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FORMULAS FOR THE L^2-MINIMAL SOLUTIONS OF THE $\partial\bar{\partial}$-EQUATION IN THE UNIT BALL OF $\mathbf{C}^N$

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Abstract.

We construct explicit integral formulas for the $\partial\bar{\partial}$-equation in the unit ball of $\mathbf{C}^n$, which yield minimal solutions in certain L^2-spaces. We also show that our formulas give some known results about divisors of holomorphic functions.

Introduction.

The aim of this paper is to give explicit solution formulas for the equation

$$i\partial\bar{\partial}u = \theta \tag{1}$$

where θ is a closed (1,1)-current in the unit ball of $\mathbf{C}^n$. The equation (1) is of interest mainly because of its connection with divisors of holomorphic functions ([6] and [7]). With every divisor there is associated a positive (1,1)-current θ, and the solutions to (1) are precisely the functions $u=\log|f|$, where f is a function that defines the divisor. Thus, in [1], [4], [9], [10], and several other papers one has constructed solutions to (1) with different properties, related to holomorphic functions in various classes.

The method used in [4], [9], and [10] goes back to Lelong [7], and consists in solving (1) in two steps. First one solves

$$idw = \theta$$

where $w=w_{1,0}+w_{0,1}$ is a 1-current and $w_{1,0}$ and $w_{0,1}$ are its components of bidegree $(1,0)$ and $(0,1)$, respectively. Then one solves

$$\bar{\partial}U = w_{0,1}$$

and notes that

$$i\partial\bar{\partial}(U+\bar{U}) = i\partial w_{0,1}+i\bar{\partial}w_{1,0} = idw = \theta$$

if $\theta=\bar{\theta}$, and one chooses w so that $\bar{w}=-w$. Thus $u=U+\bar{U}$ is a solution to (1).

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Although this method has been very successful it is not quite natural when compared to the case $n=1$, where these two steps may be replaced by a single one. For instance, the Henkin-Skoda theorem, see [4] and [9], which characterizes those divisors which are defined by functions of the Nevanlinna class, is proved in the unit disc by forming the classical Blaschke product. If the zeros are $a_0, a_1, a_2, \ldots$ counted with multiplicities then the associated current is

$$\theta = c \sum_j \delta_{a_j} d\zeta \wedge d\bar{\zeta}$$

where δ_{a_j} denotes the Dirac measure at a_j. The Blaschke condition

$$\sum_j 1-|a_j| < \infty$$

of the zero-set ensures that the integral

$$u(z) = c \int_{|\zeta|<1} \log\left|\frac{\zeta - z}{1-\zeta\bar{z}}\right| \theta$$

is convergent. Moreover, u satisfies (1) and in fact equals $\log|B|$, where B is the corresponding Blaschke product.

In section 1 we define, for $\alpha \geqq -1$, solution operators $M_{n,\alpha}$ to (1) in the unit ball of $\mathbf{C}^n$. It turns out that $M_{n,\alpha}$ give the minimal solutions in certain weighted L^2-spaces when $\alpha \geqq 0$ (Theorem 1).

In section 2 we derive explicit integral formulas for the operators for integer values of α (Theorem 2).

Finally, in section 3 we estimate some solutions on the boundary (Theorems 3 and 4), and indicate how one from these estimates can obtain the Henkin-Skoda theorem and also a theorem of Varopoulos [10], concerning divisors of H^p-functions, in a way analogous to that of one variable. Of course we obtain these theorems only in the unit ball though they are true in any strictly pseudoconvex domain of $\mathbf{C}^n$ (satisfying a necessary topological condition).

I wish to thank Bo Berndtsson who introduced me to this subject and suggested the topic of this paper.

1.

The following notations and conventions are used throughout this paper. The unit ball in C^n is denoted by B_n or just B when there is no possibility of misunderstanding. In the same way ∂B_n or ∂B denotes the unit sphere and dS surface measure on ∂B. For $\zeta, z \in \mathbf{C}^n$ and a $(0,1)$-form $w = \sum w_i d\bar{\zeta}_i$ we write $\zeta \cdot \bar{z} = \sum \zeta_i \bar{z}_i$ and $w \cdot \bar{z} = \sum w_i \bar{z}_i$, hence $|\zeta|^2 = \zeta \cdot \bar{\zeta}$.

Moreover

$$\beta = \frac{i}{2}\partial\bar{\partial}|\zeta|^2 \quad \text{and} \quad \beta_k = \frac{1}{k!}\beta \wedge \ldots \wedge \beta \ (k \text{ times}) .$$

Thus β_n equals $d\lambda$, the $2n$-dimensional Lebesque measure. We denote by $C^\infty_{p,q}(\bar{B}_n)$ the set of (p,q)-forms with coefficients in $C^\infty(\bar{B}_n)$.

From now on we assume that θ is a closed form, $\theta \in C^\infty_{1,1}(\bar{B}_n)$. Let $w \in C^\infty_{0,1}(\bar{B}_n)$ satisfy

$$(1) \qquad i\partial w = \theta \quad \text{and} \quad \bar{\partial} w = 0 .$$

We can take $w = \bar{\partial} u$, where u is an arbitrary, smooth solution to

$$(2) \qquad i\partial\bar{\partial} u = \theta .$$

Conversely, any function u, satisfying

$$(3) \qquad \bar{\partial} u = w$$

must be a solution to (2).

The idea is to start with certain (known) solution formulas for (3), for a fixed w, and then modify them in such a way that the resulting solutions will be independent of the choice of w. Then we have obtained solution formulas for (2) and it will be possible to express the solutions in formulas without any occurrences of w.

There is a solution of (3) whose boundary values are given by

$$(4) \qquad u(z) = \frac{\Gamma(n+\alpha)}{(n-1)!\,\pi^n\Gamma(\alpha+1)}\left(\frac{i}{2}\right)^n \int_{B_n}\left(\frac{1-|\zeta|^2}{1-\bar{\zeta}\cdot z}\right)^{n+\alpha} d\bar{z}\cdot\zeta \wedge w \wedge \frac{\left(\partial\bar{\partial}\log\frac{1}{1-|\zeta|^2}\right)^{n-1}}{1-\zeta\cdot\bar{z}}$$

See [2, Section 2 Example 1].

LEMMA 1.

$$\frac{(1-|\zeta|^2)^n}{(n-1)!}\left(\frac{i}{2}\right)^n d\bar{z}\cdot\zeta \wedge w \wedge \left(\partial\bar{\partial}\log\frac{1}{1-|\zeta|^2}\right)^{n-1} = (w\cdot\bar{z}-(w\cdot\bar{\zeta})\zeta\cdot\bar{z})\,d\lambda$$

If we take this for granted for a while and note that

$$w\cdot\bar{z}-(w\cdot\bar{\zeta})\zeta\cdot\bar{z} = w\cdot\bar{z}-w\cdot\bar{\zeta}+(1-\zeta\cdot\bar{z})w\cdot\bar{\zeta}$$

we can write (4) as

$$u(z) = \frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha+1)}\int_{B_n}\frac{(1-|\zeta|^2)^\alpha(w\cdot\bar{z}-w\cdot\bar{\zeta})}{(1-\bar{\zeta}\cdot z)^{n+\alpha}(1-\zeta\cdot\bar{z})}d\lambda + \frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha+1)}\int_{B_n}\frac{(1-|\zeta|^2)^\alpha w\cdot\bar{\zeta}}{(1-\bar{\zeta}\cdot z)^{n+\alpha}}d\lambda$$

We denote these terms by $K_{n,\alpha}(w)$ and $R_{n,\alpha}(w)$, respectively. Since $R_{n,\alpha}(w)$ is holomorphic, $K_{n,\alpha}(w)$ is a solution operator to (3). $K_{n,1}$ was used by Skoda in [9] to get solutions with boundary values in $L^1(\partial B)$. For $\alpha > -1$, the formulas for $K_{n,\alpha}$ was found by Charpentier [3], and Berndtsson [1]. In the latter it is also shown that $K_{n,\alpha}$ can be analytically continued to get solutions for $\alpha > -n$.

If ψ is a continuous function on $\bar{B}_n$ then

$$(5) \qquad (\alpha+1)\int_{B_n}(1-|\zeta|^2)^\alpha\psi(\zeta)\,d\lambda(\zeta) \to \tfrac{1}{2}\int_{\partial B_n}\psi(\zeta)\,dS$$

when $\alpha \to -1$. Hence

$$(6) \qquad K_{n,-1}(w) = \frac{\Gamma(n-1)}{2\pi^n}\int_{\partial B_n}\frac{w\cdot\bar{z}-w\cdot\bar{\zeta}}{(1-\bar{\zeta}\cdot z)^{n-1}(1-\zeta\cdot\bar{z})}dS\,.$$

For $z \in B_n$ we set

$$(7)\ M_{n,\alpha}(\theta) = K_{n,\alpha}(w)-\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha+1)}\int_{B_n}\frac{(1-|\zeta|^2)^\alpha}{\zeta\cdot\bar{z}}\left(\left(\frac{1}{(1-\zeta\cdot\bar{z}}\right)^{n+\alpha}-1\right)w\cdot\bar{z}\,d\lambda$$

when $\alpha > -1$, and

$$(8) \qquad M_{n,-1}(\theta) = K_{n,-1}(w)-\sum_{j=1}^{n-1}\frac{\Gamma(n-1)}{2\pi^n}\int_{\partial B_n}\frac{w\cdot\bar{z}}{(1-\zeta\cdot\bar{z})^j}dS\,.$$

Note that when α is an integer

$$(9) \qquad M_{n,\alpha}(\theta) = K_{n,\alpha}(w)-\sum_{j=1}^{n+\alpha}\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha+1)}\int_{B_n}\frac{(1-|\zeta|^2)^\alpha w\cdot\bar{z}}{(1-\zeta\cdot\bar{z})^j}d\lambda\,.$$

It is clear that $M_{n,\alpha}(\theta)$ is the boundary values of a solution to (2) since we only have subtracted terms which are anti-holomorphic, hence pluriharmonic, functions.

We define $M_{n,\alpha}(\theta)(z)$ in B_n to be the unique solution to (2) with these boundary values.

The following theorem says that the $M_{n,\alpha}(\theta)$ are well-defined, i.e. independent of the choice of w, and it also states their main properties.

THEOREM 1. a) *The operators $M_{n,\alpha}$, $\alpha \geqq -1$, are well-defined for all closed $\theta \in C^\infty_{1,1}(\bar{B}_n)$ and satisfy*

$$(10) \qquad i\partial\bar{\partial}M_{n,\alpha}(\theta) = \theta\,.$$

Let L^2_α denote $L^2((1-|\zeta|^2)^{\alpha-1}d\lambda)$ for $\alpha > 0$ and $L^2(\partial B)$ for $\alpha = 0$.

b) *For $\alpha \geqq 0$, $M_{n,\alpha}(\theta)$ is the minimal solution in L^2_α.*

c) *For $\alpha \geqq 0$ and an arbitrary smooth function u we have the decomposition*

(11)
$$u = M_{n,\alpha}(i\partial\bar{\partial}u)+\Pi_{n,\alpha}(u)$$

where $\Pi_{n,\alpha}$ *is the orthogonal projection onto the subspace* $L^2_\alpha \cap \{u; u$ pluriharmonic$\}$.

(12)
$$\Pi_{n,\alpha}(u) = \frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}\int_{B_n}\left(\frac{1}{(1-\bar{\zeta}\cdot z)^{n+\alpha}}+\frac{1}{(1-\zeta\cdot\bar{z})^{n+\alpha}}-1\right)u(\zeta)(1-|\zeta|^2)^{\alpha-1}\,d\lambda$$

for $\alpha>0$ *and*

(13)
$$\Pi_{n,0}(u) = \frac{\Gamma(n)}{\pi^n}\int_{\partial B_n}\left(\frac{1}{(1-\bar{\zeta}\cdot z)^n}+\frac{1}{(1-\zeta\cdot\bar{z})^n}-1\right)u(\zeta)dS\,.$$

d) *For integer values of* α, $\alpha\geqq -1$, *the boundary values of* $M_{n,\alpha}(\theta)$ *are given by the explicit integral formula of Theorem 2 in section 2.*

e) *Let* θ *be a closed form on* $\bar{B}_n$ *and consider it as a form on* $\bar{B}_{n+1}$, *not depending on the last variable. Then* $M_{n+1,\alpha-1}(\theta)$ *does not depend on the last variable and*

(14)
$$M_{n,\alpha}(\theta)(z) = M_{n+1,\alpha-1}(\theta)(z,z_{n+1})\,.$$

REMARK. The formulas in Theorem 2 give the boundary values of $M_{n,\alpha}(\theta)$. Part e) of Theorem 1 means that $M_{n,\alpha}(\theta)$ can be computed in the interior of B_n by the equality

$$M_{n,\alpha}(\theta)(z) = M_{n+1,\alpha-1}(\theta)(z,\sqrt{1-|z|^2})\,.$$

ADDED IN PROOF. This computation is carried out in [11].

REMARK. By the method used in [1], $M_{n,\alpha}(\theta)$ may be analytically continued to $\alpha>-n$, and by uniqueness, a) and e) of Theorem 1 remain true for these α. In fact one can verify that the kernels constructed in [1] coincide with $M_{n,1-n}$.

For the proof of Theorem 1 we need some lemmas.

LEMMA 2. *Let* ψ *be a continuous function on* $\bar{B}_{n+1}$, *independent of the last variable* ζ_{n+1}. *Then*

(15)
$$(\alpha+1)\int_{B_{n+1}}(1-|\zeta|^2)^\alpha\psi(\zeta)\,d\lambda(\zeta) = \pi\int_{B_n}(1-|\zeta|^2)^{\alpha+1}\psi(\zeta)\,d\lambda(\zeta)$$

and

(16)
$$\tfrac{1}{2}\int_{\partial B_{n+1}}\psi(\zeta)\,dS = \pi\int_{B_n}\psi(\zeta)\,d\lambda(\zeta)\,.$$

PROOF. In fact (15) is an application of Fubini's theorem, and (16) follows when $\alpha \to -1$, by (5).

LEMMA 3. *Suppose $u \in C^\infty(\bar{B}_n)$ and $w = \bar\partial u$. Then*

$$K_{n,\alpha}(w) + R_{n,\alpha}(w) = u - P_{n,\alpha}(u)$$

for $\alpha > -1$, where

$$P_{n,\alpha}(u) = \frac{\Gamma(n+\alpha+1)}{\pi^n \Gamma(\alpha+1)} \int_{B_n} \frac{(1-|\zeta|^2)^\alpha u(\zeta)}{(1-\bar\zeta\cdot z)^{n+\alpha+1}}\, d\lambda . \tag{17}$$

PROOF. See [2, Section 2 Example 1]. The constant $\Gamma(n+\alpha+1)/\pi^n\Gamma(\alpha+1)$ is not explicitly stated but it can be determined in the following way. The kernel $P_{n,\alpha}$ reproduces holomorphic functions, in particular $u \equiv 1$. Since the equality $1 = P_{n,\alpha}(1)$ must hold for $z = 0$, one only has to observe that

$$\int_{B_n} (1-|\zeta|^2)^\alpha d\lambda = \frac{\pi}{\alpha+1} \int_{B_{n-1}} (1-|\zeta|^2)^{\alpha+1} d\lambda = \dots$$
$$= \frac{\pi^n}{(\alpha+1)(\alpha+2)\dots(\alpha+n)}$$

by Lemma 2.

LEMMA 4. *If $\bar\partial w = 0$ then for $\alpha > -1$,*

$$K_{n,\alpha+1}(w) = K_{n,\alpha}(w) + R_{n,\alpha}(w) .$$

This Lemma is just formula (16) in Section 2 of [1].

PROOF OF THEOREM 1. Let $\bar\partial u = w$ and $\alpha > 0$. Now

$$w\cdot\bar z d\lambda = \frac{i}{2} d\bar z\cdot\zeta \wedge \bar\partial u \wedge \beta_{n-1} .$$

Thus if we put

$$I = \int_B \frac{(1-|\zeta|^2)^\alpha}{\zeta\cdot\bar z}\left(\left(\frac{1}{1-\zeta\cdot\bar z}\right)^{n+\alpha} - 1\right) w\cdot\bar z\, d\lambda , \tag{18}$$

then

$$I = \frac{i}{2}\int_B \frac{(1-|\zeta|^2)^\alpha}{\zeta\cdot\bar z}\left(\left(\frac{1}{1-\zeta\cdot\bar z}\right)^{n+\alpha} - 1\right) d\bar z\cdot\zeta \wedge \bar\partial u \wedge \beta_{n-1} .$$

Now we integrate by parts (the boundary integral vanishes).

$$I = \frac{i}{2}\int_B u\bar{\partial}\frac{(1-|\zeta|^2)^\alpha}{\zeta\cdot\bar{z}}\left(\left(\frac{1}{1-\zeta\cdot\bar{z}}\right)^{n+\alpha}-1\right)\wedge d\bar{z}\cdot\zeta\wedge\beta_{n-1}$$
$$= \frac{i}{2}\alpha\int_B u\frac{(1-|\zeta|^2)^{\alpha-1}}{\zeta\cdot\bar{z}}\left(\left(\frac{1}{1-\zeta\cdot\bar{z}}\right)^{n+\alpha}-1\right)d\bar{z}\cdot\zeta\wedge\bar{\partial}|\zeta|^2\wedge\bar{\partial}\beta_{n-1}$$

since

$$\frac{1}{\zeta\cdot\bar{z}}\left(\left(\frac{1}{1-\zeta\cdot\bar{z}}\right)^{n+\alpha}-1\right)$$

is holomorphic in ζ. Finally we note that

$$\frac{i}{2}d\bar{z}\cdot\zeta\wedge\bar{\partial}|\zeta|^2\wedge\beta_{n-1} = \zeta\cdot\bar{z}d\lambda$$

so

$$I = \alpha\int_B (1-|\zeta|^2)^{\alpha-1}\left(\left(\frac{1}{1-\zeta\cdot\bar{z}}\right)^{n+\alpha}-1\right)u\,d\lambda\,. \tag{19}$$

From Lemmas 3 and 4 we have

$$K_{n,\alpha}(w) = u-\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}\int_B\frac{(1-|\zeta|^2)^{\alpha-1}}{(1-\bar{\zeta}\cdot z)^{n+\alpha}}u\,d\lambda\,. \tag{20}$$

If we recall the definition (7) of $M_{n,\alpha}(i\partial\bar{\partial}u)$ (with $\theta=i\partial\bar{\partial}u$ and $w=\bar{\partial}u$) and apply (18), (19), and (20) we obtain

$$M_{n,\alpha}(\theta) = u-\Pi_{n,\alpha}(u)\,. \tag{21}$$

To show that the definition (7) is independent of the choice of w, we suppose that w_1 and w_2 are two such choices. If

$$\bar{\partial}u = w_1-w_2\,,$$

then u must be pluriharmonic. Because of (21), it is enough to show that

$$u = \Pi_{n,\alpha}(u) \tag{22}$$

for pluriharmonic u.

Now

$$\Pi_{n,\alpha}(u)(z) = P_{n,\alpha-1}(u)(z)+\bar{P}_{n,\alpha-1}(u)(z)-P_{n,\alpha-1}(u)(0)$$

is real and by Lemma 3 and (17) one sees that

$$P_{n,\alpha-1}(f)(z) = f(z) \quad\text{and}\quad P_{n,\alpha-1}(\bar{f})(z) = \bar{f}(0)$$

for a holomorphic function f. From this (22) follows.

The independence of w for $0\geqq\alpha\geqq-1$ now follows since $M_{n,\alpha}(\theta)$ is analytic in α.

We have already noted that $i\partial\bar{\partial}M_{n,\alpha}(\theta)=\theta$ and thus part a) is completely proved.

Letting $\alpha \to 0$ in (21) and (22) it is clear that $\Pi_{n,\alpha}$ is a projection onto

$$L^2_\alpha \cap \{u;\ u \text{ is pluriharmonic}\}, \qquad \text{for } \alpha \geqq 0 .$$

Let $\langle \cdot, \cdot \rangle_\alpha$ denote the inner product of L^2_α. One can easily verify that

$$\langle \Pi_{n,\alpha}(u), v \rangle_\alpha = \langle u, \Pi_{n,\alpha}(v) \rangle_\alpha \qquad \text{for } u,v \in C^\infty(\bar{B}_n)$$

which means that $\Pi_{n,\alpha}$ is self-adjoint, hence the orthogonal projection. This proves c). As a direct consequence we obtain part b). Namely, among all solutions to $i\partial\bar{\partial}u=\theta$ it is clear from c) by the Pythagorean theorem that $M_{n,\alpha}(\theta)$ must be the one with minimal norm in L^2_α.

Since the content of d) is postponed to section 2, it only remains to prove e).

Let $i\partial\bar{\partial}u=\theta$ on $\bar{B}_n$ and consider θ and $w=\bar{\partial}u$ as forms on $\bar{B}_{n+1}$ not depending on ζ_{n+1}. From Lemma 2 and (7) it is clear that

$$M_{n+1,\alpha-1}(\theta)(z,0) = M_{n,\alpha}(\theta)(z) . \tag{23}$$

It is also easily seen from the definition that, for $z \in B_n$, $M_{n+1,\alpha-1}(\theta)(z, e^{it}\sqrt{1-|z|^2})$ is independent of t. Since $M_{n+1,\alpha-1}(\theta)(z, z_{n+1})$ is harmonic in z_{n+1}, it must be constant in z_{n+1}. Together with (23) this is the content of e).

REMARK. The only fact about the operators $P_{n,\alpha}$ and $K_{n,\alpha}$ we actually use in the proof above and for rest of this paper is the relation

$$K_{n,\alpha+1}(\bar{\partial}u) = u - P_{n,\alpha}(u) \tag{24}$$

(Lemmas 3 and 4), which for $z \in \partial B_n$ can be proved directly along the lines of Theorem 16.7.2 (1) in [8].

We end this section with the proof of Lemma 1. In fact

$$\left(\partial\bar{\partial}\log\frac{1}{1-|\zeta|^2}\right)^{n-1} = \left(\frac{\partial\bar{\partial}|\zeta|^2}{1-|\zeta|^2}+\frac{\partial|\zeta|^2 \wedge \bar{\partial}|\zeta|^2}{(1-|\zeta|^2)^2}\right)^{n-1}$$
$$= \frac{(1-|\zeta|^2)(\partial\bar{\partial}|\zeta|^2)^{n-1}}{(1-|\zeta|^2)^n}+(n-1)\frac{\partial|\zeta|^2 \wedge \bar{\partial}|\zeta|^2 \wedge (\partial\bar{\partial}|\zeta|^2)^{n-1}}{(1-|\zeta|^2)^n}$$

since the exterior product of a 1-form with itself is zero. Thus

$$\frac{(1-|\zeta|^2)^n}{(n-1)!}\left(\frac{i}{2}\right)^n d\bar{z}\cdot\zeta \wedge w \wedge \left(\partial\bar{\partial}\log\frac{1}{1-|\zeta|^2}\right)^{n-1}$$
$$= (1-|\zeta|^2)\frac{i}{2}d\bar{z}\cdot\zeta \wedge w \wedge \beta_{n-1}+\left(\frac{i}{2}\right)^2 d\bar{z}\cdot\zeta \wedge w \wedge \partial|\zeta|^2 \wedge \bar{\partial}|\zeta|^2 \wedge \beta_{n-2} \cdot$$

The first term of the right hand side is equal to

$$(1-|\zeta|^2)w\cdot\bar{z}d\lambda\,,$$

and by (1) of section 2, the second term equals

$$(w\cdot\bar{z})|\zeta|^2-(w\cdot\bar{\zeta})\zeta\cdot\bar{z}\,.$$

From this Lemma 1 follows.

2.

In this section we derive explicit integral integral formulas for $M_{n,\alpha}(\theta)$. It is convenient to introduce the notation

$$\mu = \frac{i}{2}d\bar{z}\cdot\zeta \wedge dz\cdot\bar{\zeta} = \frac{i}{2}\sum \bar{z}_i d\zeta_i \wedge \sum z_j d\bar{\zeta}_j$$

and

$$\gamma = \frac{i}{2}\partial|\zeta|^2 \wedge \bar{\partial}|\zeta|^2 = \frac{i}{2}\sum \bar{\zeta}_i d\zeta_i \wedge \sum \zeta_i d\bar{\zeta}_i\,.$$

When nothing else is said, we assume that $z \in \partial B_n$. At several occasions we shall make use of the equalities

$$\frac{i}{2}d\bar{z}\cdot\zeta \wedge w \wedge \gamma \wedge \beta_{n-2} = ((w\cdot\bar{z})|\zeta|^2-(w\cdot\bar{\zeta})\zeta\cdot\bar{z})\,d\lambda \tag{1}$$

and

$$\frac{i}{2}\partial|\zeta|^2 \wedge w \wedge \mu \wedge \beta_{n-2} = (w\cdot\bar{\zeta}-(w\cdot\bar{z})\bar{\zeta}\cdot z)d\lambda\,. \tag{2}$$

These can be deduced by direct computation or may be considered as special cases of Lemma 2.4 in [1].

Recall that for $\alpha > -1$

$$K_{n,\alpha}(w) = \frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha+1)}\int_{B_n}\frac{(1-|\zeta|^2)^\alpha(w\cdot\bar{z}-w\cdot\bar{\zeta})}{(1-\bar{\zeta}\cdot z)^{n+\alpha}(1-\zeta\cdot\bar{z})}d\lambda\,.$$

Now

$$w\cdot\bar{z}-w\cdot\bar{\zeta} = (1-\bar{\zeta}\cdot z)w\cdot\bar{z}+(\bar{\zeta}\cdot z)w\cdot\bar{z}-w\cdot\bar{\zeta}\,.$$

Hence by (2)

$K_{n,\alpha}(w) =$

$$\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha+1)}\left[\int_{B_n}\frac{(1-|\zeta|^2)^\alpha w\cdot\bar z}{(1-\bar\zeta\cdot z)^{n+\alpha-1}(1-\zeta\cdot\bar z)}d\lambda-\int_{B_n}\frac{\frac{i}{2}(1-|\zeta|^2)^\alpha\partial|\zeta|^2\wedge w\wedge\mu\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^{n+\alpha}(1-\zeta\cdot\bar z)}\right].$$

Denote the second integral inside the brackets by I. Then

$$I = -\frac{i}{2(\alpha+1)}\int_{B_n}\frac{\partial(1-|\zeta|^2)^{\alpha+1}\wedge w\wedge\mu\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^{n+\alpha}(1-\zeta\cdot\bar z)}.$$

Note that

$$\frac{\mu}{(1-\bar\zeta\cdot z)^{n+\alpha}(1-\zeta\cdot\bar z)}$$

is a closed form and that

$$i\partial w = \theta .$$

Hence by Stokes' theorem (the boundary integral vanishes)

$$I = \frac{1}{2(\alpha+1)}\int_{B_n}\frac{(1-|\zeta|^2)^{\alpha+1}\theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^{n+\alpha}(1-\zeta\cdot\bar z)}.$$

Thus we have proved,

(3)
$$K_{n,\alpha}(w) = \frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha+1)}\int_{B_n}\frac{(1-|\zeta|^2)^\alpha w\cdot\bar z}{(1-\bar\zeta\cdot z)^{n+\alpha-1}(1-\zeta\cdot\bar z)}d\lambda-$$
$$-\frac{\Gamma(n+\alpha)}{2\pi^n\Gamma(\alpha+2)}\int_{B_n}\frac{(1-|\zeta|^2)^{\alpha+1}\wedge\theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^{n+\alpha}(1-\zeta\cdot\bar z)}.$$

If $n=2$ and we let $\alpha\to-1$ we obtain

$$K_{2,-1}(w) = \frac{1}{2\pi^2}\int_{\partial B_2}\frac{w\cdot\bar z}{1-\zeta\cdot\bar z}dS-\frac{1}{2\pi^2}\int_{B_2}\frac{\theta\wedge\mu}{|1-\bar\zeta\cdot z|^2}.$$

Thus by the very definition ((8) section 1)

$$M_{2,-1}(\theta)(z) = -\frac{1}{2\pi^2}\int_{B_2}\frac{\theta\wedge\mu}{|1-\bar\zeta\cdot z|^2}.$$

When $n+\alpha>1$ we need the following lemma. The proof consists of elementary but tedious computations so we postpone it to the end of this section.

LEMMA 5. *Suppose* $\alpha>-1$, *and set*

$$A_{\alpha,j,k} = \int_{B_n}\frac{(1-|\zeta|^2)^\alpha w\cdot\bar z}{(1-\bar\zeta\cdot z)^j(1-\zeta\cdot\bar z)^k}d\lambda$$

Then the following equality holds if $1 \leqq j,\ k \leqq n+\alpha-1$.

$$A_{\alpha,j,k} = A_{\alpha,j-1,k} + \frac{k}{n+\alpha-k} A_{\alpha,j-1,k+1} +$$

$$+ \frac{n+\alpha}{2(\alpha+1)(n+\alpha-k)} \int_{B_n} \frac{(1-|\zeta|^2)^{\alpha+1}\theta \wedge \beta_{n-1}}{(1-\bar{\zeta}\cdot z)^j (1-\zeta\cdot\bar{z})^k} +$$

$$+ \frac{jk}{2(\alpha+1)(\alpha+2)(n+\alpha-k)} \int_{B_n} \frac{(1-|\zeta|^2)^{\alpha+2}\theta \wedge \mu \wedge \beta_{n-2}}{(1-\bar{\zeta}\cdot z)^{j+1}(1-\zeta\cdot\bar{z})^{k+1}} -$$

$$- \frac{k}{2(\alpha+1)(n+\alpha-k)} \int_{B_n} \frac{(1-|\zeta|^2)^{\alpha+1}\theta \wedge \mu \wedge \beta_{n-2}}{(1-\bar{\zeta}\cdot z)^j (1-\zeta\cdot\bar{z})^{k+1}} -$$

$$- \frac{1}{2(\alpha+1)} \int_{B_n} \frac{(1-|\zeta|^2)^{\alpha+1}\theta \wedge \mu \wedge \beta_{n-2}}{(1-\bar{\zeta}\cdot z)^j (1-\zeta\cdot\bar{z})^k} .$$

Now, if we let α be an integer, $\alpha \geqq 0$, and apply Lemma 5 $r-1$ times to $A_{\alpha,n+\alpha-1,1}$ we get

$$A_{\alpha,n+\alpha-1,1} = \sum_{s=1}^{r} C_{r,s} A_{\alpha,n+\alpha-r,s} + \text{terms not involving } w .$$

By induction over r one shows that

$$C_{r,s} = \frac{(r-1)(r-2)\dots(r-s+1)}{(n+\alpha-1)(n+\alpha-2)\dots(n+\alpha-s+1)} = \frac{(r-1)!(n+\alpha-s)!}{(n+\alpha-1)!(r-s)!} .$$

Thus after $n+\alpha-1$ steps we find that

$$A_{\alpha,n+\alpha-1,1} = \sum_{s=1}^{n+\alpha} A_{\alpha,0,s} + \text{terms not involving } w .$$

Comparing with (9) of section 2 and (3), it is clear that $M_{n,\alpha}(\theta)$ is constituted by all terms involving θ, which occur in these $n+\alpha-1$ steps (multiplied by $\Gamma(n+\alpha)/\pi^n\Gamma(\alpha+1)$) plus the term

$$-\frac{\Gamma(n+\alpha)}{2\pi^n\Gamma(\alpha+2)} \int_{B_n} \frac{(1-|\zeta|^2)^{\alpha+1} \wedge \theta \wedge \mu \wedge \beta_{n-2}}{(1-\bar{\zeta}\cdot z)^{n+\alpha}(1-\zeta\cdot\bar{z})}$$

from (3).

The case $\alpha = -1$ is handled in exactly the same way. One only has to multiply by $\alpha+1$ and let $\alpha \to -1$ in Lemma 5 before applying it.

Checking up the constants of the terms occurring in each step, we arrive at

THEOREM 2. *Suppose* $\alpha \geqq -1$ *and* α *is an integer. For* $z \in \partial B_n$ *we have*

$$M_{n,\alpha}(\theta)(z) = \frac{1}{2\pi^n}\Bigg[\sum_{\substack{j,k\geqq 1\\ j+k\leqq n+\alpha}} A_{j,k}\int_{B_n}\frac{(1-|\zeta|^2)^{\alpha+1}\theta\wedge\beta_{n-1}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^k}+$$

$$+\sum_{\substack{j,k\geqq 1\\ j+k\leqq n+\alpha}} B_{j,k}\int_{B_n}\frac{(1-|\zeta|^2)^{\alpha+2}\theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar{\zeta}\cdot z)^{j+1}(1-\zeta\cdot\bar{z})^{k+1}}-$$

$$-\sum_{\substack{j,k\geqq 1\\ j+k\leqq n+\alpha+1}} C_{j,k}\int_{B_n}\frac{(1-|\zeta|^2)^{\alpha+1}\theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^k}\Bigg],$$

where

$$A_{j,k} = \frac{(n+\alpha)(n+\alpha-j-1)!\,(n+\alpha-k-1)!}{(\alpha+1)!\,(n+\alpha-j-k)!},$$

$$B_{j,k} = \frac{jk(n+\alpha-j-1)!\,(n+\alpha-k-1)!}{(\alpha+2)!\,(n+\alpha-j-k)!}$$

and

$$C_{j,k} = \frac{(n+\alpha-j)!\,(n+\alpha-k)}{(\alpha+1)!\,(n+\alpha-j-k+1)!}.$$

It remains to prove Lemma 5. Suppose $1\leqq j,k\leqq n+\alpha-1$ and let

$$B_{\alpha,j,k} = \int_{B_n}\frac{(1-|\zeta|^2)^{\alpha}\bar{\zeta}\cdot w}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^k}\,d\lambda.$$

We claim that

$$B_{\alpha,j,k} = \frac{k}{\alpha+1}A_{\alpha+1,j,k+1}+\frac{1}{2(\alpha+1)}\int_B\frac{(1-|\zeta|^2)^{\alpha+1}\theta\wedge\beta_{n-1}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^k} \tag{4}$$

and

$$A_{\alpha+1,j,k+1} = \frac{\alpha+1}{n+\alpha}A_{\alpha,j-1,k+1}+\frac{\alpha+1}{n+\alpha}B_{\alpha,j,k}+ \tag{5}$$

$$+\frac{j}{2(\alpha+2)(n+\alpha)}\int_B\frac{(1-|\zeta|^2)^{\alpha+2}\theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar{\zeta}\cdot z)^{j+1}(1-\zeta\cdot\bar{z})^{k+1}}-$$

$$-\frac{1}{2(n+\alpha)}\int_B\frac{(1-|\zeta|^2)^{\alpha+1}\theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^{k+1}}.$$

To prove (4), note that

$$(\bar{\zeta}\cdot w)d\lambda = \frac{i}{2}\partial|\zeta|^2\wedge w\wedge\beta_{n-1}.$$

Hence

$$B_{\alpha,j,k} = -\frac{1}{\alpha+1}\int_B \frac{\frac{i}{2}\partial(1-|\zeta|^2)^{\alpha+1}\wedge w\wedge\beta_{n-1}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^k}.$$

By Stokes' theorem (recall that $i\partial w=\theta$)

$$B_{\alpha,j,k} = \frac{k}{\alpha+1}\int_B \frac{(1-|\zeta|^2)^{\alpha+1}\frac{i}{2}d\bar{z}\cdot\zeta\wedge w\wedge\beta_{n-1}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^{k+1}} + \\ +\frac{1}{2(\alpha+1)}\int_B \frac{(1-|\zeta|^2)^{\alpha+1}\theta\wedge\beta_{n-1}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^k},$$

which is the same as (4), since

$$\frac{i}{2}d\bar{z}\cdot\zeta\wedge w\wedge\beta_{n-1} = (w\cdot\bar{z})d\lambda.$$

It is somewhat more involved to see that (5) holds. Since

$$\beta_{n-1} = \frac{1}{n-1}\,\frac{i}{2}\partial\bar{\partial}|\zeta|^2\wedge\beta_{n-2}$$

we have

$$A_{\alpha+1,j,k+1} = \frac{1}{n-1}\left(\frac{i}{2}\right)^2\int_B \frac{(1-|\zeta|^2)^{\alpha+1}d\bar{z}\cdot\zeta\wedge w\wedge\partial\bar{\partial}|\zeta|^2\wedge\beta_{n-2}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^{k+1}}.$$

Now

$$\partial\frac{d\bar{z}\cdot\zeta}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^{k+1}} = 0,$$

so that Stokes' theorem gives

$$A_{\alpha+1,j,k+1} = \frac{\alpha+1}{n-1}\frac{i}{2}\int_B \frac{(1-|\zeta|^2)^{\alpha}\gamma\wedge d\bar{z}\cdot\zeta\wedge w\wedge\beta_{n-2}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^{k+1}} + \\ +\frac{i}{4(n-1)}\int_B \frac{(1-|\zeta|^2)^{\alpha+1}d\bar{z}\cdot\zeta\wedge\bar{\partial}|\zeta|^2\wedge\theta\wedge\beta_{n-2}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^{k+1}}.$$

If we apply (1) to the first term on the right hand side we get

$$A_{\alpha+1,j,k+1} = \frac{\alpha+1}{n-1}\int \frac{(1-|\zeta|^2)^{\alpha}((w\cdot\bar{z})|\zeta|^2-(w\cdot\bar{\zeta})\zeta\cdot\bar{z})}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^{k+1}}d\lambda + \\ +\frac{i}{4(n-1)(\alpha+2)}\int_B \frac{\bar{\partial}(1-|\zeta|^2)^{\alpha+2}\wedge d\bar{z}\cdot\zeta\wedge\theta\wedge\beta_{n-2}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^{k+1}}d\lambda.$$

Now we use the equality

$$(w\cdot\bar z)|\zeta|^2-(w\cdot\bar\zeta)\zeta\cdot\bar z = -(1-|\zeta|^2)w\cdot\bar z+(1-\bar\zeta\cdot z)w\cdot\bar z+ \\ +(1-\zeta\cdot\bar z)w\cdot\bar\zeta+(\bar\zeta\cdot z)w\cdot\bar z-w\cdot\bar\zeta$$

to the first term and apply Stokes' theorem to the second term and get

$$\text{(6)}\qquad A_{\alpha+1,j,k+1} = -\frac{\alpha+1}{n-1}A_{\alpha+1,j,k+1}+\frac{\alpha+1}{n-1}A_{\alpha,j-1,k+1}+\frac{\alpha+1}{n-1}B_{\alpha,j,k}+ \\ +\frac{\alpha+1}{n-1}\int_B\frac{(1-|\zeta|^2)^\alpha((\bar\zeta\cdot z)w\cdot\bar z-w\cdot\bar\zeta)}{(1-\bar\zeta\cdot z)^j(1-\zeta\cdot\bar z)^{k+1}}d\lambda+ \\ +\frac{1}{2(n-1)(\alpha+2)}\int_B\frac{(1-|\zeta|^2)^{\alpha+1}\theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^{j+1}(1-\zeta\cdot\bar z)^{k+1}}.$$

By (2)

$$\int_B\frac{(1-|\zeta|^2)^\alpha((\bar\zeta\cdot z)w\cdot\bar z-w\cdot\bar\zeta)}{(1-\bar\zeta\cdot z)^j(1-\zeta\cdot\bar z)^{k+1}}d\lambda = \frac{1}{\alpha+1}\int_B\frac{\frac{i}{2}\partial(1-|\zeta|^2)^{\alpha+1}\wedge w\wedge\mu\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^j(1-\zeta\cdot\bar z)^{k+1}}$$

and by Stokes' theorem as above one sees that the term on the right hand side is equal to

$$-\frac{1}{2(\alpha+1)}\int_B\frac{(1-|\zeta|^2)^{\alpha+1}\theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^j(1-\zeta\cdot\bar z)^{k+1}}.$$

After substituting this term in (6) and solving for $A_{\alpha+1,j,k+1}$ we obtain (5).

Now we substitute the expression of (5) for $A_{\alpha+1,j,k+1}$ in formula (4). In the resulting equation we solve for $B_{\alpha,j,k}$ and get

$$\text{(7)}\qquad B_{\alpha,j,k} = \frac{k}{n+\alpha-k}A_{\alpha,j-1,k+1}+ \\ +\frac{jk}{2(\alpha+1)(\alpha+2)(n+\alpha-k)}\int_B\frac{(1-|\zeta|^2)^{\alpha+2}\theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^{j+1}(1-\zeta\cdot\bar z)^{k+1}}- \\ -\frac{k}{2(n+\alpha-k)}\int_B\frac{(1-|\zeta|^2)^{\alpha+1}\theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^j(1-\zeta\cdot\bar z)^{k+1}}+ \\ +\frac{n+\alpha}{2(\alpha+1)(n+\alpha-k)}\int_B\frac{(1-|\zeta|^2)^{\alpha+1}\theta\wedge\beta_{n-1}}{(1-\bar\zeta\cdot z)^j(1-\zeta\cdot\bar z)^k}.$$

By the equality

$$w\cdot\bar z = w\cdot\bar\zeta+(1-\bar\zeta\cdot z)w\cdot\bar z-(w\cdot\bar\zeta-(\bar\zeta\cdot z)w\cdot\bar z),$$

formula (2) and Stokes' theorem we have

$$(8) \qquad A_{\alpha,j,k} = B_{\alpha,j,k} + A_{\alpha,j-1,k} - \frac{1}{2(\alpha+1)} \int_B \frac{(1-|\zeta|^2)^{\alpha+1}\theta \wedge \mu \wedge \beta_{n-2}}{(1-\bar{\zeta}\cdot z)^j (1-\zeta\cdot\bar{z})^k}.$$

Finally Lemma 5 is proved by combining (7) and (8).

3.

In this section we will indicate how the theorems of Henkin-Skoda and Varopoulos (see Introduction), follows from our results in the preceding sections. It is then convenient to let the kernels from section 2 operate on positive currents, and not only on smooth forms as has been the case up to now, so our first objective is to make this legitimate.

Let $M = M_{n,1}$ and L the corresponding kernel so that

$$M\theta = \int_B L \wedge \theta .$$

We then have

THEOREM 3. *Suppose θ is a positive, closed (1,1)-current in B which satisfies*

$$(1) \qquad \sup_{r<1} \int_{|\zeta|<r} (1-|\zeta|^2)\theta \wedge \beta_{n-1} < \infty .$$

We define $M\theta$ on ∂B by

$$M\theta = \lim_{r\to 1} \int_{|\zeta|<r} L \wedge \theta .$$

The limit exists in $L^1(\partial B)$ and we have

$$\int_{\partial B} |M\theta| \leqq \text{const} \sup_{r<1} \int_{|\zeta|<r} (1-|\zeta|^2)\theta \wedge \beta_{n-1} .$$

Moreover, there is a solution U to

$$(2) \qquad i\partial\bar{\partial}U = \theta$$

which has boundary values $M\theta$, in the sense that

$$U_r \to M\theta$$

in $L^1(\partial B)$ when $r \to 1$, where

$$U_r(z) = U(rz) .$$

If $G(\zeta, z)$ and $P(\zeta, z)$ are the Green's function and the Poisson kernel, respectively, then U is explicitely given by

$$U(z) = -2\int_B G(\zeta,z)\theta \wedge \beta_{n-1} + \int_{\partial B} P(\zeta,z) M\theta(\zeta)\, dS . \tag{3}$$

As will be clear from the proof, any $M_{n,\alpha}$, $\alpha \geqq 1$, would work.

The Henkin-Skoda theorem states that a divisor D in B is defined by a function in the Nevanlinna class if and only if the associated (1,1)-current θ (see [6] and [7]) satisfies the Blaschke condition (1).

The Nevanlinna class N, is defined as the set of holomorphic functions in B such that

$$\sup_{r<1} \int_{\partial B} \log^+ |f(rz)|\, dS < \infty . \tag{4}$$

If $f \in N$ is given, the current θ, associated with the divisor D defined by f, is given by

$$\theta = i\partial\bar{\partial} \log |f| . \tag{5}$$

By Jensen's formula one sees that (4) implies (1), i.e. the "only if"-part of the theorem. In the other direction, which is the hard one, the theorem follows from Theorem 3 in the following way:

Let D and θ be given, and let g be any holomorphic function defining D, i.e. satisfying (5). Hence if U is the solution of (2), given by Theorem 3, then U-$\log|g|$ is pluriharmonic and so

$$U - \log|g| = \operatorname{Re} h$$

for some holomorphic function h in B. Now

$$f = e^h g$$

also defines D and

$$\log|f| = U .$$

Thus

$$\lim_{r\to 1} \int_{\partial B} \log^+ |f(rz)|\, dS \leqq \lim_{r\to 1} \int_{\partial B} |U_r|\, dS$$
$$= \int_{\partial B} |M\theta|\, dS \leqq \text{const} \sup_{r<1} \int_{|\zeta|<r} (1-|\zeta|^2)\theta \wedge \beta_{n-1} < \infty$$

so that $f \in N$. In fact

$$\log|f_r| \to \log|f^*| = M\theta$$

in $L^1(\partial B)$, that is f belongs to the subclass N^* of N.

Again, let D and θ be given. The theorem of Varopoulos states that if

$$(1-|\zeta|^2)\theta \wedge \beta_{n-1} + \gamma \wedge \theta \wedge \beta_{n-2}$$

is a Carleson measure, then there is, for some $p>0$, a function f in H^p that defines D.

A measure $d\tau$ in B is Carleson if

$$\int_{|1-\zeta\cdot z|\leqq t} d\tau(\zeta) \leqq Ct^n \tag{6}$$

for $z \in \partial B$ and $t>0$.

To prove the theorem we need

THEOREM 4. *If θ is as in Theorem 3 and if in addition*

$$(1-|\zeta|^2)\theta \wedge \beta_{n-1} + \gamma \wedge \theta \wedge \beta_{n-2}$$

is a Carleson measure, then

$$\exp(M\theta) \in L^p(\partial B) \tag{7}$$

for some $p>0$.

As above, let f be a holomorphic function such that

$$\log|f| = U\,.$$

Since

$$\int_B G\theta \wedge \beta_{n-1} \geqq 0$$

we have from (3),

$$\log|f(z)| \leqq \int_{\partial B} P(\zeta, z) M\theta(\zeta)\, dS\,.$$

By Jensen's inequality

$$|f(z)|^p \leqq \int_{\partial B} P(\zeta, z) \exp(p\, M\theta(\zeta))\, dS$$

and hence

$$\int_{\partial B} |f_r|^p \, dS \leqq \int_{\partial B} \exp(pM\theta(\zeta))\, dS . \tag{8}$$

Since the last integral is finite by (7), (8) means that $f \in H^p$.

PROOF OF THEOREM 3. Consider the expression for $L \wedge \theta$ given by Theorem 2. Since

$$\theta \wedge \beta_{n-1} \quad \text{and} \quad \theta \wedge \mu \wedge \beta_{n-2}$$

are positive forms and

$$\tfrac{1}{2}(1-|\zeta|^2) \leqq |1-\bar{\zeta}\cdot z| \leqq 2$$

for $\zeta \in B$ and $z \in \partial B$, the modulus of any term in $L \wedge \theta$ is up to a multiplicative constant majorized by

$$\frac{(1-|\zeta|^2)^2 \theta \wedge \beta_{n-1}}{|1-\bar{\zeta}\cdot z|^{n+1}} \quad \text{or} \quad \frac{(1-|\zeta|^2)^2 \theta \wedge \mu \wedge \beta_{n-2}}{|1-\bar{\zeta}\cdot z|^{n+2}} .$$

We claim that the inequality

$$\mu \wedge \theta \wedge \beta_{n-2} \leqq \text{const}\, (|1-\bar{\zeta}\cdot z| \theta \wedge \beta_{n-1} + \theta \wedge \gamma \wedge \beta_{n-2}) \tag{9}$$

holds. Recall that γ denotes the positive form

$$\frac{i}{2}\, \partial |\zeta|^2 \wedge \bar{\partial} |\zeta|^2 .$$

Taking this claim for granted for a while, we get the estimate

$$|L \wedge \theta| \leqq \text{const} \frac{(1-|\zeta|^2)^2 \theta \wedge \beta_{n-1}}{|1-\bar{\zeta}\cdot z|^{n+1}} + \text{const} \frac{(1-|\zeta|^2)^2 \theta \wedge \gamma \wedge \beta_{n-2}}{|1-\bar{\zeta}\cdot z|^{n+2}} . \tag{10}$$

We need the estimate

$$\int_{z \in \partial B_n} \frac{dS}{|1-\bar{\zeta}\cdot z|^{n+\alpha}} \leqq C \frac{1}{(1-|\zeta|^2)^{\alpha}} \tag{11}$$

where $\alpha > 0$ and the positive constant C only depends on α (see [8, Proposition 1.4.10]). Now, let $r < r' < 1$, and set

$$I = \left| \int_{|\zeta|<r'} L \wedge \theta - \int_{|\zeta|<r} L \wedge \theta \right| .$$

By (10) we get

$$I \leqq \int_{r \leqq |\zeta| < r'} |L \wedge \theta| \leqq \text{const} \int_{r \leqq |\zeta| < r'} \frac{(1-|\zeta|^2)^2 \theta \wedge \beta_{n-1}}{|1-\bar{\zeta}\cdot z|^{n+1}} +$$

$$+\text{const} \int_{r \leqq |\zeta| < r'} \frac{(1-|\zeta|^2)^2 \theta \wedge \gamma \wedge \beta_{n-2}}{|1-\bar{\zeta} \cdot z|^{n+2}}$$

and hence by (11) and Fubini's theorem

$$\text{(12)} \qquad \int_{\partial B} I\, dS \leqq \text{const} \int_{r \leqq |\zeta| < r'} (1-|\zeta|^2)\theta \wedge \beta_{n-1} + \\ + \text{const} \int_{r \leqq |\zeta| < r'} \theta \wedge \gamma \wedge \beta_{n-2}\,.$$

We also need

$$\text{(13)} \qquad (n-1) \int_{|\zeta| < 1} (1-|\zeta|^2)\theta \wedge \beta_{n-1} = \int_{|\zeta| < 1} \theta \wedge \gamma \wedge \beta_{n-2}\,.$$

This result goes back to Malliavin (see [9, Proposition II.2.1]).

By assumption and (13), $(1-|\zeta|^2)\theta \wedge \beta_{n-1}$ and $\theta \wedge \gamma \wedge \beta_{n-2}$ are finite measures, and hence (12) shows that

$$\int_{\partial B} I\, dS \to 0$$

when $r \to 1$, that is

$$\int_{|\zeta| < r} L \wedge \theta$$

converges in $L^1(\partial B)$ to a function $M\theta$, and by similar estimates as above it follows that

$$\int_{\partial B} |M\theta|\, dS \leqq \text{const} \int_B (1-|\zeta|^2)\theta \wedge \beta_{n-1}\,.$$

Thus the first part of Theorem 3 is proved if we can show the inequality (9).

To this end, put

$$a = \sum a_i d\zeta_i \quad \text{and} \quad b = \sum b_i d\zeta_i\,.$$

Since $\theta \wedge \beta_{n-2}$ is a positive form,

$$\langle a, b \rangle = \frac{i}{2} a \wedge b \wedge \theta \wedge \beta_{n-2}$$

is a positively (semi-) definite inner product. In particular

$$\tfrac{1}{2}\langle a+b, a+b \rangle \leqq \langle a, a \rangle + \langle b, b \rangle\,.$$

Applying this to

$$a = \sum (\bar{z}_i - \bar{\zeta}_i) d\zeta_i \quad \text{and} \quad b = \sum \bar{\zeta}_i d\zeta_i$$

we get

$$(14) \quad \tfrac{1}{2}\mu \wedge \theta \wedge \beta_{n-2} \leqq \frac{i}{2}\partial|\zeta - z|^2 \wedge \bar{\partial}|\zeta - z|^2 \wedge \theta \wedge \beta_{n-2} + \gamma \wedge \theta \wedge \beta_{n-2} .$$

Moreover,

$$(15) \qquad \partial|\zeta - z|^2 \wedge \bar{\partial}|\zeta - z|^2 \wedge \theta \wedge \beta_{n-2} \leqq \text{const}\, |\zeta - z|^2 \theta \wedge \beta_{n-1}$$

and

$$(16) \qquad |\zeta - z|^2 \leqq 4|1 - \bar{\zeta} \cdot z|$$

From (14), (15), and (16) we obtain (9).

To prove the second part of Theorem 3 we need the following well-known result from potential theory.

LEMMA 6. *If $g \in L^1(\partial B)$, $d\tau$ is a positive measure in B satisfying*

$$\int_B (1 - |\zeta|^2)\, d\tau(\zeta) < \infty ,$$

and

$$V(z) = -\int_B G(\zeta, z)\, d\tau(\zeta) + \int_{\partial B} P(\zeta, z) f(\zeta)\, dS$$

then

$$V_r \to f$$

in $L^1(\partial B)$ when $r \to 1$, and

$$\Delta V = \mu$$

in B.

Now, we define U by

$$(17) \qquad U(z) = -2\int_B G(\zeta, z)\theta \wedge \beta_{n-1} + \int_{\partial B} P(\zeta, z) M\theta(\zeta)\, dS .$$

In view of Lemma 6, Theorem 3 is completely proved when we have affirmed that

$$(18) \qquad i\partial\bar{\partial} U = \theta .$$

First we note that (18) holds if θ is smooth. Namely, in this case the is a solution V with boundary values $M\theta$, according to Theorem 1. Since

$$\theta \wedge \beta_{n-1} = 2 \sum_j \theta_{jj} d\lambda$$

it follows that both of U and V solve

$$\Delta U = 4 \sum_j \theta_{jj}$$

and have boundary values $M\theta$, so that

$$V = U ,$$

i.e. (18) holds.

Next, we suppose that θ is a current defined in a neighbourhood of $\bar{B}$. Let φ be a function in $C_0^\infty(B)$, which only depends on $|\zeta|$, and such that

$$0 \leqq \varphi \leqq 1 \quad \text{and} \quad \int \varphi \, d\lambda = 1 .$$

Put

$$\varphi_\varepsilon(\zeta) = \varepsilon^{-2n} \varphi(\zeta/\varepsilon)$$

and

$$\theta_\varepsilon = \theta * \varphi_\varepsilon .$$

Then θ_ε is positive, closed and smooth in a neighbourhood of $\bar{B}$.

Claim I. $M\theta_\varepsilon \to M\theta$ in $L^1(\partial B)$ when $\varepsilon \to 0$.

In fact, for $s<1$ we have

$$(19) \quad |M\theta_\varepsilon - M\theta| \leqq \left| \int_{|\zeta|<s} L \wedge (\theta_\varepsilon - \theta) \right| + \int_{s \leqq |\zeta| < 1} |L \wedge \theta_\varepsilon| + \int_{s \leqq |\zeta| < 1} |L \wedge \theta| .$$

By (10), (11) and Fubini's theorem we have

$$(20) \quad \int_{\partial B} \int_{s \leqq |\zeta| < 1} |L \wedge \theta_\varepsilon| \, dS \leqq$$

$$\leqq \text{const} \int_{s \leqq |\zeta| < 1} (1-|\zeta|^2) \theta_\varepsilon \wedge \beta_{n-1} + \text{const} \int_{s \leqq |\zeta| < 1} \theta_\varepsilon \wedge \gamma \wedge \beta_{n-2} .$$

Using (13) and the analogue equality, where B is replaced by $\{\zeta, |\zeta|<s\}$, we get

$$(21) \quad \int_{s \leqq |\zeta| < 1} \theta_\varepsilon \wedge \gamma \wedge \beta_{n-2} =$$

$$= (n-1) \int_{s \leqq |\zeta| < 1} (1-|\zeta|^2) \theta_\varepsilon \wedge \beta_{n-1} + (n-1) \int_{|\zeta|<s} (1-s^2) \theta_\varepsilon \wedge \beta_{n-1}$$

and combining (20) and (21),

$$(22)\quad \int_{\partial B}\int_{s\leqq|\zeta|<1}|L\wedge\theta_\varepsilon|\,dS \leqq$$

$$\leqq \text{const}\int_{s\leqq|\zeta|<1}(1-|\zeta|^2)\theta_\varepsilon\wedge\beta_{n-1}+\text{const}\int_{|\zeta|<s}(1-s^2)\theta_\varepsilon\wedge\beta_{n-1}\,.$$

One sees that both of the integrals on the right hand side of (22) are arbitrarily small, uniformly in ε, if $\varepsilon<1-s$ and $1-s$ is chosen small enough.

Now, consider the right hand side of (19). We have just seen that s can be chosen such that the middle and (similarly) the last terms have arbitrarily small $L^1(\partial B)$-norms. When $s<1$ is fixed, the first term tends to zero uniformly on ∂B when ε tends to zero, and this implies that

$$\lim_{\varepsilon\to 0}\int_{\partial B}|M\theta_\varepsilon-M\theta|\,dS = 0\,.$$

Claim II.

$$\int_{|\zeta|<1}G\theta_\varepsilon\wedge\beta_{n-1}\to\int_{|\zeta|<1}G\theta\wedge\beta_{n-1}$$

weakly in B.

In fact, if $\psi\in C_0(B)$ then

$$\int\psi(z)\,d\lambda(z)\int_{|\zeta|<1}G(\zeta,z)\theta_\varepsilon\wedge\beta_{n-1} = \int_{|\zeta|<1}h(\zeta)\theta_\varepsilon\wedge\beta_{n-1}$$

where

$$h(\zeta) = O(1-|\zeta|^2)$$

by the properties of the Green's function. Since $\theta_\varepsilon\to\theta$ weakly in a neighbourhood of $\bar{B}$, it follows that

$$\int_{|\zeta|<1}h(\zeta)\theta_\varepsilon\wedge\beta_{n-1}\to\int_{|\zeta|<1}h(\zeta)\theta\wedge\beta_{n-1}$$

which proves Claim II.

We note that Claim I implies that

$$\int_{\partial B}P(\zeta,z)M\theta_\varepsilon(\zeta)\,dS\to\int_{\partial B}P(\zeta,z)M\theta(\zeta)\,dS$$

in $L^1(B)$. Combining this with Claim II and recalling the definition (17) of U and U_ε (where U_ε corresponds to θ_ε) we conclude that

$$U_\varepsilon \to U$$

weakly in B. Since U_ε is smooth, we know that

$$i\partial\bar{\partial}U_\varepsilon = \theta_\varepsilon$$

and by continuity

(23)
$$i\partial\bar{\partial}U = \theta$$

in the case when θ was defined in a neighbourhood of $\bar{B}$. To obtain the general case, set for $t<1$,

$$\theta_t(\zeta) = t^{2n}\theta(t\zeta).$$

Hence, θ_t is defined in a neighbourhood of $\bar{B}$ and in a similar way, as was the case with θ_ε, one proves that U_t tends to U weakly when t increases to one, and thus confirms (23) for a general θ. As was noted before this completes the proof of Theorem 3.

Proof of Theorem 4. In [10], Varopoulos finds a solution which has boundary values u in BMO (∂B) if θ satisfies the hypothesis of Theorem 4. Then it follows from the John-Nirenberg theorem [5] that $\exp u \in L^p(\partial B)$ for some $p>0$.

Here we give a direct proof of Theorem 4. From (10) and the hypothesis we have that

$$|L \wedge \theta| \leqq \text{const}\frac{1-|\zeta|^2}{|1-\bar{\zeta}\cdot z|^{n+1}}d\tau(\zeta)$$

where τ is a Carleson measure. Put

$$u(z) = \int_B \frac{1-|\zeta|^2}{|1-\bar{\zeta}\cdot z|^{n+1}}d\tau(\zeta).$$

It is enough to show that

(24)
$$\exp u \in L^p(\partial B)$$

for some $p>0$. Set

$$E_s = \{z \in \partial B\ ;\ u(z)>s\}.$$

To show (24), we want to estimate $|E_s|$ (where $|\cdot|$ denotes normalized surface measure). Now

(25)
$$s|E_s| \leqq \int_{E_s} u(z)\,dS$$

and by Fubini's theorem

$$\int_{E_s} u(z)\,dS = \int_B \psi_{E_s}(\zeta)\,d\tau(\zeta) \tag{26}$$

where we, for any measurable set $E \subset \partial B$, define $\psi_E(\zeta)$ on B by

$$\psi_E(\zeta) = \int_E \frac{1-|\zeta|^2}{|1-\bar{\zeta}\cdot z|^{n+1}}\,dS\,.$$

From (11), we immediately see that

$$\psi_E(\zeta) \leqq \psi_{\partial B}(\zeta) \leqq C\,. \tag{27}$$

We also have

LEMMA 7. *If τ is a Carleson measure in B then there is a positive constant C such that*

$$\tau\{\zeta;\, \psi_E(\zeta) > \alpha\} \leqq C\frac{1}{\alpha}|E|$$

for any (measurable) set $E \subset \partial B$.

By Lemma 7 and (27) we get

$$\begin{aligned}\int_B \psi_{E_s}(\zeta)\,d\tau(\zeta) &= \int_0^\infty \tau\{\psi_{E_s} > \alpha\}\,d\alpha \\ &\leqq C\int_{|E_s|}^{C} \frac{1}{\alpha}|E_s|\,d\alpha + \tau(B)\int_0^{|E_s|} d\alpha \leqq C|E_s|\log\frac{1}{|E_s|}\,.\end{aligned} \tag{28}$$

From (25), (26), and (28)

$$s|E_s| \leqq C|E_s|\log\frac{1}{|E_s|}$$

i.e.

$$|E_s| \leqq e^{-s/C}$$

Finally,

$$\int_{\partial B} e^{pu(z)}\,dS = p\int_0^\infty |E_s|e^{ps}\,ds + \int_{\partial B} dS$$

and thus (24) holds if $p < 1/C$.

PROOF OF LEMMA 7. For any $\zeta \in B$, set

$$R_\zeta = \left\{ w \in B \; ; \; \left| 1 - \frac{\bar{w} \cdot \zeta}{|\zeta|} \right| < 2(1-|\zeta|) \right\}.$$

Note that $\zeta \in R_\zeta$. Furthermore, set

$$K_\zeta = \partial R_\zeta \cap \partial B .$$

If E is any measurable subset of ∂B, we define the maximal function of E, on ∂B by

$$M_E(z) = \sup_{t>0} t^{-n} |\{x \in \partial B \; ; \; |1-\bar{x}\cdot z| < t\} \cap E| .$$

We also need the following: (Proposition 5.1.2 in [8])

$$|2 - \bar{a}\cdot c|^{\frac{1}{2}} \leqq |1-\bar{a}\cdot b|^{\frac{1}{2}} + |1-\bar{b}\cdot c|^{\frac{1}{2}}, \quad a, b, c \in \bar{B} . \tag{29}$$

Our first objective is to show that there is a constant $c>0$, such that if $\psi_E(\zeta) > \alpha$, then $M_E > \alpha/c$ on K_ζ.

Integration by parts gives

$$\begin{aligned} \psi_E(\zeta) &= (1-|\zeta|^2) \int_E \frac{dS}{|1-\bar{\zeta}\cdot z|^{n+1}} \\ &\leqq \text{const}\,(1-|\zeta|) \int_0^\infty |\{z ; |1-\bar{\zeta}\cdot z| < s\} \cap E| \frac{ds}{s^{n+2}} . \end{aligned} \tag{30}$$

Suppose $x \in K_\zeta$. We claim that

$$\{z \in \partial B \; ; \; |1-\bar{\zeta}\cdot z| < s\} \subset \{z \in \partial B \; ; \; |1-\bar{x}\cdot z| < 12s\} . \tag{31}$$

Obviously,

$$\{z \in \partial B \; ; \; |1-\bar{\zeta}\cdot z| < s\} = \varnothing \quad \text{if } s < 1-|\zeta| . \tag{32}$$

Otherwise; if $s \geqq 1-|\zeta|$ and $|1-\bar{\zeta}\cdot z| < s$, then by (29),

$$\begin{aligned} |1-\bar{x}\cdot z| &\leqq 3\left(\left| 1 - \frac{\bar{x}\cdot\zeta}{|\zeta|} \right| + \left| 1 - \frac{\bar{\zeta}\cdot\zeta}{|\zeta|} \right| + |1-\bar{\zeta}\cdot z| \right) \\ &< 3(2(1-|\zeta|) + (1-|\zeta|) + s) < 12s . \end{aligned}$$

Thus (31) is established.

From (30), (31) and (32) we get

$$\psi_E(\zeta) \leqq \text{const}\,(1-|\zeta|) \int_{1-|\zeta|}^\infty |\{z ; |1-\bar{x}\cdot z| < 12s\} \cap E| \frac{ds}{s^{n+2}}$$

and hence by the definition of $M_E(x)$,

$$\psi_E(\zeta) \leqq \text{const}\, M_E(x)(1-|\zeta|) \int_{1-|\zeta|}^{\infty} \frac{ds}{s^2} = cM_E(x)\,.$$

Since x was arbitrary in K_ζ, we have shown

$$K_\zeta \subset \{z \in \partial B\ ;\ M_E(z) > \alpha/c\}\,. \tag{33}$$

The set $\{\psi_E > \alpha\}$ is covered by the union of all sets R_ζ such that $\psi_E(\zeta) > \alpha$. By a covering lemma very similar to Lemma 5.2.3 in [8], there is a disjoint subsequence

$$R_{\zeta_1}, R_{\zeta_2}, \ldots, R_{\zeta_k}, \ldots$$

such that

$$\{\psi_E > \alpha\} \subset \bigcup \tilde{R}_{\zeta_i}$$

where

$$\tilde{R}_{\zeta_i} = \left\{ w \in B\ ;\ \left|1 - \frac{\bar{w}\cdot\zeta_i}{|\zeta_i|}\right| < 36(1-|\zeta_i|) \right\}.$$

Similarly,

$$\tilde{K}_{\zeta_i} = \partial \tilde{R}_{\zeta_i} \cap \partial B\,.$$

Note that there are positive constants c_1 and c_2, such that

$$c_1 t^n \leqq \{x \in \partial B\ ;\ |1 - \bar{x}\cdot z| < t\} \leqq c_2 t^n\,.$$

Thus

$$|\tilde{K}_{\zeta_i}| \leqq \text{const}\, |K_{\zeta_i}|\,. \tag{34}$$

Note that the Carleson condition on τ implies that

$$\tau(\tilde{R}_{\zeta_i}) \leqq \text{const}\, |\tilde{K}_{\zeta_i}|\,. \tag{35}$$

From (34), (35) and the disjointness we have

$$\begin{aligned} \tau(\{\psi_E > \alpha\}) &\leqq \tau(\cup \tilde{R}_{\zeta_i}) \leqq \sum \tau(\tilde{R}_{\zeta_i}) \\ &\leqq \text{const} \sum |\tilde{K}_{\zeta_i}| \leqq \text{const} \sum |K_{\zeta_i}| = \text{const}\, |\cup K_{\zeta_i}|\,. \end{aligned} \tag{36}$$

By (33) and (36) we obtain

$$\tau(\{\psi_E > \alpha\}) \leqq \text{const} \left\{ z \in \partial B\ ;\ M_E(z) > \frac{\alpha}{c} \right\}. \tag{37}$$

Finally, according to Theorem 5.2.4 in [8], the maximal function M is of weak type (1,1), i.e.

$$|\{M_E > \alpha\}| \leqq \text{const}\,\frac{1}{\alpha}|E|$$

and from this and inequality (37), Lemma 7 follows.

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