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DERIVATIONS, DYNAMICAL SYSTEMS, AND SPECTRAL RESTRICTIONS

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Abstract.

Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system with G locally compact abelian and consider a closed *-derivation δ commuting with α. If $\mathfrak{A}$ is α-prime and $\hat{G}/\Gamma(\alpha)$ compact, then $D(\delta)$ contains the spectral subspaces $\mathfrak{A}^\alpha(K)$ of α corresponding to compacts $K \subset \hat{G}$ if, and only if, δ generates a bounded perturbation of a one-parameter subgroup ϱ of α_G. Alternatively if $\mathfrak{A}$ is abelian and $G = \mathbf{R}$ the spectral condition $D(\delta) \supseteq \mathfrak{A}^\alpha(K)$ implies that δ generates a group β obtained from α by a rescaling of the corresponding flow.

1. Introduction.

In this paper we consider a C*-dynamical system $(\mathfrak{A}, G, \alpha)$, consisting of a C*-algebra $\mathfrak{A}$ and a continuous action α of a locally compact abelian group G as *-automorphisms of $\mathfrak{A}$, together with a closed symmetric derivation δ of $\mathfrak{A}$ which commutes with the action α of G. If the domain $D(\delta)$ of δ contains the spectral subspaces $\mathfrak{A}^\alpha(K)$ of α corresponding to compact subsets K of the dual $\hat{G}$ of G, then δ is automatically the generator of a strongly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$. This is a consequence of two standard observations and a positivity argument developed in [3]. First one remarks that the restriction δ_K of δ to $\mathfrak{A}^\alpha(K)$ is a closed everywhere defined, hence bounded, operator which generates a uniformly continuous one-parameter group β^K on $\mathfrak{A}^\alpha(K)$. Second by differentiation of the function

$$t \mapsto \beta_{-t}^{K-K}(\beta_t^K(A)^*\beta_t^K(A)) ,$$

where $A \in \mathfrak{A}^\alpha(K)$, and exploitation of the derivation property one deduces that

$$0 \leqq \beta_t^K(A)^*\beta_t^K(A) = \beta_t^{K-K}(A^*A) .$$

Third, one uses the regularization and positivity argument of [3, Lemma 1.8], together with the norm density of $\{\mathfrak{A}^\alpha(K)\}_{K \subseteq \hat{G}}$, to conclude that β^K are the restriction to $\mathfrak{A}^\alpha(K)$ of a one-parameter group of *-automorphisms β of $\mathfrak{A}$.

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Our main result establishes that if $\mathfrak{A}$ is α-prime and the quotient of $\hat{G}$ by the Connes spectrum $\Gamma(\alpha)$ is compact, then the group constructed in the above manner is in fact a bounded perturbation of a one-parameter subgroup ϱ of α_G. Alternatively if $\mathfrak{A}$ is abelian and $G = \mathbf{R}$, then β is obtained from α by a rescaling of the corresponding flow, where the scale depends on the orbit. We also construct an example with $\mathfrak{A}$ non-abelian showing the restriction on $\Gamma(\alpha)$ is essential for our conclusion.

Finally we note that there have been many recent investigations of similar questions for compact abelian G (see for example [3], [4], and the references contained therein). Moreover Davies has recently shown [5] that the spectral condition $D(\delta) \supseteq \mathfrak{A}^\alpha(K)$ can be used to deduce that δ is a generator for a large class of non-compact, non-commutative G. The first two steps in his proof follow the outline given above but restrictions on G come from the regularity-positivity argument used in the third step (see [3, Remark 3]).

2. Derivations and subgroups.

Our principal result is the following theorem.

THEOREM 2.1. *Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system with G a locally compact abelian group. Assume that $\mathfrak{A}$ is α-prime and $\hat{G}/\Gamma(\alpha)$ is compact, where $\hat{G}$ is the dual of G and $\Gamma(\alpha)$ denotes the Connes spectrum of α. Further let δ be a closed symmetric derivation which commutes with α.*

The following conditions are equivalent:

1. $$D(\delta) \supseteq \bigcup_{K \text{ compact}} \mathfrak{A}^\alpha(K),$$
   where $\mathfrak{A}^\alpha(K)$ denotes the spectral subspaces of α,
2. *There is a strongly continuous one-parameter subgroup ϱ of α_G with generator δ_ϱ such that $\delta - \delta_\varrho$ has bounded closure.*

PROOF 1 $\Rightarrow$ 2. First from condition 1 it follows that δ generates a strongly continuous one-parameter group of *-automorphisms β of $\mathfrak{A}$, by the argument sketched in the introduction. Second define an action γ of $G \times \mathbf{R}$ by

$$\gamma_{(g,t)} = \alpha_g \circ \beta_t .$$

The Connes spectrum $\Gamma(\gamma)$ of γ is a closed subgroup of $(G \times \mathbf{R})\hat{} = \hat{G} \times \mathbf{R}$, and the rest of the proof follows from analysis of $\Gamma(\gamma)$.

The basic idea is best illustrated by the case $G = \mathbf{R}$. Then $\Gamma(\gamma)$ is a closed subgroup of $\mathbf{R}^2$ and one argues that $\Gamma(\gamma) \neq \{0\}$ because $\Gamma(\alpha) \neq \{0\}$. But $\Gamma(\gamma) \cap (I \times \mathbf{R})$ is compact for each closed interval I because of the domain assumption on δ. Consequently

$$\Gamma(\gamma) \subseteqq \{(p, \lambda p) \; ; \; \varrho \in \mathbf{R}\} \quad \text{for some } \lambda \in \mathbf{R} .$$

It then follows that $\delta - \lambda\delta_\alpha$ is bounded and ϱ is identified as the subgroup $t \mapsto \alpha_{\lambda t}$.

Now let us return to the general case.

First define π_1 as the projection of $\hat{G} \times \mathbf{R}$ onto $\hat{G}$ and let $H = \pi_1(\Gamma(\gamma))$. If K is a compact subset of $\hat{G}$ with $K \cap H = \emptyset$ then $\mathrm{Sp}(\gamma) \cap K \times \mathbf{R}$ is compact because δ is bounded on $\mathfrak{A}^\alpha(K)$. Now by definition

$$\Gamma(\gamma) = \bigcap_{\mathscr{B} \in \mathscr{H}^\gamma(\mathfrak{A})} \mathrm{Sp}(\gamma|_{\mathscr{B}}) ,$$

where $\mathscr{H}^\gamma(\mathfrak{A})$ denotes the non-zero γ-invariant hereditary subalgebras of $\mathfrak{A}$. Therefore since

$$(\mathrm{Sp}(\gamma) \cap K \times \mathbf{R}) \cap \Gamma(\gamma) = \emptyset$$

and $\mathfrak{A}$ is γ-prime [4], there is a $\mathscr{B} \in \mathscr{H}^\gamma(\mathfrak{A})$ such that

$$\mathrm{Sp}(\gamma|_{\mathscr{B}}) \cap (\mathrm{Sp}(\gamma) \cap K \times \mathbf{R}) = \emptyset ,$$

i.e.,

$$\mathrm{Sp}(\gamma|_{\mathscr{B}}) \cap K \times \mathbf{R} = \emptyset .$$

Thus $\mathrm{Sp}(\alpha|_{\mathscr{B}}) \cap K = \emptyset$ which implies that $\Gamma(\alpha) \subset H$. Note that π_1 is injective on $\Gamma(\gamma)$, because $\mathrm{Sp}(\gamma) \cap \{0\} \times \mathbf{R}$ is bounded.

Next for each $\sigma \in \Gamma(\alpha)$, let $\varrho(\sigma) \in \mathbf{R}$ be such that $(\sigma, \varrho(\sigma)) \in \Gamma(\gamma)$. The map $\varrho \colon \Gamma(\alpha) \to \mathbf{R}$ is a continuous homomorphism since $\Gamma(\gamma)$ is closed and ϱ is locally bounded.

Let $\hat{\varrho} \colon \mathbf{R} \to \hat{\Gamma}(\alpha) = G/\Gamma(\alpha)^\perp$ be the dual map. Since $\Gamma(\alpha)^\perp = (\hat{G}/\Gamma(\alpha))\hat{}$ and the group $\hat{G}/\Gamma(\alpha)$ is compact by assumption it follows that $\Gamma(\alpha)^\perp$ is discrete. Therefore any one-parameter subgroup of $G/\Gamma(\alpha)^\perp$ lifts to a one-parameter group of G. in particular $\hat{\varrho}$ lifts to G. Let ϱ^* denote this lifting and define g by $g(t) = -\varrho^*(t)$. Then g is a continuous homomorphism of $\mathbf{R}$ into G.

Now define

$$I = \{(g(t), t) \; ; \; t \in \mathbf{R}\} .$$

We claim that $\mathrm{Sp}(\gamma)/I^\perp$ is bounded in $\hat{I} \cong \mathbf{R}$. If it is not bounded there exists a sequence $(\sigma_n, \varrho_n) \in \mathrm{Sp}(\gamma)$ such that the characters

$$t \mapsto \langle (g(t), t), (\sigma_n, \varrho_n) \rangle$$

of $\mathbf{R}$ are unbounded. Since replacing (σ_n, ϱ_n) by $(\sigma_n, \varrho_n) + (\sigma, \varrho)$ with $(\sigma, \varrho) \in \Gamma(\gamma)$ does not change the character, and $\hat{G}/\Gamma(\alpha)$ is compact, we can assume that σ_n varies over some compact subset K of $\hat{G}$, with $K + \Gamma(\alpha) = \hat{G}$. This shows that (ϱ_n) must be unbounded. Since $\mathrm{Sp}(\gamma) \cap K \times \mathbf{R}$ is bounded this is a contradiction.

Since $\mathrm{Sp}\,(\alpha_{g(\cdot)}\circ\beta.)$ is the closure of $\mathrm{Sp}\,(\gamma)/I^{\perp}$ this implies that $t \mapsto \alpha_{g(t)}\circ\beta_t$ is uniformly continuous. Thus condition 2 is satisfied with $\varrho_t=\alpha_{-g(t)}$.

2 ⇒ 1. This is straightforward.

COROLLARY 2.2. *Let α be a strongly continuous one-parameter group of *-automorphisms of the C*-algebra $\mathfrak{A}$, with infinitesimal generator δ_α and assume that $\mathfrak{A}$ is α-prime and $\Gamma(\alpha)\neq\{0\}$. Further let δ be a closed symmetric derivation which commutes with α.*

The following conditions are equivalent

1. $D(\delta) \supseteq D(\delta_\alpha)$,
2. $D(\delta) \supseteq \bigcup_{K \text{ compact}} \mathfrak{A}^\alpha(K)$,
3. *there exist $\lambda \in \mathbf{R}$ and a bounded symmetric derivation δ' of A such that*

$$\delta = \lambda\delta_\alpha + \delta' .$$

PROOF 1 ⇒ 2 is trivial and 3 ⇒ 1 by definition. But 2 ⇒ 3 is a special case of Theorem 2.1.

One can also obtain analogues of Theorem 2.1 and Corollary 2.2 for abelian C*-algebras.

THEOREM 2.3. *Let $\mathfrak{A}=C_0(X)$ be an abelian C*-algebra with spectrum X and α a strongly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$ with infinitesimal generator δ_α. Further let S_t denote the one-parameter group of homeomorphisms of X corresponding to α*, i.e.

$$(\alpha_t f)(x) = f(S_t x), \quad f \in C_0(X),\ x \in X ,$$

and introduce X_0 by

$$X_0 = \{x \in X \ ;\ S_t x = x \text{ for all } t\} .$$

Finally let δ be a closed symmetric derivation on $\mathfrak{A}$.

The following two conditions are equivalent:

1. a. *δ commutes with α,*

   b. $D(\delta) \supseteq \bigcup_{K \text{ compact}} \mathfrak{A}^\alpha(K)$,
   where $\mathfrak{A}^\alpha(K)$ denotes the spectral subspaces of α.

2. a. *δ is a generator.*
   b. *there exists a continuous real-valued S-invariant function l on $X \setminus X_0$ such that the flow T_t defined by $\exp\{t\delta\}$ on X is given by*

$$T_t x = S_{l(x)t} x \quad \text{if } x \in X \setminus X_0$$
$$= x \quad \text{if } x \in X_0 ,$$

   c. *the function l is bounded on the set $\{x \in X; p(x) > \varepsilon\}$ for any $\varepsilon > 0$ where*

$$p(x) = \inf\{t > 0 \; ; \; S_t x = x\} ,$$

with the convention that $\inf \varnothing = \infty$.

PROOF 1 $\Rightarrow$ 2. It follows from the argument sketched in the introduction that δ generates a strongly continuous one-parameter group of *-automorphisms β. Now define $\gamma_{(s,t)} = \alpha_s \circ \beta_t$ for $(s,t) \in \mathbf{R}^2$.

Next suppose $\mathfrak{A}$ is γ-prime, then Sp (γ) is a closed subgroup of $\mathbf{R}^2$. If Sp $(\gamma) = \{0\}$, then $\mathfrak{A} = \mathbf{C}$ and $\beta_t = \alpha_{\lambda t}$ for any $\lambda \in \mathbf{R}$. If, however, Sp $(\gamma) \neq \{0\}$, then Sp $(\gamma) \cong \mathbf{R}$, or $\mathbf{C}$, and there is a $\lambda \in \mathbf{R}$ such that

$$\mathrm{Sp}\,(\gamma) \subseteq \{(\varrho, \lambda\varrho) \; ; \; \varrho \in \mathbf{R}\}$$

since the other possibilities Sp $(\gamma) \cong \mathbf{R}^2$, $\mathbf{Z}^2$, or $\mathbf{R} \times \mathbf{Z}$ and Sp $(\gamma) \subset \{0\} \times \mathbf{R}$ are excluded by condition 1b as in the proof of Theorem 2.1. In this case the group $t \mapsto \alpha_{\lambda t} \circ \beta_{-t}$ has a trivial spectrum, and hence $\beta_t = \alpha_{\lambda t}$.

Now let us return to the general case.

If $S^\gamma_{(s,t)} = S_s \circ T_t$ denotes the group of homeomorphisms of X corresponding to γ, let Y denote the closure of a γ-orbit in X. Then the action $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$, on $C_0(Y)$ induced by α, β, γ, satisfy the same properties as α, β, γ, on $C_0(X)$. (For example let $f \in C_0(Y)$ have compact $\dot{\alpha}$-spectrum and let f_1 be an extension of f to a function in $C_0(X)$. Since there is a continuous function g whose Fourier transform $\hat{g}$ has compact support and which satisfies

$$f(x) = \int dt g(t) f(S_t x), \qquad x \in Y,$$

the function f_2 defined by

$$f_2(x) = \int dt g(t) f_1(S_t x), \qquad x \in X ,$$

is an extension of f with compact α-spectrum. Hence $f_2 \in D(\delta_\beta)$, which implies that $f \in D(\delta_{\dot{\beta}})$ by restriction. Here we gave used δ_β and $\delta_{\dot{\beta}}$ to denote the generators of β and $\dot{\beta}$ respectively.) But $\mathfrak{A} = C_0(Y)$ is $\dot{\gamma}$-prime and hence there exists a $\lambda \in \mathbf{R}$ such that

$$T_s x = S_{\lambda s} x, \qquad x \in Y,$$

by the reasoning of the previous paragraph. Thus if Y is the closure of the γ-orbit through x one has $T_t x = x$ for $x \in X_0$ and one can define a unique real function l over $X \setminus X_0$ such that

$$T_t x = S_{l(x)t} x$$

for $x \in X \setminus X_0$. We next argue that l is continuous. For this purpose we shall use the fact that $(t, x) \to S_t x$ (respectively $T_t x$) is jointly continuous, which is shown by using the strong continuity of α (respectively β) and local compactness of X.

Let $x_\mu \in X \setminus X_0$ converge to $x \in X \setminus X_0$. If $l(x_\mu)$ is unbounded, for a suitable subnet of μ choose $t_\mu \in \mathbf{R}$ such that $l(x_\mu)t_\mu$ converges to a non-zero λ and t_μ converges to zero. Then

$$x = \lim T_{t_\mu t} x_\mu = \lim S_{l(x_\mu)t_\mu t} X_\mu = S_{\lambda t} x$$

which implies that $x \in X_0$, a contradiction. On the other hand if $l(x_\mu)$ is bounded we may suppose it converges to some limit $\lambda \in \mathbf{R}$ and then

$$T_t x = \lim S_{l(x_\mu)t} x_\mu = S_{\lambda t} x \, .$$

Hence $\lambda = l(x)$ by definition. Thus l is continuous on $X \setminus X_0$.

Finally suppose that for some $\varepsilon > 0$ the function l is not bounded on $X^\varepsilon = \{x;\ p(x) > \varepsilon\}$. Then there is a sequence of positive integers n_k such that the sets

$$X_k^\varepsilon = \{x \in X^\varepsilon \ ; \ n_k < |l(x)| < n_k + 1\}$$

are non empty. Since X^ε is an open S-invariant set and l is a continuous S-invariant function it follows that X_k^ε is open and S-invariant. Next let $f_k \in C_0(X)$ be such that $\operatorname{supp} f_k \subset X_k^\varepsilon$, $\|f_k\| = 1$, and $\mathrm{Sp}_\alpha(f_k)$ is in a small sub-interval of $[1, 1 + 1/2\pi\varepsilon]$. The existence of such functions follows because

$$\mathrm{Sp}\,(S|_{X_k^\varepsilon}) \supset \mathbf{Z}/2\pi p(x) \qquad \text{for } x \in X_k^\varepsilon \, ,$$

with the convention $\mathbf{Z}/2\pi\infty = \mathbf{R}$. Then $f = \sum f_k/\sqrt{n_k}$ converges in $C_0(x)$ and

$$\mathrm{Sp}_\alpha(f) \subset [1, 1 + 1/2\pi\varepsilon]$$

but $f \notin D(\delta)$ because $\|\delta(f_k)\| \sim \sqrt{n_k}$ (see Lemma 2.4 below).

$2 \Rightarrow 1$. It follows from Theorem 2.1 and Corollary 2.3 of [1] that the flow T in condition 2b determines a strongly continuous one-parameter group of $*$-automorphisms β of $\mathfrak{A}$ which commutes with α therefore the generator δ commutes with α.

Now we show that condition 2 implies condition 1b. Let $f \in C_0(X)$ with $\mathrm{Sp}_\alpha(f) \subset [-n, n]$ for some $n > 0$. Then for $x \in X$ with $p(x) < 1/2\pi n$ one has

$$f(S_t x) = f(x), \quad t \in \mathbf{R} .$$

Thus the β-spectrum of f restricted to the S-orbit through x is $\{0\}$, if not empty. Let

$$l = \sup\{|l(x)| \; ; \; p(x) \geqq 1/2\pi n\} .$$

Then it follows that the β-spectrum of f restricted to the S-orbit through x is contained in $[-nl, nl]$, if not empty, for each $x \in X$ with $p(x) \geqq 1/2\pi n$. Therefore $\mathrm{Sp}_\beta(f) \subset [-nl, nl]$.

REMARK. One can strengthen the last conclusion. It follows from the properties of l specified in conditions 2b and 2c that the associated flow T defines a strongly continuous group of *-automorphisms β of $\mathfrak{A}$ by the relation

$$(\beta_t f)(x) = f(T_t x), \quad f \in \mathfrak{A} ,$$

and the generator δ of β then satisfies condition 1.

For example the continuity of T can be established as follows. Let $x_\mu \in X$ and $t_\mu \in \mathbf{R}$ be convergent nets with limits x and t. One must show that $T_{t_\mu} x_\mu \to T_t x$. But if $p(x) > 0$, then $l(x_\mu) \to l(x)$ and this follows from continuity of S. Next suppose $p(x) = 0$, that is $x \in X_0$. If there is an $\varepsilon > 0$ such that $p(x_\mu) > \varepsilon$, then we may assume $l(x_\mu)$ has a limit l. Then $T_{t_\mu} x_\mu$ converges to $S_{lt} x = x$. If on the other hand there is a $\delta > 0$ such that $p(x_\mu) < \delta$, then there are $l_\mu \in [0, \delta]$ such that

$$S_{l(x_\mu) t_\mu} x_\mu = S_{l_\mu} x_\mu$$

and we may assume the l_μ converge to a limit l. Then $T_{t_\mu} x_\mu$ converges to $S_l x = x$. Thus we conclude that $T_{t_\mu} x_\mu$ has a convergent subnet with limit $T_t x$, and continuity of T follows immediately.

Theorem 2.3 is similar to Corollary 2.3 of [1] but the strong locality condition in the latter result is replaced by the spectral condition $D(\delta) \supseteqq \mathfrak{A}^\alpha(K)$, K compact. Thus by comparison of the two statements one concludes that the spectral condition implies strong locality of δ with respect to δ_α, whenever δ commutes with α.

Comparing Theorems 2.1 and 2.3, when $\mathfrak{A}$ is α-prime and $G = \mathbf{R}$, the absence of assumptions on the Connes spectrum $\Gamma(\alpha)$ in Theorem 2.3 might incline one to think that such assumptions are irrelevant in Theorem 2.1. But this is not the case. One can construct examples with commuting one-parameter automorphism groups α, β such that $D(\delta_\alpha) \subset D(\delta_\beta)$, but $D(\delta_\alpha) \neq D(\delta_\beta)$ and hence δ_β cannot be decomposed in the form $\delta_\beta = \lambda\delta_\alpha + \delta$ with δ bounded. We conclude with an example of this nature.

Let M_2 denote the algebra of 2×2 matrices and consider a direct product $\mathfrak{A}$

$=\bigotimes_{n=1}^{\infty}\mathfrak{A}_n$ of copies $\mathfrak{A}_n$ of M_2. Further let $\alpha_t^{(n)}$ be the one-parameter automorphism group of $\mathfrak{A}_n$ implemented by the unitary group

$$U_t^{(n)} = \begin{pmatrix} 1 & 0 \\ 0 & \exp\{-i\lambda_n(\alpha)t\} \end{pmatrix},$$

where $\lambda_n(\alpha) \in \mathbf{R}$, and let $\alpha = \bigotimes_{n=1}^{\infty}\alpha_t^{(n)}$. Similarly giving a sequence $\lambda_n(\beta) \in \mathbf{R}$ one can construct an automorphism group $\beta = \bigotimes_{n=1}^{\infty}\beta_t^{(n)}$. We will argue that the $\lambda_n(\alpha)$, $\lambda_n(\beta)$, can be chosen such that $D(\delta_\alpha) \subset D(\delta_\beta)$, $D(\delta_\alpha) \neq D(\delta_\beta)$, and $\Gamma(\alpha) = \{0\}$ but $\Gamma(\beta)$ can be quite arbitrary (cf. Remark 1 below).

Assume $\lambda_n(\beta)$, and hence β, are given and set

$$\mu_k = \sum_{n=1}^{k} |\lambda_n(\beta)| .$$

Next choose $\lambda_n(\alpha) > 0$ such that

$$\lambda_k(\alpha) > 2^{k+1}M\mu_k, \qquad k = 1, 2, \ldots ,$$

and

$$\lambda_k(\alpha) > 2M\sum_{j=1}^{k-1}\lambda_j(\alpha), \qquad k = 2, 3, \ldots ,$$

where M will subsequently be chosen suitably large. Define $\sigma_n^{\pm}, \sigma_n^{z} \in \mathfrak{A}_n$ by

$$\sigma_n^{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_n^{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_n^{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then

$$\alpha_t^{(n)}(\sigma_n^{\pm}) = e^{\pm i\lambda_n(\alpha)t}\sigma_n^{\pm}, \qquad \alpha_t^{(n)}(\sigma_n^{z}) = \sigma_n^{z} .$$

Now let $\{x_n\}$ be a bounded sequence in

$$\bigcup_{k=1}^{\infty}\bigotimes_{n=1}^{k}\mathfrak{A}_n$$

such that $\|\delta_\alpha(x_n)\| < C$. We aim to show that $\|\delta_\beta(x_n)\|$ is bounded. For this purpose consider x as a sum of reduced monomials in $\sigma_k^{\pm}$ with coefficients in the algebra generated by the σ_n^{z}, i.e. in the fixed point algebra $\mathfrak{A}^\alpha$ of α. Let x_{nk}^{+}, respectively x_{nk}^{-}, be the sum of those monomials which contain a factor σ_k^{+}, respectively σ_k^{-}, but do not contain $\sigma_j^{\pm}$, for $j > k$, and let x_{n0} denote the term without any $\sigma_j^{\pm}$ factors. Then $\mathrm{Sp}_\alpha(x_{n0}) = \{0\}$ if $x_{n0} \neq 0$ and

$$\mathrm{Sp}_\alpha(x_{nk}^{+}) \subset \left[\lambda_k(\alpha) - \sum_{j=1}^{k-1}\lambda_j(\alpha),\ \lambda_k(\alpha) + \sum_{j=1}^{k-1}\lambda_j(\alpha)\right]$$

$$\subset \left[\left(1-\frac{1}{2M}\right)\lambda_k(\alpha), \left(1+\frac{1}{2M}\right)\lambda_k(\alpha)\right]$$

$$\mathrm{Sp}_\alpha(x_{nk}^-) \subset \left[-\left(1+\frac{1}{2M}\right)\lambda_k(\alpha), -\left(1-\frac{1}{2M}\right)\lambda(\alpha)\right]$$

if $x_{nk}^\pm \neq 0$. Now

$$\left(1-\frac{1}{2M}\right)\lambda_k(\alpha)-\left(1+\frac{1}{2M}\right)\lambda_{k-1}(\alpha) > \left(1-\frac{1}{2M}\right)\lambda_k(\alpha)-\left(1+\frac{1}{2M}\right)\frac{1}{2M}\lambda_k(\alpha)$$

$$= \left(1-\frac{1}{M}-\frac{1}{4M^2}\right)\lambda_k(\alpha)$$

and hence for large M the α-spectra of the $\{x_{nk}^+\}_{k\geq 1}$ are disjoint. Similarly the α-spectra of the $\{x_{nk}^-\}_{k\geq 1}$ are disjoint. Hence $x-x_{nk}^+$ and x_{nk}^+ have disjoint α-spectra as do $x-x_{nk}^-$ and x_{nk}^-. To proceed in estimating $\delta_\alpha(x_n)$ and $\delta_\beta(x_n)$ we need two lemmas.

LEMMA 2.4. *Let α be a strongly continuous one-parameter group of *-automorphisms of a* C*-*algebra $\mathfrak{A}$ and let $I=[a-b,a+b]$ for $b>0$. Then there exists an $N\geqq 1$, independent of a and b, such that*

$$\|(\delta_\alpha - iaI)(x)\| \leqq Nb\|x\|, \quad x \in \mathfrak{A}^\alpha(I)$$

PROOF. Let $g \in C^\infty(\mathbf{R})$ be a function with compact support satisfying $g(t)=1$ for $t \in \langle -3/2, 3/2\rangle$ and let

$$f(t) = \frac{1}{2\pi}\int dp\, g\left(\frac{p-a}{b}\right)e^{-itp}$$

$$= \frac{b}{2\pi}\int dp\, g(p)e^{-it(a+pb)}.$$

Then $f \in \mathscr{S}(\mathbf{R})$ and

$$x = \int dt\, f(t)\alpha_t(x), \quad x \in \mathfrak{A}^\alpha(I).$$

But

$$f'(t) = -iaf(t)-\frac{ib^2}{2\pi}\int dp\, pg(p)e^{-it(a+pb)}$$

and hence

$$
\begin{aligned}
(\delta_\alpha - iaI)(x) &= -\int dt\, f'(t)\alpha_t(x) - ia \int dt\, f(t)\alpha_t(x) \\
&= \int dt\, \alpha_t(x) \frac{ib^2}{2\pi} \int dp\, pg(p) e^{-it(a+pb)}
\end{aligned}
$$

Thus setting

$$
G(t) = \left| \frac{1}{2\pi} \int dp\, pg(p) e^{-itp} \right|
$$

one obtains

$$
\begin{aligned}
\|(\delta_\alpha - iaI)(x)\| &\leqq \int dt\, b^2 G(tb) \|x\| \\
&= b \int dt\, G(t) \|x\| = bN\|x\|
\end{aligned}
$$

and N depends only on g.

LEMMA 2.5. *Assume* $\mathrm{Sp}_\alpha(x) \subset [a-b, a+b]$ *for* $b \geqq 0$ *and* $\mathrm{Sp}_\alpha(y) \subset \langle -\infty, a-(1+\varepsilon)b \rangle \cup \langle a+(1+\varepsilon)b, \infty \rangle$ *with* $\varepsilon > 0$. *It follows that*

$$
\|x\| \leqq C\|x+y\|
$$

for some C *depending only on* ε.

PROOF. Let $g \in C^\infty(\mathrm{R})$ be such that

$$
\begin{aligned}
g(p) &= 1 \quad \text{for } |p| \leqq 1 \\
&= 0 \quad \text{for } |p| \geqq 1+\varepsilon .
\end{aligned}
$$

Next for $b > 0$ define f by

$$
f_b(t) = \frac{1}{2\pi} \int dp\, g\left(\frac{p-a}{b}\right) e^{-itp} .
$$

Then one has

$$
(*) \qquad
\begin{aligned}
x &= \int dt\, f_b(t)\alpha_t(x) \\
0 &= \int dt\, f_b(t)\alpha_t(y) .
\end{aligned}
$$

Therefore

$$\begin{aligned}\|x\| &= \left\|\int dt\, f_b(t)\alpha_t(x+y)\right\| \\ &\leqq \int dt|f_b(t)|\,\|x+y\| \\ &\leqq \int dt|\hat{g}(t)|\,\|x+y\|\end{aligned}$$

where $\hat{g}$ denotes the Fourier transform of g.

If $b=0$ and $\mathrm{Sp}_\alpha(x)=\{a\}$, then (*) is valid for small $b>0$ and hence in both cases one has $\|x\|\leqq C\|x+y\|$.

Now let us return to estimating $\delta_\alpha(x_n)$ and $\delta_\beta(x_n)$.

First applying Lemma 2.5 to $x=\delta_\alpha(x_{nk}^{\pm})$ and $y=\delta_\alpha(x_n-x_{nk}^{\pm})$ with M chosen large enough that $\varepsilon=(1-1/M-1/4M^2)>0$ one deduces that

$$\|\delta_\alpha(x_{nk}^{\pm})\| \leqq C\|\delta_\alpha(x_n)\|$$

for some C depending only on M. Next applying Lemma 2.4 to $x_{nk}^{\pm}$ with $a=\lambda_k(\alpha)$ and $b=\lambda_k(\alpha)/2M$ one concludes that

$$\begin{aligned}\|\delta_\alpha(x_{nk}^{\pm})\| &\geqq \lambda_k(\alpha)\|x_{nk}^{\pm}\|-(N\lambda_k(\alpha)/2M)\|x_{nk}^{\pm}\| \\ &= \lambda_k(\alpha)(1-N/2M)\|x_{nk}^{\pm}\|\,.\end{aligned}$$

But $\mathrm{Sp}_\beta(x_{nk}^{\pm})\subset[-\mu_k,\mu_k]$ and hence

$$\|\delta_\beta(x_{nk}^{\pm})\| \leqq N\mu_k\|x_{nk}^{\pm}\|$$

by Lemma 2.4. Combining these last two estimates with M chosen large enough that $\varkappa=N/2M<1$ gives

$$\begin{aligned}\|\delta_\beta(x_{nk}^{\pm})\| &\leqq (\mu_k/\lambda_k(\alpha))N/(1-\varkappa)\|\delta_\alpha(x_{nk}^{\pm})\| \\ &\leqq C2^{-k}\varkappa(1-\varkappa)^{-1}\,\|\delta_\alpha(x_n)\|\,.\end{aligned}$$

Therefore

$$\begin{aligned}\|\delta_\beta(x_n)\| &\leqq \sum_{k\geqq 1}\|\delta_\beta(x_{nk}^{\pm})\| \\ &\leqq C2^{-k}\varkappa(1-\varkappa)^{-1}\,\|\delta_\alpha(x_n)\|\,.\end{aligned}$$

Since the x_n form a core for δ_α this implies that δ_β is relatively bounded with respect to δ_α and in particular $D(\delta_\beta)\supseteqq D(\delta_\alpha)$. But the inclusion is strict because $\lambda_k(\beta)/\lambda_k(\alpha)\to 0$ as $k\to\infty$.

REMARKS. 2. The Connes spectrum $\Gamma(\gamma)$ of a product action of the above type can be identified, by a straightforward argument, as the intersection of the closures of the sets

$$\left\{\sum_{n=k}^{\infty} \varepsilon_n \lambda_n(\gamma)\ ;\ \varepsilon_n = \pm 1, 0, \varepsilon_n = 0 \text{ except for a finite number of}\right\}_{k \geqq 1}.$$

2. In the example constructed above, $\mathrm{Sp}(\alpha)$ is discrete and the fixed point algebra $\mathfrak{A}^\alpha$ of α is commutative and has totally disconnected spectrum. Hence any closed *-derivation δ commuting with α is a generator. This follows because one automatically has $\delta(\mathfrak{A}^\alpha) = \{0\}$ and $D(\delta)$ contains any other eigenspace of α (which is one-dimensional over $\mathfrak{A}^\alpha$). Thus δ is a generator on each eigenspace and it follows that it is a generator on $\mathfrak{A}$.

3. Lemmas 2.4 and 2.5 also allow one to conclude that the C^∞-elements of the automorphism group α coincide with the analytic elements if and only if δ_α is bounded. For example assume δ_α is not bounded and choose $p_n > 0$ with $p_n + 2 < p_{n+1}$ such that there exist $x_n \in \mathfrak{A}^\alpha([p_n, p_n + 1])$ with $\|x_n\| = 1$. By iterating Lemma 2.4 one obtains

$$\|\delta_\alpha^m(x_n)\| < (p_n + N)^m .$$

Hence if $\lambda_n \geqq 0$ are chosen such that

$$(*) \qquad \sum_{n \geqq 1} (p_n + N)^m \lambda_n < +\infty$$

for $m = 1, 2, \ldots$ then

$$x = \sum_{n \geqq 1} \lambda_n x_n$$

is a C^∞-element of δ_α. But applying Lemma 2.5 to $\delta_\alpha^m(x - \lambda_n x_n)$ and $\lambda_n \delta_\alpha^m(x_n)$ and using Lemma 2.4 one concludes that

$$\begin{aligned}\|\delta_\alpha^m(x)\| &\geqq C^{-1}\lambda_n \|\delta_\alpha^m(x_n)\| \\ &> C^{-1}\lambda_n (p_n - N)^m ,\end{aligned}$$

for n sufficiently large. Therefore if $t \geqq 0$

$$\sum_{m \geqq 0} \frac{t^m}{m!} \|\delta_\alpha^m(x)\| \geqq C^{-1}\lambda_n \exp\{t(p_n - N)\}$$

for all large n. Now x fails to be an analytic element of α if the right hand side of this last inequality diverges as $n \to \infty$ for all $t > 0$. But this can be arranged by suitable choice of λ_n without violating (*), e.g. one can choose $\lambda_n \sim \exp\{-\sqrt{p_n}\}$.

Similarly one can show that the classes of C^∞-elements, quasi-analytic elements [2], and analytic elements of α are distinct whenever δ_α is unbounded.

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