MATH. SCAND. 56 (1985), 29-42

INTERPOLATION OF MARCINKIEWICZ SPACES

MICHAEL CWIKEL AND PER NILSSON

Abstract.

For each concave non-negative function $\varrho(t)$, the Marcinkiewicz space M_ϱ consists of all measurable functions f such that

$$\|f\|_{M_\varrho} = \sup_{t>0} (\varrho(t))^{-1} \int_0^t f^*(s)\,ds < \infty .$$

Interpolation spaces with respect to couples $(M_{\varrho_0}, M_{\varrho_1})$ of such spaces are considered. It is shown that for certain choices of ϱ_0 and ϱ_1 these interpolation spaces can be characterized by a monotonicity property with respect to the K-functional, that is $(M_{\varrho_0}, M_{\varrho_1})$ is a Calderón couple. A necessary and sufficient condition is given for interpolation from a couple of weighted L^∞ spaces to certain couples $(M_{\varrho_0}, M_{\varrho_1})$ to be characterized by K-functionals.

0. Introduction.

Let (X, Σ, μ) be a measure space and $\varrho(t)$ be a positive function on $[0, \infty)$. We define the Marcinkiewicz space M_ϱ to consist of all (equivalence classes of) measurable functions f on X such that $\int_0^t f^*(s)ds \leqq C\varrho(t)$ for all $t>0$ and some constant C. As usual f^* denotes the non-increasing rearrangement of f. M_ϱ is normed by

$$\|f\|_{M_\varrho} = \sup_{t>0} (\varrho(t))^{-1} \int_0^t f^*(s)\,ds .$$

As important special cases of Marcinkiewicz spaces we have

$$L^1(\varrho(t)=1), \quad L^\infty(\varrho(t)=t), \quad L^{p,\infty} = \text{Weak } L^p(\varrho(t)=t^{1-1/p},\ 1<p<\infty),$$
$$L^1+L^\infty(\varrho(t)=1+t) \quad \text{and} \quad L^1 \cap L^\infty(\varrho(t)=\min(1,t)).$$

In particular, the weak L^p spaces arise naturally in connection with the Marcinkiewicz interpolation theorem ([2, p. 6]) and its generalizations. Since $\int_0^t f^*(s)\,ds$ is always a concave function of t and tends to zero as $t \to 0$ we may equivalently replace ϱ by its greatest concave minorant $\tilde{\varrho}$ on $(0, \infty)$ and obtain $M_\varrho = M_{\tilde{\varrho}}$. We may also take $\tilde{\varrho}(0) = \lim_{t\to 0} \tilde{\varrho}(t)$. In view of these remarks we need

Received May 17, 1983.

only consider functions ϱ which are non-negative, concave and continuous on $[0, \infty)$, and consequently also non-increasing.

In this paper we consider the problem of characterizing the interpolation spaces with respect to certain couples of Marcinkiewicz spaces $(M_{\varrho_0}, M_{\varrho_1})$. We refer to [2] or [12] for basic notions and terminology pertaining to the theory of interpolation spaces. More specifically we are concerned with the question of whether all interpolation spaces A with respect to $(M_{\varrho_0}, M_{\varrho_1})$ can be characterized by the following property: If $a \in A$ and $b \in M_{\varrho_0} + M_{\varrho_1}$ and if for all $t > 0$,

$$K(t, b;\, M_{\varrho_0}, M_{\varrho_1}) \leqq K(t, a;\, M_{\varrho_0}, M_{\varrho_1}),$$

then $b \in A$.

If all interpolation spaces with respect to $(M_{\varrho_0}, M_{\varrho_1})$ can be characterized in this way, then $(M_{\varrho_0}, M_{\varrho_1})$ is termed a *Calderón couple* or *K-monotonic couple.* (See, e.g., [9], [10] for recent results concerning such couples.)

Our main result here is that, for each non-negative concave function ϱ, each of the couples (M_ϱ, L^∞) and (L^1, M_ϱ) are Calderón couples. The proofs proceed via a reduction to the well known special case of the couple (L^1, L^∞) which was treated by both Calderón [6] and Mitjagin [13] (Thus above it would be more accurate, if rather cumbersome, to speak of a "Calderón-Mitjagin couple".) Note that in general $(M_{\varrho_0}, M_{\varrho_1})$ is not Calderón, as can be seen from the example of the couple $(L^1 \cap L^\infty, L^1 + L^\infty)$ investigated by Ovčinnikov [15]. On the other hand there exist couples (such as $(L^{p_0,\infty}, L^{p_1,\infty})$ or $(M_{\varrho_0}, M_{\varrho_1})$, where ϱ_0 and ϱ_1 satisfy condition (3.1) below) which are Calderón couples, though not of the form (M_ϱ, L^∞) or (L^1, M_ϱ). (See [16], [17], [8].)

We also briefly consider the description of interpolation spaces for operators mapping a couple of weighted L^∞ spaces to the couple (M_ϱ, L^∞) or the couple (L^1, M_ϱ). We give necessary and sufficient conditions on ϱ for the corresponding interpolation spaces to be characterized in terms of K-functionals (i.e., for the couples (M_ϱ, L^∞) and (L^1, M_ϱ) to have the "universal right K property" in the terminology of [11, Section 4]). This complements some earlier results of Peetre [16], [17] (cf. also [8]) and shows, in a sense to be made precise below, that, unlike the couples $(L^{p_0,\infty}, L^{p_1,\infty})$ and $(L^{p,\infty}, L^\infty)$ treated in [16] (and implicitly in [8]) and similarly to the couple (L^1, L^∞), each of the couples (M_ϱ, L^∞) and (L^1, M_ϱ) is essentially different from a couple of weighted L^∞ spaces, for a certain class of functions ϱ.

REMARKS. (1). The above results lead us naturally to the following questions:

(i) What are necessary and sufficient conditions on ϱ_0 and ϱ_1 for $(M_{\varrho_0}, M_{\varrho_1})$ to be Calderón, or for it to have the unversal right K property? (The latter property implies the former.)

(ii) Does there exist a rearrangement invariant Banach space B such that (L^1, B) or (B, L^∞) is not a Calderón couple? (For a rather complete answer to similar questions in the context of weighted Banach lattices see [10].)

(2). A commonly encountered alternative version of Marcinkiewicz space is $M(\psi)$ which consists of all functions f for which $\sup_{t>0} \psi(t) f^*(t) < \infty$. (Cf. [12], [17].) Here $\psi(t)$ is some positive concave function on $(0, \infty)$. The spaces $M(\psi)$ and M_ϱ coincide if and only if ϱ satisfies condition (3.1) below and $\varrho(t)$ is equivalent to $t/\psi(t)$. (See [17], cf. also [12], p. 115.) The important quasinormed space Weak L^1 corresponds to $M(\psi)$ for $\psi(t)=t$ but of course cannot be obtained as M_ϱ for any choice of ϱ.

Our results are presented as follows. In Section 1 of the paper we show that (M_ϱ, L^∞) is a Calderón couple. The corresponding result for (L^1, M_ϱ) is given in Section 2. The remaining results concerning interpolation for operators mapping a couple of weighted L^∞ spaces to the couple (M_ϱ, L^∞) or (L^1, M_ϱ) are presented in Section 3.

ACKNOWLEDGEMENT. We thank Jaak Peetre for some helpful discussions.

1. The couple (M_ϱ, L^∞).

THEOREM 1. *For any non-negative concave function ϱ, the couple (M_ϱ, L^∞) is a Calderón couple.*

PROOF. It suffices to show (cf. [6], [18], etc) that for any functions $f, g \in M_\varrho + L^\infty$, if

$$K(t, g;\ M_\varrho, L^\infty) \leqq K(t, f;\ M_\varrho, L^\infty) \quad \text{for all } t>0\,, \tag{1.1}$$

then there exists a linear operator T bounded on both M_ϱ and L^∞ with norms independent of f and g such that $Tf=g$. Since, in the notation of [14],

$$\begin{aligned} M_\varrho &= (L^1, L^\infty)_{L^\infty_{1/\varrho};K} \\ &= \left\{ h \in L^1 + L^\infty \;\middle|\; \|K(\cdot, h;\ L^1, L^\infty)\|_{L^\infty_{1/\varrho}} = \sup_{t>0} (1/\varrho(t)) K(t, h;\ L^1, L^\infty) < \infty \right\}, \end{aligned}$$

we can apply the Brudnyĭ-Krugljak reiteration theorem ([3], [4], [14]) to obtain that the K-functional $K(t, h;\ M_{\varrho_0}, M_{\varrho_1})$ is equivalent to the expression

(1.2)

$$K(t, K(\cdot, h;\ L^1, L^\infty);\ L^\infty_{1/\varrho_0}, L^\infty_{1/\varrho_1}) \sim \sup_{s>0} \left(\min\,(1/\varrho_0(s), t/\varrho_1(s)) \int_0^s h^*(u)\,du\right).$$

In particular

$$K(t,h;\, M_\varrho, L^\infty) \sim \sup_{s>0}\left[\min\,(1/\varrho(s), t/s)\int_0^s h^*(u)\,du\right].$$

Since $(1/s)\int_0^s h^*(u)\,du$ and $\varrho(s)/s$ are both non-increasing functions of s it follows that for each $s'>0$,

$$K(s'/\varrho(s'), h;\, M_\varrho, L^\infty) \sim \sup_{0<s<s'}\,(1/\varrho(s))\int_0^s h^*(u)\,du\,. \tag{1.3}$$

Now suppose that f and g are any functions in $M_\varrho+L^\infty$ which satisfy (1.1). For the construction of the operator T, appealing to [6, Lemma 2, p. 277], we see that we can assume that the measure space is $(0,\infty)$ equipped with Lebesgue measure and that $f=f^*$, $g=g^*$. (Note that Lemma 2 of [6] is valid even without the restrictions of σ-finiteness and the requirement that f_2 vanishes on the set where $f_2\leqq\alpha$, via arguments as in [7, pp. 232–233].)

Now, introducing the notation $m(t,h)=(1/\varrho(t))\int_0^t h(u)\,du$, we have, in view of (1.1) and (1.3), that

$$\sup_{0<s<t} m(s,g) \leqq C\sup_{0<s<t} m(s,f) \quad \text{for all } t>0 \tag{1.4}$$

where C is an absolute constant. Now if $m(s,f)$ happens to be a non-decreasing function of s for all $s>0$ then (1.4) implies that $\int_0^t g^*(u)\,du\leqq C\int_0^t f^*(u)\,du$ for all $t>0$ and by the Calderón-Mitjagin theorem ([6], [7, pp. 232–233]) there exists an operator T with norm C on L^1 and on L^∞ and therefore also on M_ϱ such that $Tf=g$. The rest of our proof amounts essentially to a reduction to this simple special case. The main step will be to construct a linear operator V which is bounded on both M_ϱ and L^∞ such that $m(s, Vf)$ is non-decreasing and (1.4) holds with f replaced by Vf.

Let $n(s,f)=\sup_{0<t\leqq s} m(t,f)$. This is a finite continuous non-decreasing function for all $s>0$. The set

$$W = \{s>0 \mid n(s,f)>m(s,f)\}$$

is thus open and can be expressed as the union of a countable or finite collection of disjoint open intervals $\{I_1, I_2, \ldots\}$. (Alternatively W is empty and the proof of the theorem proceeds trivially as above.) We set $I_\nu=(\alpha_\nu,\beta_\nu)$ for each I_ν in the above collection. The functions $m(s,f)$ and $n(s,f)$ clearly coincide for all $s\notin W$. In particular $m(\alpha_\nu,f)=n(\alpha_\nu,f)$ if $\alpha_\nu>0$. For all $s\in I_\nu$, $m(s,f)\leqq m(a_\nu,f)$. (Otherwise, for some such s, $n(s,f)=\sup_{\alpha_\nu\leqq t\leqq s} m(t,f)$ $>n(\alpha_\nu,f)$ and, at the point $t\neq\alpha_\nu$ where this supremum is attained, $m(t,f)$ $=n(t,f)$ which is a contradiction.) Consequently, on I_ν, $n(s,f)$ assumes the constant value $m(\alpha_\nu,f)$ which coincides with $m(\beta_\nu,f)$ if $\beta_\nu<\infty$. If one of the intervals I_ν has left endpoint zero we shall reserve the notation $I_0 = (0, \beta_0)$ for this interval. By a similar argument to that used above $n(s,f)$ assumes a constant value on I_0 equal to $\theta = \limsup_{t\to 0} m(t,f)$. (If $n(s,f) > \theta$ then $n(s,f) = \sup_{\varepsilon \leqq t \leqq s} m(t,f)$ for some suitably small $\varepsilon > 0$ and at the point t where the supremum is attained $m(t,f) = n(t,f)$, contradicting $t \in W$.) For later use we define a seminorm τ on $M_\varrho + L^\infty$ by

$$\tau(h) = \limsup_{n\to\infty} \varrho(\gamma_n)^{-1} \int_0^{\gamma_n} |h(x)|\,dx$$

where the sequence $(\gamma_n)_{n=1}^\infty$ is in I_0 and satisfies $\lim_{n\to\infty}\gamma_n = 0$ and $\lim_{n\to\infty} m(\gamma_n, f) = \theta = \tau(f)$.

We now define the operator V by

$$Vh = h\chi_{(0,\infty)\setminus W} + \sum_{\nu \neq 0} \left(\varrho(\alpha_\nu)^{-1} \int_0^{\alpha_\nu} h(x)\,dx\right)\varrho'\chi_{I_\nu}$$

where ϱ' denotes the derivative of ϱ which necessarily exists a.e. on $(0,\infty)$ and is a non-increasing function. If the collection of intervals I_ν includes I_0 with left endpoint zero then we must add an extra term of the form $\varrho' l(h)\chi_{I_0}$ to the formula for Vh, where l is a continuous linear functional on $M_\varrho + L^\infty$, whose existence is guaranteed by the Hahn-Banach theorem, such that $|l(h)| \leqq \tau(h)$ for all $h \in M_\varrho + L^\infty$ and $l(f) = \tau(f) = \theta$. (It may even happen that $W = (0,\infty) = I_0$ so that $\varrho' l(h)\chi_{I_0}$ is the only non zero term in the above expression for Vh.)

For any $h \in M_\varrho$ we have the a.e. pointwise estimate

$$\begin{aligned} |Vh| &\leqq |h| + \varrho' \sup_{t>0} \varrho(t)^{-1} \int_0^t |h(x)|\,dx \\ &\leqq |h| + \|h\|_{M_\varrho}\varrho'\,. \end{aligned}$$

So

$$\|Vh\|_{M_\varrho} \leqq \|h\|_{M_\varrho} + \|h\|_{M_\varrho}\|\varrho'\|_{M_\varrho} \leqq 2\|h\|_{M_\varrho}\,.$$

We next show that V is also bounded on L^∞. We first recall that for a.e. $t > 0$, it follows from the concavity and non-negativity of ϱ that

$$\varrho'(t) \leqq \varrho(t)/t\,. \tag{1.5}$$

Thus, for each $h \in L^\infty$, $|Vh|$ restricted to an interval $I_\nu = (\alpha_\nu, \beta_\nu)$ with $\nu \neq 0$ is dominated by

$$\varrho'(\alpha_\nu)(\alpha_\nu/\varrho(\alpha_\nu))\|\mathrm{h}\|_{L^\infty} \leqq \|h\|_{L^\infty}\,.$$

If $\nu = 0$ we either have $\lim_{t\to 0}\varrho(t)/t = \infty$ in which case $l(h) = 0$, or $\lim_{t\to 0}\varrho(t)/t < \infty$ in which case

$$l(h) \leqq \|h\|_{L^\infty}/\lim_{t\to 0}(\varrho(t)/t) \qquad \text{and} \qquad \varrho'(t) \leqq \lim_{t\to 0}\varrho(t)/t$$

so that $|Vh| \leqq \|h\|_{L^\infty}$ on I_0. Combining these estimates, we obtain that $\|Vh\|_{L^\infty} \leqq \|h\|_{L^\infty}$ for all $h \in L^\infty$.

We now turn to calculating $m(s, Vf)$. We first observe that for each of the intervals I_v with $\alpha_v > 0$, $\beta_v < \infty$

$$\begin{aligned}\int_{I_v} Vf\,dx &= (\varrho(\beta_v) - \varrho(\alpha_v))(\varrho(\alpha_v))^{-1}\int_0^{\alpha_v} f(x)\,dx \\ &= \varrho(\beta_v)m(\alpha_v, f) - \varrho(\alpha_v)m(\alpha_v, f) \\ &= \varrho(\beta_v)m(\beta_v, f) - \varrho(\alpha_v)m(\alpha_v, f) \\ &= \int_0^{\beta_v} f\,dx - \int_0^{\alpha_v} f\,dx = \int_{I_v} f\,dx\,.\end{aligned}$$

Similarly, if $\alpha_0 = 0$, $\beta_0 < \infty$

$$\int_{I_0} Vf\,dx = \varrho(\beta_0)\theta = \varrho(\beta_0)m(\beta_0, f) = \int_{I_0} f\,dx\,.$$

Thus for any $s \notin W$ it follows that $\int_0^s Vf\,dx = \int_0^s f\,dx$, and so $m(s, Vf) = m(s, f) = n(s, f)$. If however $s \in W$, then $s \in I_v = (\alpha_v, \beta_v)$ for some v and

$$\begin{aligned}\int_0^s Vf dx &= \int_0^{\alpha_v} Vf\,dx + \int_{\alpha_v}^s Vf\,dx \\ &= \int_0^{\alpha_v} f\,dx + (\varrho(s) - \varrho(\alpha_v))m(\alpha_v, f) \\ &= \varrho(\alpha_v)m(\alpha_v, f) + (\varrho(s) - \varrho(\alpha_v))m(\alpha_v, f) \\ &= \varrho(s)m(\alpha_v, f) \qquad \text{for } \alpha_v > 0\,, \\ &= \varrho(s)n(s, f)\,.\end{aligned}$$

Similarly, if $\alpha_v = 0$, $\int_0^s Vf\,dx = \varrho(s)\theta = \varrho(s)n(s, f)$. Thus in either case $m(s, Vf) = n(s, f)$ and indeed this equality holds for all $s > 0$. Since $n(s, f)$ is non-decreasing we have $m(s, g) \leqq Cn(s, f) = Cm(s, Vf) \leqq Cm(s, (Vf)^*)$ for all $s > 0$. Consequently

$$\int_0^s g^*(x)\,dx \leqq C\int_0^s (Vf)^*(x)\,dx \qquad \text{for all } s > 0$$

and an application of the Calderón-Mitjagin theorem, much as before, yields an operator V_1 with norm C on L^1 and L^∞ such that $V_1(Vf) = g$. Clearly, $T = V_1V$ is an operator with all the properties we require and the proof is complete.

2. The couple (L^1, M_ϱ).

THEOREM 2. *For any non-negative concave function* ϱ, *the couple* (L^1, M_ϱ) *is a Calderón couple.*

PROOF. We shall use several ideas similar to those in the proof of Theorem 1. Again we begin with arbitrary non-increasing non-negative functions $f=f^*$, $g=g^*$ in L^1+M_ϱ on $(0,\infty)$ satisfying a K-functional inequality

$$K(t,g;\, L^1, M_\varrho) \leqq K(t,f;\, L^1, M_\varrho) \quad \text{for all } t>0 . \tag{2.1}$$

We have to construct a linear operator T bounded on L^1 and on M_ϱ with norms independent of f and g such that $Tf=g$. In this case the formula (1.2) implies that

$$K(t,h;\, L^1, M_\varrho) \sim \sup_{s>0}\left(\min\,(1, t/\varrho(s)) \int_0^s h^*(u)\,du\right).$$

Thus, for every $s'>0$,

$$K(\varrho(s'),h;\, L^1, M_\varrho) \sim \sup_{s\geqq s'}\,(\varrho(s')/\varrho(s)) \int_0^s h^*(u)\,du$$

and it follows that

$$\sup_{s\geqq t} m(s,g) \leqq C \sup_{s\geqq t} m(s,f) \quad \text{for all } t>0 , \tag{2.3}$$

where, as before

$$m(s,h) = (1/\varrho(s)) \int_0^s h(s)\,ds ,$$

and $C>1$ is an absolute constant.

Much as in the proof of the preceding theorem, our strategy will be essentially to find a way of "converting" $m(s,f)$ to a monotonic (in this case non-increasing) function of s so that we can remove the suprema in (2.3) and then apply the Calderón-Mitjagin theorem to deduce the existence of the required operator T. More precisely, we shall construct a linear operator V which maps each of the spaces L^1 and M_ϱ continuously into themselves with bounds not exceeding 5, and having the property that, for all $s>0$, either

$$2m(s,Vf) \geqq \sup_{t\geqq s} m(t,f) \tag{2.4a}$$

or

$$m(s,Vf) \geqq C^{-1} m(s,g) . \tag{2.4b}$$

Indeed from (2.4a), (2.3) and (2.4b) we can deduce that

$$m(s,g) \leqq 2Cm(s,Vf) \leqq 2Cm(s,(Vf)^*) \quad \text{for all } s>0 .$$

Then we obtain T with bounds on L^1 and M_ϱ not exceeding 10C by an argument identical to that in the previous theorem.

Thus it remains to construct the operator V.

Let $N(s,f)=\sup_{t\geqq s} m(t,f)$. Thus is clearly a finite continuous non-increasing function for all $s>0$. The set

$$W = \{s>0 \mid N(s,f)>m(s,f)\}$$

is thus open and can be expressed as the union $W=\bigcup_{\nu\in\Psi} I_\nu$ of disjoint open intervals I_ν, where Ψ is a finite or countable index set. One of the intervals I_ν may be semi-infinite and if so we shall denote it by $I_\infty=(\alpha_\infty,\infty)$. Similarly there may be an interval with left endpoint zero. This will not require special treatment unless $\varrho(0)>0$, in which case we shall denote it by $I_0=(0,\beta_0)$. All other intervals I_ν will have endpoints denoted by α_ν, β_ν, that is, $I_\nu=(\alpha_\nu,\beta_\nu)$.

The functions $m(s,f)$ and $N(s,f)$ clearly coincide for all positive $s\notin W$. In particular, for each I_ν with $\beta_\nu<\infty$, $m(\beta_\nu,f)=N(\beta_\nu,f)$. Furthermore, using a minor modification ("mirror image") of the corresponding argument in the proof of Theorem 1, it follows that $N(s,f)$ assumes the constant value $m(\beta_\nu,f)$ on I_ν and, if $\alpha_\nu>0$, $m(\alpha_\nu,f)=m(\beta_\nu,f)$. Similarly, if W contains an interval $I_\infty=(\alpha_\infty,\infty)$, then $N(s,f)$ assumes a constant value on I_∞ which equals $\theta=\limsup_{t\to\infty} m(t,f)$. Furthermore there exists a sequence $(\gamma_k)_{k=1}^\infty$ in I_∞ such that $\gamma_k\to\infty$ and $\lim_{k\to\infty} m(\gamma_k,f)=\theta$. Let τ be the seminorm on L^1+M_ϱ defined by

$$\tau(h) = \limsup_{k\to\infty} \int_0^{\gamma_k} |h(x)|\,dx/\varrho(\gamma_k) .$$

Let W_1 be the subset of W which is the union of all those intervals I_ν with

$$\nu\in\Phi = \{\nu\in\Psi \mid \nu\neq 0,\ \nu\neq\infty,\ \varrho(\beta_\nu)\geqq 2\varrho(\alpha_\nu)\}$$

together with whichever of the intervals I_∞ and I_0 happen to appear in W. We define the operator V by

$$Vh = h\chi_{(0,\infty)\setminus W_1} + \sum_{\nu\in\Phi}\left[\frac{\int_{I_\nu} h(x)\,dx}{\varrho(\beta_\nu)-\varrho(\alpha_\nu)}\right]\varrho'\chi_{I_\nu} .$$

If W contains the interval $I_\infty=(\alpha_\infty,\infty)$, then we add the term $V_\infty h=\varrho' l(h)\chi_{I_\infty}$ to the above formula, where l is a linear functional satisfying $l(f)=\tau(f)=\theta$ and $|l(h)|\leqq\tau(h)$ for all $h\in L^1+M_\varrho$. If W contains the interval $I_0=(0,\beta_0)$ (meaning also that $\varrho(0)>0$) and if I_0 is distinct from I_∞ then we add the term

$$V_0 h = \left(g+\beta_0^{-1}\int_{I_0}(Cf-g)\,dx\right)\int_{I_0} h\,dx\chi_{I_0}\Big/\int_{I_0} Cf\,dx$$

to the formula for Vh. (Note that $\int_{I_0}(Cf-g)\,dx>0$ since, by (2.3), $m(\beta_0,f) = N(\beta_0,f)\geqq C^{-1}m(\beta_0,g)$.)

To show that V is bounded on L^1 we first observe that for all $v\in\Phi$ and $h\in L^1$

$$\int_{I_v}|Vh|\,dx = \frac{\varrho(\beta_v)-\varrho(\alpha_v)}{\varrho(\beta_v)-\varrho(\alpha_v)}\left|\int_{I_v} h(x)\,dx\right| \leqq \int_{I_v}|h(x)|\,dx\,.$$

If I_∞ is present then either $\lim_{t\to\infty}\varrho(t)=\infty$ and $\tau(h)=0$, or $\lim_{t\to\infty}\varrho(t)=\delta<\infty$ and

$$\int_{I_\infty}|Vh|\,dx \leqq (\delta-\varrho(\alpha_\infty))\tau(h) \leqq \delta\cdot\|h\|_{L^1}/\delta\,.$$

If I_0 is present then

$$\int_{I_0}|V_0 h|\,dx = \left|\int_{I_0} h\,dx\right| \leqq \int_{I_0}|h|\,dx\,.$$

Combining these estimates we deduce that $\|Vh\|_{L^1}\leqq 2\|h\|_{L^1}$.

We next verify the boundedness of V on M_ϱ. For any $h\in M_\varrho$, $\tau(h)\leqq\|h\|_{M_\varrho}$ and, for all v in Φ,

$$\left|\int_{I_v} h\,dx/(\varrho(\beta_v)-\varrho(\alpha_v))\right| \leqq \int_0^{\beta_v}|h(x)|\,dx/(\tfrac{1}{2}\varrho(\beta_v)) \leqq 2\|h\|_{M_\varrho}\,.$$

Furthermore

$$\begin{aligned}\|V_0 h\|_{M_\varrho} &\leqq \frac{\varrho(\beta_0)}{C\int_{I_0} f\,dx}\|h\|_{M_\varrho}\left\|\left[g+\beta_0^{-1}\int_{I_0}(Cf-g)\,dx\right]\chi_{I_0}\right\|_{M_\varrho}\\ &\leqq (Cm(\beta_0,f))^{-1}\|h\|_{M_\varrho}\left(\sup_{0\leqq s\leqq\beta_0} m(s,g)+\int_{I_0}|Cf|\,dx/\varrho(\beta_0)\right).\end{aligned}$$

Since $\varrho(0)>0$, $m(s,g)$ is continuous and bounded on $[0,\beta_0]$ and attains its maximum value at some $s_0\in(0,\beta_0]$. By (2.3)

$$m(s_0,g) \leqq CN(s_0,f) = Cm(\beta_0,f)\,.$$

Thus $\|V_0 h\|_{M_\varrho}\leqq 2\|h\|_{M_\varrho}$.

We deduce that

$$\begin{aligned}\|Vh\|_{M_\varrho} \leqq \|h\chi_{(0,\infty)\setminus W_1}\|_{M_\varrho}+2\|h\|_{M_\varrho}\|\varrho'\chi_{W_1}\|_{M_\varrho}+\\ +\|V_0 h\|_{M_\varrho} \leqq 5\|h\|_{M_\varrho}\,.\end{aligned}$$

Finally we must show that for all $s>0$, $m(s, Vf)$ satisfies at least one of the estimates (2.4a) and (2.4b). For each interval I_ν with either $\nu=0$ or $\nu \in \Phi$ we have $\int_{I_\nu} Vf\,dx=\int_{I_\nu} f\,dx$. Also Vf and f coincide for all $s \in (0,\infty)\setminus W_1$. Thus

$$\int_0^s Vf\,dx = \int_0^s f\,dx \qquad \text{for all } s \in (0,\infty)\setminus W_1\,. \tag{2.5}$$

In particular for all $\nu \in \Phi$ with $\alpha_\nu>0$ we have $\alpha_\nu \in (0,\infty)\setminus W_1$. Thus, for $s \in I_\nu$,

$$\begin{aligned} m(s, Vf) &= (\varrho(s))^{-1}\left(\int_0^s Vf\,dx\right) = (\varrho(s))^{-1}\left(\int_0^{\alpha_\nu} f\,dx+\int_{\alpha_\nu}^s Vf\,dx\right) \\ &= (\varrho(s))^{-1}\left(\varrho(\alpha_\nu)m(\alpha_\nu, f)+\frac{(\varrho(s)-\varrho(\alpha_\nu))}{(\varrho(\beta_\nu)-\varrho(\alpha_\nu))}(\varrho(\beta_\nu)m(\beta_\nu, f)-\varrho(\alpha_\nu)m(\alpha_\nu, f))\right). \end{aligned}$$

Since $m(\alpha_\nu, f)=m(\beta_\nu, f)$, we deduce that

$$m(s, Vf) = (\varrho(s))^{-1}[(\varrho(\alpha_\nu)+\varrho(s)-\varrho(\alpha_\nu))m(\alpha_\nu, f)] = m(\alpha_\nu, f) = N(s, f)$$

implying, a fortiori, that (2.4a) holds for $s \in I_\nu$.

If $\alpha_\nu=0$ and $\varrho(\alpha_\nu)=\varrho(0)=0$ then we obtain similarly that

$$\begin{aligned} m(s, Vf) = (\varrho(s))^{-1}\int_0^s Vf\,dx &= (\varrho(s))^{-1}\frac{\varrho(s)}{\varrho(\beta_\nu)}\varrho(\beta_\nu)m(\beta_\nu, f) \\ &= m(\beta_\nu, f) = N(s, f)\,, \end{aligned}$$

and again (2.4a) follows. If an interval $I_0=(0,\beta_0)$ appears in W_1 (so what $\varrho(0)>0$) then for each $s \in I_0$,

$$\int_0^s Vf\,dx \geqq C^{-1}\int_0^s g\,dx\,,$$

proving that (2.4b) holds on I_0. On I_∞,

$$\begin{aligned} \int_0^s Vf\,dx &= \int_0^{\alpha_\infty} f\,dx+(\varrho(s)-\varrho(\alpha_\infty))l(f) \\ &= \varrho(\alpha_\infty)m(\alpha_\infty, f)+(\varrho(s)-\varrho(\alpha_\infty))\theta\,. \end{aligned}$$

But $N(s, f)=\theta=m(\alpha_\infty, f)$ so $m(s, Vf)=\theta=N(s, f)$ and, as before, (2.4a) follows.

We have thus shown that either (2.4a) or (2.4b) holds for each $s \in W_1$. Also, if $s \notin W$ then, by (2.5), $m(s, Vf)=m(s, f)=N(s, f)$. This leaves us to consider only those intervals $I_\nu \subset W$ for which $\nu \neq 0$, $\nu \neq \infty$ and $\nu \notin \Phi$. For such intervals $\varrho(\beta_\nu)<2\varrho(\alpha_\nu)$ and therefore

$$N(s, f) = m(\alpha_\nu, f) \leqq (\varrho(s)/\varrho(\alpha_\nu))m(s, f) < 2m(s, f) = 2m(s, Vf)\,,$$

showing that here (2.4a) holds and that V has all the required properties and so completing the proof.

3. The description of interpolation spaces for operators mapping a couple of weighted L^∞ spaces to a couple of Marcinkiewicz spaces.

We begin by recalling the following definitions.

DEFINITION 3.1. *Let* $\bar{A}=(A_0, A_1)$ and $\bar{B}=(B_0, B_1)$ be two compatible couples of Banach spaces. Let A and B be normed intermediate spaces with respect to $\bar{A}$ and to $\bar{B}$ respectively. A and B are *relative interpolation spaces* if all linear operators defined on A_0+A_1 which map A_j boundedly into B_j for $j=0,1$ also map A boundedly into B. A and B are *relative K-spaces* if whenever $a \in A$ and $b \in B_0+B_1$ and $K(t,b;\ \bar{B}) \leqq K(t,a;\ \bar{A})$ for all $t>0$, then it follows that $b \in B$ with $\|b\|_B \leqq C\|a\|_A$ for some constant C independent of a and b. $\bar{A}$ and $\bar{B}$ are *relative Calderón couples*, if all relative interpolation spaces A and B are relative K-spaces. The couple $\bar{B}$ has the *universal right K property* if, for all couples $\bar{A}$ of weighted L^∞ spaces on every given measure space, the couples $\bar{A}$ and $\bar{B}$ are relatively Calderón.

It was shown by Peetre [17] that, if for some constant C

$$\int_0^u \varrho(s)/s\, ds \leqq C\varrho(u) \qquad \text{for all } u>0 \tag{3.1}$$

for $\varrho=\varrho_0$ and $\varrho=\varrho_1$, then $(M_{\varrho_0}, M_{\varrho_1})$ has the universal right K property. For the reader's convenience, we sketch a proof of this fact. We use Theorem 4.2 of [11, p. 30], and the notation introduced there. Thus in our case $\bar{J}$ is the couple couple $(M_{\varrho_0}, M_{\varrho_1})$ and $\bar{I}$ is the couple $(L^\infty_{w_0}, L^\infty_{w_1})$ with weight functions $w_j(s) = s/\varrho_j(s)$, $j=0,1$. Given any $y \in \sum(\bar{J})$ we need to find $x \in \sum(\bar{I})$ such that $K(t,x;\ \bar{I}) \sim K(t,y;\ \bar{J})$ for all $t>0$ and $y=Tx$ for some bounded operator T: $\bar{I} \to \bar{J}$.

In fact, we simply take x to be the non increasing rearrangement of y, $x=y^*$ and let T be a map bounded on L^1 and L^∞ and therefore on M_{ϱ_0} and M_{ϱ_1} with norms one such that $Ty^*=y$. (Cf. [6, Lemma 2] and the remarks concerning that lemma in Section 1 above.) Then for any $f \in L^\infty_{w_j}$,

$$\|Tf\|_{M_{\varrho_j}} \leqq \|f\|_{M_{\varrho_j}} \leqq C\|f\|_{L^\infty_{w_j}}$$

in view of (3.1). Thus $K(t,y;\ \bar{J}) \leqq CK(t,x;\ \bar{I})$ and also

$$
\begin{aligned}
K(t,x;\bar{I}) &\sim \sup_{s>0}\left(\min\left(\frac{s}{\varrho_0(s)},\frac{st}{\varrho_1(s)}\right)y^*(s)\right)\\
&\leq \sup_{s>0}\left(\min\left(\frac{1}{\varrho_0(s)},\frac{t}{\varrho_1(s)}\right)\int_0^s y^*(u)\,du\right)\\
&\sim K(t,y;\bar{J}) \quad \text{for all } t>0\,,
\end{aligned}
$$

which completes the proof.

In this section we observe that, at least for the couples (M_ϱ, L^∞) and (L^1, M_ϱ), condition (3.1) is also necessary for the universal right K property to hold. (This also shows incidentally that one cannot hope for simpler proofs of Theorems 1 and 2 resembling the argument for $(L^{p_0,\infty}, L^{p_1,\infty})$ in [16].)

Specifically we show that (M_ϱ, L^∞) has the above property if and only if ϱ satisfies (3.1). As for the couple (L^1, M_ϱ) we have $L^1 = M_{\varrho_0}$ where $\varrho_0(s)=1$ does not satisfy (3.1). Correspondingly we shall show that (L^1, M_ϱ) does not have the unversal right K property for any choice of ϱ, except in the trivial case where ϱ is bounded above and below so that $M_\varrho = L^1$.

To treat the couple (M_ϱ, L^∞), we let $g(x)=\sqrt{\varrho(x)/x}$ and $f(x)=\int_0^x g(s)\,ds$. By the monotonicity of $\varrho(x)$ and $\varrho(x)/x$ it follows readily that $\sqrt{x\varrho(x)} \leqq f(x) \leqq 2\sqrt{x\varrho(x)}$. Also

$$K(t,g;\, M_\varrho, L^\infty) \sim K(t,f;\, L^\infty_{1/\varrho}, L^\infty_{1/x})\,.$$

Now

$$f \in L^\infty_{1/\sqrt{x\varrho}} = [L^\infty_{1/\varrho}, L^\infty_{1/x}]^{\frac{1}{2}}$$

so if (M_ϱ, L^∞) has the universal right K property it should follow that $g \in [M_\varrho, L^\infty]^{\frac{1}{2}}$ with $\|g\|_{[M_\varrho, L^\infty]^{\frac{1}{2}}}$ bounded by some absolute constant C. (Here $[M_\varrho, L^\infty]^{\frac{1}{2}}$ is the complex interpolation space obtained by Calderón's second method ([2], [5].) Let $(g_n)_{n=1}^\infty$ be an increasing sequence of non-negative, non-increasing simple functions converging pointwise to g. Then each $g_n \in M_\varrho \cap L^\infty$ and, by [5, Section 13.6] and [1],

$$\|g_n\|_{[M_\varrho, L^\infty]^{\frac{1}{2}}} = \|g_n\|_{[M_\varrho, L^\infty]_{\frac{1}{2}}} = \|g_n\|_{(M_\varrho)^{\frac{1}{2}}(L^\infty)^{\frac{1}{2}}} \leqq C \quad \text{for all } n\,.$$

Thus $\|g_n^2\|_{M_\varrho} \leqq C^2$ and, for each $t>0$, $\int_0^t (g_n(u))^2\,du \leqq C^2\varrho(t)$. By monotone convergence $\int_0^t (g(u))^2\,du \leqq C^2\varrho(t)$ which by Schwarz's inequality implies that ϱ satisfies (3.1) as required.

We now turn to the couple (L^1, M_ϱ). We begin by recalling the "Hölder-like" inequality

$$\int_0^\infty FG\,dx \leqq \left(\int_0^\infty F^*\varrho'\,dx+\varrho(0)\|F\|_{L^\infty}\right)\|G\|_{M_\varrho}, \tag{3.2}$$

which can be proved using obvious minor modifications of the argument in [12, p. 115]. (If $\varrho(0)=0$ the term $\varrho(0)\|F\|_{L^\infty}$ is taken to be zero, even if F is unbounded.)

Now consider the function $g(x)=\varrho'(x)/2\sqrt{\varrho(x)}$. This is non-increasing and non-negative and the function

$$v(x) = \int_0^x g(s)\,ds = \sqrt{\varrho(x)}-\sqrt{\varrho(0)}$$

is in the space $L^\infty_{1/\sqrt{\varrho}}=[L^\infty,L^\infty_{1/\varrho}]^{\frac{1}{2}}$ with norm 1.

Also $K(t,g;\,L^1,M_\varrho)\sim K(t,v;\,L^\infty,L^\infty_{1/\varrho})$. Suppose then that (L^1,M_ϱ) has the universal right K property. This would imply that $g\in[L^1,M_\varrho]^{\frac{1}{2}}$ with norm bounded by an absolute constant C. Now, letting $(g_n)_{n=1}^\infty$ be an increasing sequence of non-negative non-increasing simple functions converging to g, we can again invoke [5, 13.6] and [1] to deduce that

$$g_n \in [L^1,M_\varrho]_{\frac{1}{2}} \subset (L^1)^{\frac{1}{2}}(M_\varrho)^{\frac{1}{2}}$$

and in fact

$$g_n \leqq (g_{n,1})^{\frac{1}{2}}(g_{n,2})^{\frac{1}{2}}$$

where $\|g_{n,1}\|_{L^1}$ and $\|g_{n,2}\|_{M_\varrho}$ are both bounded by constants which can be arbitrarily close to C. Consequently, for any non-negative non-increasing function f,

$$\begin{aligned}\int_0^\infty fg\,dx &= \lim_{n\to\infty}\int_0^\infty fg_n\,dx \\ &\leqq \lim_{n\to\infty}\left(\int_0^\infty g_{n,1}\,dx\right)^{\frac{1}{2}}\left(\int_0^\infty f^2g_{n,2}\,dx\right)^{\frac{1}{2}} \\ &\leqq C\left(\int_0^\infty f^2\varrho'\,dx+\varrho(0)\|f\|^2_{L^\infty}\right),\end{aligned} \tag{3.3}$$

where we have used (3.2) in the last step.

We shall now show that an estimate of the form (3.3) does not hold for any choice of contant C. This contradicts our assumption above and shows that, on the contrary, (L^1,M_ϱ) does not have the universal right K property.

Recall that we are excluding the trivial case $M_\varrho=L^1$. Thus either $\lim_{t\to\infty}\varrho(t)=\infty$ or $\lim_{t\to 0}\varrho(t)=\varrho(0)=0$ (or both).

In the first case we take $f=1/[\max(1,(\log\varrho)\sqrt{\varrho})]$. Then we obtain $\int_0^\infty fg\,dx=\infty$ although $\int_0^\infty f^2\varrho'\,dx+\varrho(0)\|f\|^2_{L^\infty}$ is finite.

In the second case we take $f=(\sqrt{\varrho}\log(1/\varrho))^{-1}\chi_{(0,a]}$, where $a>0$ is sufficiently small to ensure that $\varrho(a)<1$ and also that $\sqrt{t}\log(1/t)$ is an increasing function of t for $0<t<\varrho(a)$. Thus f will be non-negative and non-increasing and, as before, $\int_0^\infty fg\,dx=\infty$ and $\int_0^\infty f^2\varrho'\,dx<\infty$. This shows that in all cases (3.3) cannot hold and completes our argument.

REFERENCES

1. J. Bergh, *On the relation between the two complex methods of interpolation*, Indiana Univ. Math. J. 28 (1979), 775–778.
2. J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, (Grundlehren Math. Wiss. 223), Springer-Verlag, Berlin - Heidelberg - New York, 1976.
3. Ju. A. Brudnyĭ and N. Ja. Krugljak, *Real interpolation functors*, Dokl. Akad. Nauk. SSSR 256 (1981), 14–17. ((Russian) = Soviet Math. Dokl. 23 (1981), 5–8.)
4. Ju. A. Brudnyĭ and N. Ja. Krugljak, *Real interpolation functors*, book manuscript, 1981 (Russian.)
5. A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), 113–190.
6. A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. 26 (1966), 273–299.
7. M. Cwikel, *Monotonicity properties of interpolation spaces*, Ark. Mat. 14 (1976), 213–236.
8. M. Cwikel, *Monotonicity properties of interpolation spaces* II, Ark. Mat. 19 (1981), 123–136.
9. M. Cwikel, *K-divisibility of the K-functional and Calderón couples*, Ark. Mat., 22 (1984), 39–62.
10. M. Cwikel and P. Nilsson, *Interpolation of weighted Banach lattices*, preprint.
11. M. Cwikel and J. Peetre, *Abstract K and J spaces*, J. Math. Pures Appl. 60 (1981), 1–50.
12. S. G. Krein, Ju. I. Petunin, and E. M. Semenov, *Interpolation of linear operators*, (Trans. Math. Monographs 54), American Mathematical Society, Providence, R.I., 1982.
13. B. S. Mitjagin, *An interpolation theorem for modular spaces*, Mat. Sb. 66 (1965), 472–482 (Russian).
14. P. Nilsson, *Reiteration theorems for real interpolation and approximation spaces*, Ann. Mat. Pura Appl. 132 (1982), 291–330.
15. V. I. Ovčinnikov, *On estimates of interpolation orbits*, Mat. Sb. 115 (1981), 642–652 (Russian).
16. J. Peetre, *Banach couples* I, Technical report, Lund, 1971.
17. J. Peetre, *Generalizing Ovčinnikov's theorem*, Technical report, Lund, 1981.
18. G. Sparr, *Interpolation of weighted L_p spaces*, Studia Math. 62 (1978), 229–271.

DEPARTMENT OF MATHEMATICS
TECHNION - ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA
ISRAEL

AND

MATEMATISKA INSTITUTIONEN
CHALMERS TEKNISKA HÖGSKOLA OCH GÖTEBORGS UNIVERSITET
41296 GÖTEBORG
SWEDEN