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ON THE POSTAGE STAMP PROBLEM WITH THREE STAMP DENOMINATIONS, III

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The present paper is an immediate continuation of Selmer [7] and Selmer - Rödne [8]. All references to theorems and formulas from sections 1–13 are automatically to [7] or [8].

14. The sets of h_0- and $(h_0 - 1)$-representable numbers.

Let $A_k' = A_k \cup \{0\}$. The set (1.2) of h-representable numbers (*at most* h addends) may then in standard terminology be denoted by hA_k'. Our aim in the present section is to determine the sets h_0A_3' and $(h_0-1)A_3'$.

We shall rely heavily on the results in Rödseth [6], and use his notation, with one exception: He operates with an integer r, $0 \leqq r < a_3$. To avoid confusion with our use of r, we shall replace his r by l.

Rödseth's Lemma 4 states that

$$t_{-l}^* = x_v(a_3-1)+y_v(a_3-a_2), \quad (x_v,y_v)\in X_v \cup Y_v.$$

We consider the numbers (all $\equiv l \pmod{a_3}$):

$$(14.1) \qquad (h_0-t)a_3-t_{-l}^* = (h_0-t-x_v-y_v)a_3+y_va_2+x_v \geqq 0,$$

and claim that these belong to h_0A_3' for $t \geqq 0$. This is trivial if $h_0-t-x_v-y_v \geqq 0$, since the coefficient sum $\Sigma = h_0-t \leqq h_0$. It remains to show that the set

$$S_l = \{l, l+a_3, l+2a_3, \ldots, y_va_2+x_v-a_3\} \subset h_0A_3'.$$

And this is proved by Rödseth, since S_l is just the sequence (4.1) of [6].

On the other hand, the numbers (14.1) do *not* belong to h_0A_3' if $t = -t' < 0$. Assume to the contrary that

$$(h_0+t')a_3-t_{-l}^* = x_3a_3+x_2a_2+x_1, \quad \Sigma x_i = h' \leqq h_0.$$

As in section 3, we conclude that

$$t_{-l}^*-t'a_3 = (h_0-h')a_3+x_1(a_3-1)+x_2(a_3-a_2)$$

has a representation by $\bar{A}_3 = \{a_3-a_2, a_3-1, a_3\}$, cf. (2.15). (Rödseth uses $A_3^* = \bar{A}_3 \cup \{0\}$.) But this is a contradiction, since t_{-l}^* is defined as the smallest integer in its residue class $\pmod{a_3}$ with a representation by $\bar{A}_3$.

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Letting (x_v, y_v) run through all lattice points of $X_v \cup Y_v$, we get all residue classes $l \pmod{a_3}$, and have the following

THEOREM 14.1.

$$h_0 A_3' = \bigcup_{(x_v,y_v)\in X_v \cup Y_v} \{(h_0 - t - x_v - y_v)a_3 + y_v a_2 + x_v \geqq 0, t = 0,1,\ldots\}.$$

For use in the next section, we shall also determine $(h_0-1)A_3'$. Clearly

$$(h_0-1)A_3' \subset h_0 A_3' - a_3 = \{n - a_3 | n \in h_0 A_3'\}.$$

If A_3 is *pleasant*, it suffices to use regular representations, and clearly

$$(h_0-1)A_3' = (h_0 A_3' - a_3) \cap \mathsf{N}_0$$

(where $\mathsf{N}_0 = \{0,1,2,\ldots\}$). For non-pleasant A_3, however, we get problems with the number n_0 of (11.13):

(14.2)

$$n_0 = a_3 - r - 1 = (f-1)a_2 + a_2 - 1 = n_{h_0-1}(A_3) + 1 \notin (h_0-1)A_3',$$

where $n_0 + a_3 = 2fa_2 + r - 1 \in h_0 A_3'$, since $1 \leqq r \leqq a_2 - f - 1$ by (4.3). (For pleasant A_3, it follows from (2.8) that $n_0 + a_3 = n_{h_0}(A_3) + 1 \notin h_0 A_3'$.)

We shall show that n_0 is usually the *only* exception:

THEOREM 14.2. *For A_3 non-pleasant, with $r \neq 1$ and $s \neq q$, we have*

$$(h_0-1)A_3' = (h_0 A_3' - a_3) \cap \mathsf{N}_0 \setminus \{a_3 - r - 1\}. \tag{14.3}$$

To prove this, we replace h_0 by $h_0 - 1$ in the arguments leading to Theorem 14.1. The only critical point is whether now $S_l \subset (h_0-1)A_3'$.

To show that $S_l \subset h_0 A_3'$, Rödseth used his Lemma 5, which states that for $1 \leqq i \leqq v$, we have

$$x_{i-1} + y_{i-1} + Q_i - 1 \leqq h_0 \quad \text{if} \quad P_i \leqq s_i \tag{14.4}$$

$$x_i + y_i + R_i - 1 \leqq h_0 \quad \text{if} \quad P_i > s_i. \tag{14.5}$$

If these relations hold with *strict inequalities*, it follows that $S_l \subset (h_0 - 1)A_3'$.

We note that Rödseth's division algorithm for a_3/a_2 is the same as the one leading to our Theorem 6.1. In particular, we have $a_3 = q_1 a_2 - s_1$, hence $q_1 = q$, $s_1 = s$, and $v > 0$ for a non-pleasant A_3, when $s \geqq q$ by (2.10).

Studying Rödseth's proof of his Lemma 5, we observe the following facts:

1) For $i=1$, when $P_1 = q_1 \leqq s_1$, we have equality in (14.4) if and only if (x_0, y_0) is the upper right corner of Y_0:

$$(x_0, y_0) = (s_0 - 1, P_1 - P_0 - 1) = (a_2 - 1, f - 1). \tag{14.6}$$

Then $y_0 a_2 + x_0$ is just the number n_0 of (14.2).

2) For $i > 1$, hence $Q_i > 1$, a necessary condition for equality in (14.4) or (14.5) is $s_i = s_{i-1} - 1$ or $s_{i+1} = s_i - 1$, respectively. But then such a relation must hold from the start:

$$s = s_1 = s_0 - 1 = a_2 - 1, \text{ hence } r = 1$$

(cf. the recurrence relation $s_{j+1} = q_{j+1}s_j - s_{j-1}$, $q_{j+1} \geqq 2$). If $r \neq 1$, we thus have strict inequalities in (14.4–5) for all $i > 1$.

In Rödseth's proof of $S_l \subset h_0 A'_3$, he divides S_l into "subsequences" between $y_{i-1}a_2 + x_{i-1}$ and

$$y_i a_2 + x_i = y_{i-1}a_2 + x_{i-1} + Q_i \left[\frac{x_{i-1}}{s_i}\right] a_3.$$

We have noted that the case $i = 1$ needs a special treatment. Since $s_1 = s$, $Q_1 = 1$, we must consider the numbers $za_3 + y_0 a_2 + x_0$, $0 \leqq z < [x_0/s]$. Using the "a_3-transfer" $a_3 = qa_2 - s$ of section 11, this may be written as

$$za_3 + y_0 a_2 + x_0 = (y_0 + zq)a_2 + x_0 - zs, \tag{14.7}$$

with positive constant term, and a coefficient sum

$$\Sigma = x_0 + y_0 - z(s - q) \leqq x_0 + y_0.$$

If $x_0 + y_0 < h_0$, then also $\Sigma < h_0$ for all z. If $x_0 + y_0 = h_0$, corresponding to the corner (14.6), then $\Sigma < h_0$ for $z > 0$ if $s > q$, but $\Sigma = h_0$ for all z when $s = q$.

If $s = q$, then $v = 1$ by Theorem 7.1, and the "subsequence" just completed covers the whole of S_l. If $v > 1$, we have seen that the remaining subsequences yield no problems if $r \neq 1$.

This completes the proof of (14.3), and also shows that if $s = q$, then

(14.8)

$$(h_0 - 1)A'_3 = (h_0 A'_3 - a_3) \cap \mathrm{N}_0 \setminus \left\{ ta_3 - r - 1 \,|\, t = 1, 2, \ldots, \left[\frac{a_2 - 1}{s}\right] \right\}.$$

Here $ta_3 - r - 1 = n_0 + (t-1)a_3 = n_0 + za_3$, with $0 \leqq z < [x_0/s] = [(a_2 - 1)/s]$. Note that we may use also $z = [x_0/s]$ in (14.7), but the resulting number is then contained in $h_0 A'_3$ but not in $h_0 A'_3 - a_3$.

We finally treat the case $r = 1$. A modification of Rödseth's method then seems to become rather complicated. However, we can settle the case directly by a more elementary application of a_3-transfers. With $r = 1$, the only such transfers which may reduce the coefficient sum are of the form

(14.9) $$ea_3 = (ef+1)a_2 - (a_2 - e), \quad e = 1, 2, \ldots .$$

As in section 11, we start with the *regular* representations

(14.10) $$n = e_3 a_3 + e_2 a_2 + e_1, \quad e_1 \leqq a_2 - 1, \quad e_2 \leqq f - 1.$$

For $r = 1$, it is unnecessary to consider $e_2 = f$, since already $fa_2 + 1$ gives a new a_3.

For the n of (14.10), we shall decide if $n \in h_0 A_3'$. If $\Sigma_e = \Sigma e_i \leqq h_0$, we are finished. If $\Sigma_e > h_0$, we must try a transfer (14.9) with $e \leqq e_3$. The transfer is possible only if it yields a non-negative constant term, that is, if $e_1 \geqq a_2 - e$.

Similarly, we shall decide if $n' \in (h_0 - 1)A_3'$, where

(14.11) $$n' = n - a_3 = (e_3 - 1)a_3 + e_2 a_2 + e_1 \quad (e_3 > 0),$$

with $\Sigma_e' = \Sigma_e - 1$, hence no problem if $\Sigma_e \leqq h_0$. If an a_3-transfer (14.9) is necessary and possible in (14.10), and yields a new $\Sigma \leqq h_0$, then the *same* transfer gives $\Sigma' \leqq h_0 - 1$ in (14.11), provided it is possible, that is, if $e \leqq e_3 - 1$. It is easily seen that this combination of conditions *fails* only in the case

(14.12) $$n = e_3 a_3 + (f-1)a_2 + a_2 - e_3, \quad \Sigma = h_0 + 1.$$

Thus $n' = n - a_3 \notin (h_0 - 1)A_3'$ if $n' = e_3 a_3 - e_3 - 1$.

For the n of (14.12), we must use $e = e_3$ in (14.9), and get $n = (e_3 + 1)fa_2$, hence

$$n \in h_0 A_3' \Leftrightarrow (e_3 + 1)f \leqq h_0 = a_2 + f - 2 \Leftrightarrow e_3 \leqq \left[\frac{a_2 - 2}{f}\right].$$

We have thus shown that if $r = 1$, then

(14.13)
$$(h_0 - 1)A_3' = (h_0 A_3' - a_3) \cap N_0 \setminus \left\{ t(a_3 - 1) - 1 \,|\, t = 1, \ 2, \ \ldots, \ \left[\frac{a_2 - 2}{f}\right] \right\}.$$

For $t = 1$, we get $t(a_3 - 1) - 1 = a_3 - 2 = n_0$.

No problems arise if we have $s = q$ and $r = 1$ *simultaneously*. Then $s = q = a_2 - 1$, $f = q - 1 = a_2 - 2$, and the "subtrahends" $\{\cdot\}$ in (14.8) and (14.13) both consist of n_0 only.

The results (14.3), (14.8) and (14.13) imply that, but for the specified exceptions with $t > 1$ for $r = 1$ or $s = q$, *the integers* $\geqq a_3$ *with a representation in at most* h_0 *addends from* A_3 *have such a representation containing* a_3.

In particular, $[0, n_{h_0}(A_3)] \subset h_0 A_3'$. It then follows from (14.3) that

(14.14)
$$r \neq 1,\ s \neq q \Rightarrow [0, n_{h_0}(A_3) - a_3] \setminus \{a_3 - r - 1\} \subset (h_0 - 1)A_3'.$$

This was first observed numerically for a large number of bases A_3, and gave the impetus for the investigations in this section.

As in Rödseth [6], let $\Lambda(n)$ denote the number of addends in a *minimal* representation of n by a given basis A_k. We clearly have

$$\Lambda(n_h(A_k) - (x+1)a_k + 1) \geqq h - x,\ \Lambda(n_h(A_k) - xa_k) \geqq h - x,$$

since otherwise addition of $(x+1)a_k$ or xa_k would yield a contradiction. This raises the question whether there are integers $x > 0$ such that for the interval of length a_k:

(14.15) $$[n_h(A_k) - (x+1)a_k + 1,\ n_h(A_k) - xa_k] \subset (h - x)A_k'.$$

We have just seen that this holds with $x = 1$ if $k = 3$, $h = h_0$, A_3 non-pleasant, $r \neq 1$, $s \neq q$. Already for $x = 2$, however, it is easy to find counterexamples:

$$A_3 = \{1, 7, 11\},\ h_0 = 6,\ n_6(A_3) = 48;\ \Lambda(17) = 5.$$

We have made the interesting observation that for *Frobenius-dependent* A_3 with $r > 1$, (14.15) holds also with larger x:

PROPOSITION 14.1. *Let* A_3 *be Frobenius-dependent, with* $r > 1$. *In the notation* (5.8), *let*

$$(p-1)a_2 \leqq n \leqq n_{h_0}(A_3),\ x = \left[\frac{n_{h_0}(A_3) - n}{a_3}\right].$$

Then

$$n \in (h_0 - x)A_3'.$$

A proof will be published elsewhere.

15. The cases with $n_h(A_4) = n_h(A_3)$.

In (3.3), we raised the question of *basis extensions which do not increase the h-range*. We shall solve this question completely in the case

(15.1) $$n_h(A_4) = n_h(A_3 \cup \{a_4\}) = n_h(A_3),\ a_4 > a_3.$$

Even if A_4 enters the formulation, the results depend entirely on the properties of A_3.

We see from (3.4) that the *regular h*-range g_h always increases by a basis extension (assuming admissible bases). The same argument shows that if A_3 is pleasant, then

$$n_h(A_4) \geqq g_h(A_4) > g_h(A_3) = n_h(A_3),$$

so that we may assume *non-pleasant* A_3 in (15.1).

If $a_4 > n_{h_0}(A_3)+1$, then A_4 is *not admissible* for $h = h_0$ (where $h_0 = a_2 + f - 2$ refers to A_3). If then $h = h_0' > h_0$ is the smallest h for which A_4 is admissible, we trivially have $n_h(A_4) = n_h(A_3)$ for $h < h_0'$. On the other hand, it follows from (2.14) that

$$n_{h_0'}(A_4) \geqq a_4 + n_{h_0'-1}(A_3) = a_4 + n_{h_0'}(A_3) - a_3 > n_{h_0'}(A_3).$$

Similarly, it follows from (2.13–14) that

$$n_{h'}(A_4) \geqq n_{h'}(A_3),\ h' \geqq h_0' \Rightarrow n_h(A_4) > n_h(A_3),\ h > h'.$$

We may therefore restrict the problem (15.1) to the case

(15.2) $$n_{h_0}(A_4) = n_{h_0}(A_3),\ a_3 < a_4 \leqq n_{h_0}(A_3) + 1.$$

Note that a similar simplification does not apply to larger bases, since the analogue of (2.14) does not necessarily hold for $k > 3$.

We already know one case of (15.2), resulting from the basis A_{h+2} of section 3:

(15.3) $$a_2 = h_0 + 1, a_3 = h_0 + 2, a_4 = \alpha a_2 + a_3,\ 1 \leqq \alpha \leqq h_0 - 1.$$

To solve the general problem, we note that

$$n_{h_0}(A_4) = n_{h_0}(A_3) \Leftrightarrow n_{h_0}(A_3) + 1 \notin h_0 A_4'$$

(15.4) $$\Leftrightarrow n_{h_0}(A_3) + 1 - \delta a_4 \notin (h_0 - \delta)A_3',\ \delta = 1, 2, \ldots, h_0.$$

In most cases, it suffices to consider $\delta = 1$. Since

$$N = n_{h_0}(A_3) + 1 - a_4 \in [0, n_{h_0}(A_3) - a_3] \subset (h_0 A_3' - a_3) \cap \mathbf{N}_0,$$

(15.4) *fails* already for $\delta = 1$ if N does not belong to the exceptions in (14.3), (14.8) or (14.13). These cases have the common exception n_0 of (14.2), and $N = n_0$ does in fact lead to a general solution of (15.2):

(15.5) $$a_4 = \hat{a}_4 = n_{h_0}(A_3) - a_3 + r + 2 = n_{h_0}(A_3) - n_{h_0-1}(A_3)$$
$$\Rightarrow n_{h_0}(A_4) = h_{h_0}(A_3).$$

This is clear since we cannot use $\delta \geqq 2$ in (15.4):

$$2\hat{a}_4 > n_{h_0}(A_3)+1 \Leftrightarrow n_{h_0}(A_3) > 2a_3-2r-3,$$

which always holds by (2.8). – Note that $\hat{a}_4 = a_3$ if A_3 is pleasant.

If $a_4 \neq \hat{a}_4$, a necessary condition for (15.2) is that N equals one of the exceptions in (14.8) or (14.13), with $t>1$ (since $t=1$ corresponds to n_0).

We start with (14.13), hence $r=1$. Then $n_{h_0}(A_3)$ is given by (2.28), and we find that we must choose

$$(15.6) \qquad a_4 = a_3+\tau(a_3-1),\ \tau = 1,2,\ldots,\left[\frac{a_2-2}{f}\right]-1$$

(while $\tau = [(a_2-2)/f]$ corresponds to $\hat{a}_4$). We shall see that this is also *sufficient* for (15.2) to hold.

We consider a representation

$$(15.7) \qquad n_{h_0}(A_3)+1 = x_4a_4+x_3a_3+x_2a_2+x_1,$$

and must show that $\Sigma x_i > h_0$. This is trivial if $x_4 = 0$, so we can assume $x_4 > 0$, and observe that

$$h_{h_0}(A_3)+1 \equiv 0,\ a_4 \equiv a_3 \equiv 1 \pmod{a_3-1 = fa_2}.$$

With $x_2 = \varkappa f + x_2'$, $0 \leqq x_2' < f$, (15.7) then gives

$$x_4+x_3+x_1 \equiv (f-x_2')a_2,\ \text{hence}\ x_4+x_3+x_1 \geqq (f-x_2')a_2$$

$$x_4+x_3+x_2+x_1 \geqq x_4+x_3+x_2'+x_1 \geqq (f-x_2')a_2+x_2'$$
$$\geqq a_2+f-1 = h_0+1,$$

as required. – In particular, we get the known case (15.3) from (15.5–6) with $f=1$.

We next consider (14.8), hence $s=q$, $a_3 = q(a_2-1)$. By (2.29), we now have two possibilities for $n_{h_0}(A_3)$:

$$n_{h_0}(A_3) = \left(\left[\frac{a_2-1}{s}\right]+2\right)a_3-r-\begin{cases}2, & \text{if } s\nmid(a_2-1)\\ 3, & \text{if } s\mid(a_2-1).\end{cases}$$

These two cases must be considered separately.

If $s\nmid(a_2-1)$, we find that we must choose

$$(15.8) \qquad a_4 = (\tau+1)a_3,\ \tau = 1,2,\ldots,\left[\frac{a_2-1}{s}\right]-1$$

(while $\tau = [(a_2-1)/s]$ corresponds to $\hat{a}_4$). Again, this is also sufficient for (15.2) to hold:

We consider a representation (15.7). Since $a_3|a_4$, we get

$$x_2a_2+x_1 \equiv n_{h_0}(A_3)+1 \equiv -r-1 = -a_2+f \pmod{a_3 = q(a_2-1)},$$

from which we draw two conclusions:

$$\begin{aligned} &1)\ x_2a_2+x_1 \geqq a_3-r-1 \\ &2)\ x_2a_2+x_1 \equiv x_2+x_1 \equiv f-1 \pmod{a_2-1}. \end{aligned}$$

Assuming $\Sigma x_i \leqq h_0$ in (15.7), hence $x_4 > 0$, we get $x_2+x_1 < h_0 = (f-1) + (a_2-1)$, so $x_2+x_1 = f-1$, and

$$x_2a_2+x_1 \leqq (f-1)a_2 = a_3-r-a_2,$$

contradicting the first conclusion.

If $s|(a_2-1)$, hence $m = (a_2-1)/s$ an integer, we find that we must choose

$$a_4 = (\tau+1)a_3-1,\ \tau = 1, 2, \ldots, \frac{a_2-1}{s}-1 = m-1.$$

Now (15.4) holds for $\delta = 1$, and we examine $\delta = 2$:

$$n_{h_0}(A_3)+1-2a_4 = (m-2\tau)a_3-r = (m-2\tau-1)a_3+fa_2.$$

If $\tau \geqq [\frac{1}{2}(m+1)]$, this expression is negative, and an examination of (15.4) for $\delta \geqq 2$ is unnecessary, so (15.2) holds. If $\tau < [\frac{1}{2}(m+1)]$, however, the right hand side belongs to $(h_0-2)A_3'$, and (15.4) fails for $\delta = 2$. Thus (15.2) is satisfied only if

$$(15.9)\quad a_4 = (\tau+1)a_3-1,\ \tau = [\tfrac{1}{2}(m+1)], \ldots, m-1;\ m = \frac{a_2-1}{s}.$$

Summing up, we have the following

THEOREM 15.1. *For non-pleasant A_3, the equality* (15.2) *holds if and only if we have one of the cases:*

(15.5)	*for arbitrary* A_3,
(15.6)	*for* $r = 1$,
(15.8–9)	*for* $s = q$.

Based on computations by Mossige, this result was conjectured long before a proof was found. The cases $r = 1$ or $s = q$ are also proved in Krätzig-Berle [4, Kap. 4], the "if" part along the lines above, the "only if" part by explicit representations for $n_{h_0}(A_3)+1$ from h_0A_4' in the remaining cases.

16. The cases with $n_h(A_3 \cup \{a\}) = n_h(A_3)$, $a < a_3$.

In analogy with (3.3), it is quite natural to ask for cases when

(16.1)
$$n_h(A_k^*) = n_h(A_{k-1} \cup \{a\}) = n_h(A_{k-1}),\ 1 < a < a_{k-1},\ a \notin A_{k-1},$$

assuming admissible bases.

We need a particular result for the similar problem regarding regular h-ranges:

(16.2)
$$1 < a < a_2 \Rightarrow g_h(A_k^*) > g_h(A_{k-1}).$$

The proof is simple: It follows from Hofmeister [1, Satz 1] that the constant term of the regular representation for $g_h(A_\kappa)$ equals a_2-2 for all admissible A_κ. We conclude that the constant term a_2-1 of $g_h(A_{k-1})+1$ has a regular representation in at most a_2-2 addends 1 and $a \leqq a_2-1$.

In particular, $g_h(A_3^*) > g_h(A_2)$, and hence also $n_h(A_3^*) > n_h(A_2)$. The first possibility for (16.1) thus occurs when $k = 4$:

(16.3)
$$n_h(A_4^*) = n_h(A_3 \cup \{a\}) = n_h(A_3),\ 1 < a < a_3,\ a \neq a_2.$$

As in the preceding section, a study of this equality depends entirely on the properties of A_3.

If $h = h_0^*$ is the smallest h for which A_4^* is admissible, we clearly have $h_0^* \leqq h_0$ (where again $h_0 = a_2+f-2$ refers to A_3). To be "fair" to A_3, we restrict the examination of (16.3) to $h \geqq h_0$.

Before doing this, we just mention the analogous problem for *regular* h-ranges. By (16.2), we must then assume $a_2 < a < a_3$, and it is not difficult to prove that for $h \geqq h_0$:

(16.4)
$$g_h(A_4^*) = g_h(A_3) \Leftrightarrow a = fa_2+\rho,\ 0 \leqq \rho < r.$$

(My original proof is reproduced in Krätzig-Berle [4, p. 27].)

Similar arguments show that (16.3) is impossible with pleasant A_3. With $n_h(A_4^*) \geqq g_h(A_4^*)$ and $n_h(A_3) = g_h(A_3)$, equality in (16.3) could only occur under the conditions of (16.4). But by (2.8–9), we then have

$$n_h(A_3)+1 = (h-h_0+2)a_3-r-1 = (h-h_0)a_3+1\cdot a+fa_2+r-\rho-1,$$

with a coefficient sum $\leqq h$ except in the one case $r = a_2-1$, $\rho = 0$, hence $f \geqq 2$. But then

$$n_h(A_3)+1 = (h-h_0)a_3+2a+a_2-2,\ \Sigma \leqq h.$$

In what follows, we may thus assume *non-pleasant* A_3 in (16.3).

Since A_3 and A_4^* have a *common largest element* a_3, it is possible to use

Meures' result (2.16), which in combination with (2.13) shows that for $h \geqq h_0 - 1$:

$$n_h(A_k) \leqq ha_k - g(\bar{A}_k) - 1,$$

with equality if $h \geqq h_1$ ("stabilization", cf. section 3). For non-pleasant A_3, we know that $h_1 = h_0$. For A_4^*, we put $h_1 = h_1^*$. With

$$\bar{A}_3 = \{a_3 - a_2, a_3 - 1, a_3\}, \ \bar{A}_4^* = \bar{A}_3 \cup \{a_3 - a\},$$

we thus get, for $h \geqq h_0$:

$$n_h(A_3) = ha_3 - g(\bar{A}_3) - 1, \ n_h(A_4^*) \leqq ha_3 - g(\bar{A}_4^*) - 1.$$

Since trivially $n_h(A_4^*) \geqq n_h(A_3)$, this shows that

(16.5) $$g(\bar{A}_4^*) = g(\bar{A}_3) \Rightarrow n_h(A_4^*) = n_h(A_3) \text{ for } h \geqq h_0$$

(16.6) $$h \geqq h_1^* : n_h(A_4^*) = n_h(A_3) \Rightarrow g(\bar{A}_4^*) = g(\bar{A}_3).$$

We obviously have $g(\bar{A}_4^*) \leqq g(\bar{A}_3)$. With strict inequality, $g(\bar{A}_3)$ has a representation by $\bar{A}_4^*$:

$$g(\bar{A}_3) = x_1(a_3 - a) + x_2(a_3 - a_2) + x_3(a_3 - 1) + x_4 a_3.$$

It follows that

$$n_{h_0}(A_3) + 1 = h_0 a_3 - g(\bar{A}_3) = (h_0 - \Sigma x_i)a_3 + x_1 a + x_2 a_2 + x_3$$

has a representation by A_4^* with coefficient sum $h_0 - x_4 \leqq h_0$, provided that $\Sigma x_i \leqq h_0$. We thus have the following partial converse of (16.5):

(16.7) $$g(\bar{A}_3) \in h_0 \bar{A}_4^* \Rightarrow n_h(A_4^*) > n_h(A_3) \text{ for } h \geqq h_0.$$

We only proved this for $h = h_0$ above, but the general result with $h \geqq h_0$ then follows immediately from (2.13–14).

There is one trivial case of equality in (16.3):

(16.8) $$f = 1, a_2 = h_0 + 1, a_3 = h_0 + r + 1, a = a_2 - tr \geqq 2$$

(16.9) $$\Rightarrow n_h(A_4^*) = n_h(A_3) \text{ for } h \geqq h_0.$$

This follows from (16.5), since $\bar{A}_3$ and $\bar{A}_4^*$ are "equivalent" as Frobenius bases:

$$\bar{A}_3 = \{r, a_3 - 1, a_3\}, \ \bar{A}_4^* = \{r, (t+1)r, a_3 - 1, a_3\}.$$

The second element of $\bar{A}_4^*$ is a multiple of the first one.

We assume that A_3 is non-pleasant. If it is also non-dependent, it follows from Theorem 10.1 that

$$n_{h_0}(A_4^*) \geqq n_{h_0}(A_3) \geqq (h_0+1)a_2 - a_3.$$

Let $1 < a < a_2$. We then get $h_1^* \leqq h_0$ by Theorem 3.1, and can combine (16.5–6) to an equivalence for non-dependent A_3. And for Frobenius-dependent A_3, Krätzig-Berle [4, p. 23] shows very simply that we always have $n_h(A_4^*) > n_h(A_3)$ except in the already settled cases (16.8), hence

(16.10) $$1 < a < a_2 : g(\bar{A}_4^*) = g(\bar{A}_3) \Leftrightarrow n_h(A_4^*) = n_h(A_3).$$

Based on extensive computations by Mossige, I conjectured the following results:

THEOREM 16.1. *Let $a_2 < a < a_3$. Then*

$$n_h(A_4^*) > n_h(A_3) \text{ for } h \geqq h_0.$$

THEOREM 16.2. *Let $1 < a < a_2$. In addition to* (16.8), *there is one more case of equality in* (16.9):

$$f = 1,\ a_2 = h_0+1,\ a_3 = h_0+r+1,\ a = tr+1$$
$$h_0 = \tau r + \rho,\ 0 \leqq \rho < r-1,\ \tau \geqq \rho$$
$$r \equiv -1 \ (\mathrm{mod}\ \rho+1),\ t = 1, 2, \ldots, \left[\frac{\tau+1}{\rho+1}\right].$$

Both theorems were proved in the Master's thesis [2] of my student Kirfel. He used the methods of Rödseth [5] for determining the Frobenius number $g(\bar{A}_3)$. A shortened version [3] is submitted for publication.

Another student of mine, Krätzig-Berle, gave an independent and very elegant proof of Theorem 16.1 in her Diplomarbeit [4, Satz 3.1]. Using the inequalities of Theorems 10.2–5, she could determine a h_0-representation by A_4^* of $n_{h_0}(A_3)+1$.

We note that the bases A_3 of Theorem 16.2 satisfy the conditions (8.1–2), and so $n_h(A_3)$ can be determined explicitly by (8.3). It is fairly straightforward (cf. [4, Satz 2.3]) to show that this h-range is not increased when extending the basis with $a = tr+1$. The hard problem is of course to show that all *other* cases (except (16.8)) lead to an increase of the h-range.

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