz, xy, y^2)	5
14	(x^2, xy, xz)	7
15	(x^2, y^2, xy)	7

Table 1.

Here $\lambda^3 \neq -1, 0, 8$.

The specializations among these orbits are also discussed in detail in [2] and exhibit a very nice symmetry.

This symmetry is broken when one passes from the classification of nets of conics to the classification of the corresponding map germs in A.

For instance, we shall show in the next paragraph that a germ f such that j^2f is of type $k<9$ or $k=10$ is always unimodular; if the type k is 9, 11 or 12 it can be either unimodular or of a higher modality; and finally if $k \geqq 13$ then f is at least bimodular.

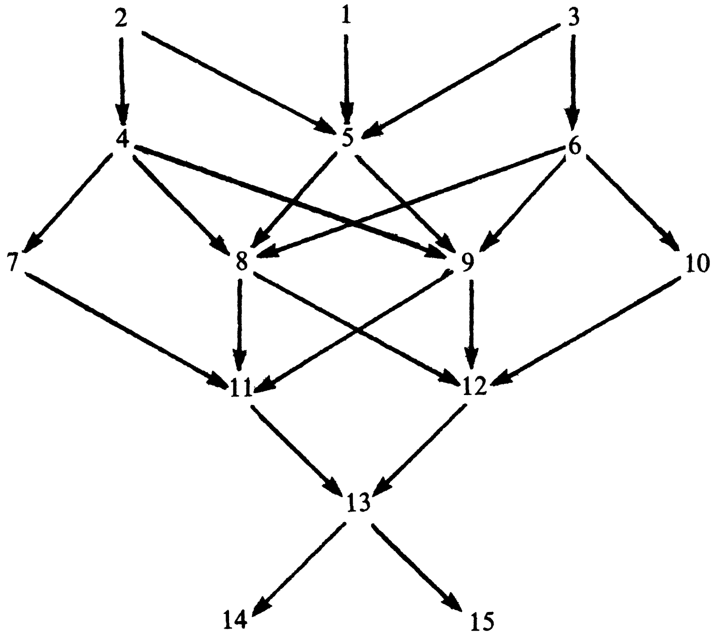

Table 2.

The last thing we need for the classification is an extension of Proposition 1.3 [1].

The situation considered there is the following.

Let $\pi\colon J^m(n,p) \to J^k(n,p)$ be the natural projection and denote its kernel by P_k^m. If we identify these jet spaces with vector spaces of polynomial mappings (via some fixed coordinate systems) we also have a natural inclusion

$$J^k(n,p) \subset J^m(n,p) \quad \text{for } k \leqq m\,.$$

For a k-jet f we consider the affine linear subspace in $J^m(n,p)$

$$J_k^m(f) = \pi^{-1}(f) = f + P_k^m$$

and try to classify the elements of $J_k^m(f)$.

These equivalence classes are precisely the orbits of the subgroup

$$G(f) = \varrho^{-1}(S(f)) \subset K^m\,,$$

where $S(f) \subset K^k$ is the stabilizer subgroup of f and $\varrho\colon K^m \to K^k$ is the natural projection between (truncated) contact groups.

A linear subspace $C(f) \subset P_k^m$ is called a *complete transversal* for f if the linear space $f+C(f)$ intersects all the orbits in $J_k^m(f)$ transversally. (Here, by intersection we mean nonempty intersection.)

In [1] we showed how to construct a complete transversal for a weighted homogeneous jet f satisfying also an additional condition (Proposition 1.3).

In what follows we shall treat the general case. Consider the subgroup $K_r \subset K$ of the contact group K defined by $K_r = R_r \cdot C_r$, where R_r is the group of analytic isomorphisms h: $(\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ such that $j^r h = \mathrm{id}$ and C_r is the group of invertible pxp matrices A over $\mathscr{E}_n$ such that $j^{r-1}(A - \mathrm{id}) = 0$.

It is easy to see that

LEMMA 1.2. i. *$TK_r f = \mathfrak{m}^{r+1} J_f + \mathfrak{m}^r \cdot I_f \cdot \theta(f)$ for any map germ f.*
ii. *If f and g are k-jets such that $j^{k-r}f = j^{k-r}g$ then $T(K_r^k f) = T(K_r^k g)$.*

A more subtle fact is the following.

LEMMA 1.3. *If f, g are k-jets such that $j^{k-r}f = j^{k-r}g$ and $g \in f + T(K_r^k f)$, then f and g are K_r^k-equivalent.*

PROOF. Consider the line $L \subset J^k(n, p)$ determined by the jets f and g and note that the wellknown Lemma (3.1) [3] of Mather applies and shows that L is contained in a K_r^k-orbit.

COROLLARY 1.4. *Any complement T to*

$$E_k^m(f) = j^m(\mathfrak{m}^{m-k+1} J_f + \mathfrak{m}^{m-k} \cdot I_f \cdot \theta(f)) \cap P_k^m$$

in P_k^m is a complete transversal for f.

PROOF. Any element in $J_k^m(f_2)$ can be written as a sum $f+a+b$, with $a \in E_k^m(f)$, $b \in T$. Take $f_1 = f+b$, $g_1 = f_1 + a$ and apply (1.3) to the m-jets f_1, g_1 with $r = m-k$.

This last result is not completely satisfactory since as m increases $E_k^m(f)$ decreases in low dimensions.

Let $F^{k+1}(f) = E_k^{k+1}(f)$ and more generally

$$F^m(f) = j^m(\mathbf{C}[x_1, \ldots, x_n] \cdot F^{k+1}(f)) \subset P_k^m .$$

(Here again we identify the jets with polynomials using a fixed coordinate system.)

With these notations, the main result is the following.

PROPOSITION 1.5. *Any complement T spanned by homogeneous polynomials to $F^m(f)$ in P_k^m is a complete transversal for f.*

PROOF. First we prove that $f+T$ intersects all the orbits in $J_k^m(f)$. We shall denote by T_d the vector subspace of T spanned by all the homogenous polynomial mappings in T of degree d.

If $T+F^m(f)=P_k^m$ then we have by comparing the homogeneous components of degree $k+1$

$$T_{k+1}+F^{k+1}(f) = P_k^{k+1} .$$

Using 1.4. it follows that any $(k+1)$ jet $f_1 \in J_k^{k+1}(f)$ is equivalent with one jet of the form $f+t_1=\bar{f}_1$ for some $t_1 \in T_{k+1}$. Moreover we have

$$T_{k+2}+F^{k+2}(\bar{f}_1) = P_{k+1}^{k+2} ,$$

since $F^{k+2}(\bar{f}_1)\supset j^{k+2}(\langle x_1,\ldots,x_n\rangle\cdot F^{k+1}(f))$. Hence again by 1.4 we see that any jet $f_2 \in J_{k+1}^{k+2}(\bar{f}_1)$ is equivalent with one jet of the form

$$\bar{f}_1+t_2 = f+t_1+t_2 \quad \text{for some } t_2 \in T_{k+2} .$$

If we apply this trick $(m-k)$ times we find that any jet f_{m-k} in $J_k^m(f)$ is equivalent to a jet of the form

$$g = f+t_1+\ldots+t_{m-k} \quad \text{for some } t_i \in T_{k+i}$$

and hence $f+T$ meets any orbit in $J_k^m(f)$. We still have to show that

$$TK^m g \cap P_k^m+T = P_k^m$$

for any jet g as above (since the first vector space is precisely $T_g(G(j^k g)\cdot g)$).

Note that

$$T_m+TK^m g \cap P_k^m \supset T_m+j^m((x_1,\ldots,x_n)^{m-k-1}F^{k+1}(f)) = P_{m-1}^m .$$

Hence we can take $(m-1)$-jets and proceed by descending induction on m.

REMARK 1.6. One can avoid the assumption that T is spanned by homogeneous polynomials by using in the above argument the obvious projections between jet spaces instead of direct-sum decompositions.

It is clear that all the above results on complete transversals work equally well in the differentiable case over the real numbers $\mathbf{R}$.

2. The classification in the case $n=3$.

The basic idea in doing the classification of the unimodular maps germs in the case $n=3$ is to use the classification of the nets of conics presented in section 1 and for each type of 2-jet to try to classify the corresponding germs by computing explicit complete transversal as in Proposition 1.5.

For $k<9$ or $k=10$ this task is straightforward and the results are contained in the following:

Type of $j^2 f$	Normal form of f	Conditions
1	$(x^2+\lambda yz, y^2+\lambda xz, z^2+\lambda xy)$	$\lambda^3 \neq 0, -1, 8$
2	$(xy+z^p, x^2+yz, y^2+xz)$	$p>2$
3	$(xy, x^2+yz, (x+z)^2+y^p$	$p>2$
4	$(x^2+yz, xy+z^p, xz+y^q)$	$p \geqq q > 2$
5	(x^2+yz, xy, z^2+y^p)	$p>2$
6	(x^2+yz, y^2, z^2)	
7	$(xy+z^p, yz+x^q, xz+y^r)$	$p \geqq q \geqq r > 2$
8	$(x^2+y^p+z^q, yz+z^2, xy)$	$p \geqq q > 2$ or $q=\infty$
10	(x^2, y^2, z^2)	

Table 3

Here $q=\infty$ means that the corresponding term z^q is omitted. As a sample of proof let us take the case $j^2 f=(xy, yz, xz)$. If e_i, $i=1,2,3$ denote the standard basis of $\mathbf{C}^3$, then a complete transversal is spanned by $z^k e_1$, $x^k e_2$, $y^k e_3$ with $k \geqq 3$.

Hence we can write

$$f = (xy+z^p A(z), yz+x^q B(x), xz+y^r C(y))$$

where $A(0)$, $B(0)$, $C(0) \neq 0$. The cooordinates change $\bar{x}=\alpha x$, $\bar{y}=\beta y$, $\bar{z}=\gamma z$, where α, β, γ are solutions of the system

$$\alpha \cdot \beta = A \cdot \gamma^p, \quad \beta\gamma = B\alpha^q, \quad \alpha\gamma = C \cdot \beta^r$$

shows then that we can take $A=B=C=1$. And by symmetry we can take $p \geqq q \geqq r > 2$.

See also the final remarks (2.5) and (2.6).

The first difficult case is when $j^2 f$ has type 9 that is $j^2 f=(xz, x^2+z^2, yz)$. A complete transversal is spanned now by $xy^{k-1}e_1$, $y^k e_1$, $y^k e_2$ with $k>2$ and hence we can write

$$f = (xz+xy^p A(y)+y^q B(y), x^2+z^2+y^r C(y), yz)$$

where $A(0), B(0), C(0) \neq 0$. With these notations we have the following result.

PROPOSITION 2.1. i) *If $q>3$ and $r>3$ then the map germ f is at least bimodular.*

ii) *In all the other cases* (i.e. *when $q=3$ or $r=3$*) *the map germ f is unimodular and is $\mathscr{K}$-equivalent to one of the following germs*

$$(xz+y^3, x^2+z^2+\lambda y^3, yz) \qquad \lambda \in \mathbf{C}$$

$$(xz+xy^{p-1}+y^p, x^2+z^2+y^3, yz) \qquad p>3$$

$$(xz+y^p, x^2+z^2+y^3, yz) \qquad p>3$$

$$(xz+xy^p, x^2+z^2+y^3, yz) \qquad p>1\ .$$

PROOF. i) Consider the weighted homogeneous map germ

$$g = (xz+axy^2+by^4, x^2+z^2+cy^4, yz)$$

with $wt(x)=wt(z)=2$, $wt(y)=1$.

An easy computation shows that

$$\dim\left(\langle xy^2e_1, y^4e_1, y^4e_2\rangle \cap TKg\right) \leqq 1$$

and hence even the 4-jet of g is at least bimodular. Moreover any map germ f as above with $q,r>3$ has a 4-jet to which some germ g specializes.

ii) The case $q=3$ produces the first one-parameter family and the case $q>3$, $r=3$ produces the last 3 series.

The next case is type 11 for j^2f i.e.

$$j^2f = (xz, yz, x^2)\ .$$

A complete transversal is spanned by $xy^{k-1}e_1$, y^ke_1, y^ke_3, z^ke_3 for $k\geqq 3$ and hence we can write any such germ in the form

$$f = (xz+xy^pA(y)+y^qB(y), yz, x^2+y^rC(y)+z^sD(z))$$

where $A(0)$, $B(0)$, $C(0)$, $D(0)\neq 0$.

With these notations we have the following.

PROPOSITION 2.2. i) *If $q>3$ and $r>3$, then the map germ f is at least bimodular.*

ii) *In all other cases the map germ f is unimodular and is $\mathcal{K}$-equivalent to one of the following germs*

$$(xz+xy^2+y^3, yz, x^2+y^3+\lambda z^p) \qquad \lambda \in \mathbf{C}\setminus\{0\},\ p>2$$

$$(xz+xy^2+y^3, yz, x^2+z^p) \qquad p>2$$

$$(xz+y^3, yz, x^2+zp) \qquad p>2$$

$$(xz+y^p, yz, x^2+y^3+z^q) \qquad p>2,\ q>2$$

$$(xz+xy^p, yz, x^2+y^3+z^q) \qquad p>1,\ q>2\ .$$

PROOF. i) This point is clear using the specialization 9 → 10 in Table (given explicitly for instance by the family (xz, yz, x^2+tz^2)) and the bimodular family constructed in the proof of Proposition 2.1.i.

ii) We shall discuss first the case $q=3$. Then j^3f can be put in one of the following normal forms:

$$f_1 = (xz+y^3, yz, x^2+az^3), \qquad f_2 = (xz+y^3, yz, x^2+y^3+az^3)$$

$$f_3 = (xz+xy^2+y^3, yz, x^2+az^3), \quad f_4 = (xz+xy^2+y^3, yz, x^2+y^3+\lambda z^3)$$

with $a=0$ or 1 and $\lambda \in \mathbf{C}$.

For $a=1$ and $\lambda \neq 0$ a simple computation shows

$$TK(f_i + \text{terms of degree} > 3) \supset \mathfrak{m}^4 \mathscr{E}_{3,3}$$

and hence the germs f_i are 3-determined. In this way we get 3 normal forms and a family with $\lambda \neq 0$.

Suppose now that $a=\lambda=0$. An easy computation shows that

$$F^4(f_i) + \langle z^4 e_3 \rangle = P_3^4$$

and hence a complement T to $F^m(f_i)$ in P_3^m as in Proposition 1.5 is given by

$$T = \langle z^4 e_3, z^5 e_3, \ldots, z^m e_3 \rangle .$$

It follows that any $f \in J_3^m(f_i)$ can be written as $f = f_i + \varrho(z) \cdot e_3$.

Let $p = \operatorname{ord} \varrho(z) \geqq 4$. When $i=1,2,3$ we can use the fact that f_i are weighted homogeneous and we get $f \sim f_i + z^p e_3$.

In the case $i=4$ we obtain a sequence of families, namely $f \sim f_4 + \lambda z^p e_3$, $\lambda \neq 0$. Suppose now $q>3$ and $r=3$.

There are 4 possibilities for the 3-jet of f in this case, namely

$$g_1 = (xz, yz, x^2+y^3) \qquad g_2 = (xz, yz, x^2+y^3+z^3)$$

$$g_3 = (xz+xy^2, yz, x^2+y^3) \quad g_4 = (xz+xy^2, yz, x^2+y^3+z^3) .$$

The jets g_i $(i=1,2,3)$ are weighted homogeneous and can be treated by similar methods as above and g_4 is not weighted homogeneous, but $TK(g_4 + \text{terms of degree} > 3) \supset \mathfrak{m}^4 \mathscr{E}_{3,3}$ and hence g_4 is 3-determined.

The next case to be treated is $j^2 f = (y^2, z^2, xy)$. A complete transversal is spanned by $x^k e_1$, $x^{k-1} z e_1$, $x^k e_e$ for $k \geqq 3$ and hence we can write such a germ in the form

$$f = (y^2 + x^p A(x) + x^q z B(x),\ z^2 + x^r C(x), yz) .$$

We have the following result.

PROPOSITION 2.3. i) *If $p>3$ and $r>3$, then the map germ f is at least bimodular.*

ii) *In all the other cases the map germ f is unimodular and is $\mathscr{K}$-equivalent to one of the following germs*

$$(y^2+x^3+x^2z, z^2+x^3, xy)$$

$$(y^2+x^3+x^2z, z^2, xy)$$

$$(y^2+x^3, z^2, xy)$$

$$(y^2+x^p, z^2+x^3, xy) \qquad p>2$$

$$(y^2+x^pz, z^2+x^3, xy) \qquad p>1$$

$$(y^2+x^p+x^{p-1}z, z^2+x^3, xy) \qquad p>2$$

$$(y^2+x^{p+2}+x^pz, z^2+x^3, xy) \qquad p>1\,.$$

The proof of this proposition is similar to and easier than the proof of Proposition 2.2 and hence we give no more details.

The classification of the unimodulars is ended by the following result.

PROPOSITION 2.4. *Any map germ $f\in A$ such that j^2f is a net of conics of type $k\geqq 13$ is at least bimodular.*

PROOF. Using the specializations given in Table 2, it is enough to prove the case $k=13$, i.e.

$$f_0 \;=\; j^2f \;=\; (x^2+yz, xy, y^2)\,.$$

A complete transversal to f_0 in $J_2^3(f_0)$ is spanned by z^3e_1, xz^2e_1, xz^2e_2, z^3e_2, xz^2e_3, z^3e_3, and a simple computation shows that $\operatorname{cod}_0(f_0)\geqq 2$. Hence by Lemma 1.4.i [1] any jet $g\in J_2^3(f_0)$ has modality at least 2.

REMARKS. (2.5) The fact that our lists of normal forms of unimodular germs do not contain any overlaps can be seen by using the numerical invariants introduced in [1] i.e. the Hilbert-Samuel function of I_f = the ideal generated by the components of f, the contact codimension of f and

$$\sigma(d) \;=\; \min\{k\;;\; TKf\supset \mathfrak{m}^k\cdot\mathscr{E}_{3,3}\}\,.$$

In one case these invariants are not enough. Namely, the germs $f=(y^2+x^3+x^2z, z^2, xy)$ and $g=(y^2+x^3, z^2+x^3, xy)$ which occur in the list of Proposition 2.3 have all the above invariants identical.

Nevertheless we can distinguish them by a simple *algebraic* property: the ideal I_f contains the square of a germ $h\in\mathscr{E}_3$ with $j^1h\neq 0$ (we can take $h=z$), but I_g does not have this property. This implies that the algebras $Q(f)=\mathscr{E}_3/I_f$ and $Q(g)$ are not C-isomorphic and hence f is not $\mathscr{K}$-equivalent to g.

(2.6) The proofs above show in fact that the listed germs are unimodular in the set

$$B = \{f \in A \subset \mathscr{E}_{3,3} : df(0)=0\} .$$

If the map germ $f \in A$ has $\operatorname{rk} df(0) \geqq 2$, then f is simple, but if $\operatorname{rk} df(0)=1$ we know that f can have modality>1 [1].

Suppose we have a convergent sequence of such map germs

$$f^n = (L_1^n + Q_1^n + \ldots, L_2^n + Q_2^n + \ldots, L_3^n + Q_3^n + \ldots)$$

such that $\Sigma f^n = \Sigma^{2,2}$ (a necessary condition for modality>1 by [1]) and $f^\infty = \lim f^n \in B$.

For each n there is a linear form L^n and constants a_i^n such that $L_i^n = a_i^n \cdot L^n$ for $i=1,2,3$.

The point $a^n = (a_1^n : a_2^n : a_3^n) \in P^2$ is well-defined and we can assume (passing to a subsequence if necessary) that $\lim a^n = a^\infty$ exists and for instance $a_1^\infty \neq 0$.

Then obvious linear target coordinate changes produce a sequence f^n as above with the additional properties that $L_2^n = L_3^n = 0$ and $L_1^n \mid Q_i^n$ for $i=2,3$ (use $\Sigma f^n = \Sigma^{2,2}$).

If the corresponding limit f^∞ has the form $(Q_1 + \ldots, Q_2 + \ldots, Q_3 + \ldots)$, then it follows that the two quadratic forms Q_2 and Q_3 have a common linear factor, this property being a closed one.

A careful examination of the listed normal forms which have 2 components of the 2-jet divisible by a linear factor shows that a $\Sigma^{2,2}$-germ of modality >1 never specializes to a germ listed above.

In this computations a central role is played by the explicit list of unimodulars in the case $n=2$.

It is an interesting open question whether there is any general principle in connection with this phenomenon.

3. Basic invariants of unimodular map germs in A.

In the classification of equidimensional unimodular map germs we have obtained the following 3 types of families of map germs $\mathbf{C}^3 \to \mathbf{C}^3$:

$$f_\lambda = (x^2 + \lambda yz, y^2 + \lambda zx, z^2 + \lambda xy) \qquad \lambda^3 \neq -1.0, 8$$

$$g_\lambda = (xz + y^3, x^2 + z^2 + \lambda y^3, yz) \qquad \lambda \in \mathbf{C}$$

$$h_{\lambda,p} = (xz + xy^2 + y^3, yz, x^2 + y^3 + \lambda z^p)\ \lambda \in \mathbf{C} \setminus \{0\},\ p>2$$

The basic invariants of these families are described by the following.

PROPOSITION 3.1. *The basic invariants for the above families are*

i) *for* f_λ: $\varphi: \mathbf{C} \setminus \{\lambda \mid \lambda^3 = -1, 0, 8\} \to \mathbf{C} \setminus \{0\}$

$$\varphi(\lambda) = \frac{\lambda^3(8-\lambda^3)^3}{(1+\lambda^3)^3}.$$

(ii) *for* g_λ: $\varphi: \mathbf{C} \to \mathbf{C}$, $\varphi(\lambda) = \lambda^2$.

iii) *for* $h_{\lambda,p}$: $\varphi: \mathbf{C} \setminus \{0\} \to \mathbf{C} \setminus \{0\}$, $\varphi(\lambda) = \lambda$.

PROOF. i) The components of f_λ are precisely the partial derivatives of the cubic $C_\lambda = 1/3\ (x^3 + y^3 + z^3 + 3\lambda xyz)$. Moreover it is shown in [2] that two nets of conics f_λ and f_μ are equivalent iff the corresponding cubic forms c_λ and c_μ are equivalent. And the basic invariant for cubic forms is known to be

$$j(\lambda) = \frac{\lambda^3(8-\lambda^3)^3}{64(1+\lambda^3)^3}.$$

ii) The germ g_λ is weighted homogeneous if we set $wt(x) = wt(z) = 3$, $wt(y) = 2$.

Let $I_\lambda = (g_\lambda^1, g_\lambda^2, g_\lambda^3)$ be the ideal generated by the components of g_λ and note that

$$\Delta^2 I_\lambda = (x^2, z^2, xz, yz, xy^2, y^3) = J,$$

the ideal generated by the 2×2 minors in the Jacobian matrix of g_λ together with I_λ as in [1].

Suppose $g_\lambda \sim g_\mu$. Then there exists a $\mathbf{C}$-algebra isomorphism $u: \mathscr{E}_3 \to \mathscr{E}_3$ such that

$$u(I_\lambda) = I_\mu.$$

In particular this gives $u(J) = J$. Moreover we can write:

$$u(x) = a_1 x + a_2 y + a_3 z + \ldots$$

$$u(y) = b_1 x + b_2 y + b_3 z + \ldots$$

$$u(z) = c_1 x + c_2 y + c_3 z + \ldots$$

Since $x^2, z^2 \in J$ we have $u(x)^2$, $u(z)^2 \in J$ and this gives $a_2 = c_2 = 0$. Similarly $u(y) \cdot u(z) \in J$ gives $c_1 = 0$.

Now we have a filtration on $R = \mathscr{E}_3$ given by the weights

$$R_\alpha = \{f \in R\ ;\ \text{all the terms of } f \text{ have } wt \geqq \alpha\}.$$

By our computation above it follows $wt(u(x)) \geqq wt(x)$ and similarly for y, z. Hence $u(R_\alpha) = R_\alpha$ and we have a map

$$\bar{u}: \frac{I_\lambda \cap R_\alpha + R_{\alpha+1}}{R_{\alpha+1}} \xrightarrow{\sim} \frac{I_\mu \cap R_\alpha + R_{\alpha+1}}{R_{\alpha+1}}.$$

We shall use this isomorphism for $\alpha=6$. $\bar{u}(xz+y^3)=(a_1x+a_3z)c_3z+b_2^3y^3$ should be a linear combination of g_μ^1 and g_μ^2. It follows that $a_3=0$ and $a_1c_3=b_2^3$. Next

$$\bar{u}(x^2+z^2+\lambda y^3) = a_1^2x^2+c_3^2z^2+\lambda b_2^3y^3 = kg_\mu^2$$

gives

$$a_1^2 = c_3^2 = \frac{\lambda b_2^3}{\mu}.$$

Finally $a_1=\pm c_3$, $b_2^3=\pm a_1^2$, $\dfrac{\lambda}{\mu}=\pm 1$.

On the other hand the coordinate change: $y \to -y, z \to -z$ shows that $g_\lambda \sim g_{-\lambda}$.

iii) This case is a little bit more difficult since the germ $h_{\lambda,p}$ is not weighted homogeneous. In fact it is the sum of a weighted homogeneous with two terms of higher orders which occur on the same component and moreover this can be done in two distinct ways:

(1) $wt(x)=wt(y)=1$, $wt(z)=2$
(2) $wt(x)=1/2$, $wt(y)=1/3$, $wt(z)=1/p$.

As in (ii) above let

$$I_\lambda-(h_{\lambda,p}^1, h_{\lambda,p}^2, h_{\lambda,p}^3) \subset R = \mathscr{E}_3$$

and note that again

$$\Delta^2 I_\lambda = (x^2, z^2, xz, xy^2, y^3) = J.$$

It follows that we can get the same information about an isomorphism $u\colon R \to R$, $u(I_\lambda)=I_\mu$ as above.

Using the first system of weights (1), the induced linear isomorphism $\bar{u}$ in weight $\alpha=3$ gives as above $b_1=0$ and $a_1c_3=a_1b_2^2=b_2^3$.

Next, it is easy to see that the orders of $u(x)$ and $u(y)$ with respect to z (i.e. the order of the power series $u(x)$ $(0,0,z)$, $wt(z)=1$) are at least $p-1$. Moreover since $yz \in I_\lambda$, we can omit in the series $u(x)$, $u(y)$, $u(z)$ all the terms which are multiples of yz.

With these two remarks, it follows that u respects the filtration on R induced by the second system of weights (2).

The corresponding linear isomorphism u for $\alpha=1$ gives $a_1^2=b_2^3=\dfrac{\lambda}{\mu}c_3^p$. (Here $R_{\alpha+1}$ is replaced by $\bigcup_{\beta>\alpha}R_\beta$). We finally get $a_1=b_2=c_3=1$, $\lambda=\mu$.

To end up, we explain now how the present classification yields the last result in the introduction.

By du Plessis' main theorem in [4], $\mathscr{A}$-finite determinacy of equidimensional map germs $(K^n,0) \to (K^n,0)$ holds in general iff $n \leqq {}^2\sigma(n,n)$, where the last number is precisely the codimension of the set B of map germs $(\mathbf{C}^n,0) \to (\mathbf{C}^n,0)$ with $\mathscr{K}$-modaility $\geqq 2$.

But our classification of equidimensional map germs of $\mathscr{K}$-modality $\leqq 1$ shows that any element of B is in the closure of 5 families of $\mathscr{K}$-orbits with $\mathscr{K}$-modality $\geqq 2$, namely the families constructed in Proposition 3.5.ii and Lemma 3.10 [1] and in Lemma 1.1, Proposition 2.2.i and Proposition 2.4 here.

Explicit computations with these 5 families of map germs show that ${}^2\sigma(n,n) = 13$ for $n \geqq 2$. Since trivially ${}^2\sigma(1,1) = \infty$, the proof of the stated result is completed.

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