MATH. SCAND. 56 (1985), 145–162

BETTI NUMBERS OF MONOID ALGEBRAS. APPLICATIONS TO 2-DIMENSIONAL TORUS EMBEDDINGS

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Introduction.

The starting point of this paper is the rather elementary observation (1.2), which leads to a formula (1.3) for the Betti numbers of a monoid algebra in terms of the combinatorial properties of the monoid, see [2]. The rest of the paper is concerned with the application of this formula to the case of 2-dimensional torus embeddings, see [3]. More specifically: In section 1 we give a method for computing the Betti numbers $\beta_i = \dim_k \mathrm{Tor}_i^A(k,k)$, when A is the monoid algebra over k of a commutative monoid Λ with cancellation law, and no non-trivial inverses. Proposition 1.3 relates the Betti numbers to the local homology of the simplicial set associated to $\Lambda_+ = \Lambda - \{1\}$ ordered such that $\lambda \leqq \lambda \cdot \mu$, when $\lambda, \mu \in \Lambda$.

In section 2 this is used to compute the Betti numbers of 2-dimensional torus embeddings A. In particular we prove that the Betti series

$$B(t) = \sum_{n \geqq 0} \beta_n t^n$$

of A is a rational function $P(t)/Q(t)$. The main result of this paper is, in fact, the explicit computation of the denominator $Q(t)$, see Corollary 2.20.

1. Betti numbers of monoid algebras.

Fix a field k and let Λ be a commutative monoid with cancellation law, i.e. such that $\lambda \cdot \mu = \lambda \cdot \mu'$ implies $\mu = \mu'$. Let $A = k(\Lambda)$ and put $\mathfrak{m} = \Lambda_+ \cdot A$ where $\Lambda_+ = \Lambda - \{1\}$. Assume $A/\mathfrak{m} = k$, that is assume Λ has no non-trivial subgroups. Put $\beta_i = \dim_k \mathrm{Tor}_i^A(k,k)$, the ith Betti number of A. Then the power series $B(t) = \sum_{n \geqq 0} \beta_n t^n$ is called the Betti serties of A. The purpose of this first paragraph is to give a method for computing the Betti series of A using only combinatorial properties of Λ_+.

Received April 12, 1983.

Let Λ_+ be ordered as follows: $\lambda_1 \leqq \lambda_2$ if and only if there exists a $\mu \in \Lambda$ such that $\mu \cdot \lambda_1 = \lambda_2$. There is a natural presheaf (projective system)

$$F\colon \Lambda_+ \to \mathrm{Ab}$$

defined by $F(\lambda) = A$, where $F(\lambda_1 \leqq \lambda_2)\colon F(\lambda_2) \to F(\lambda_1)$ is multiplication by $\mu = \lambda_2/\lambda_1$.

LEMMA 1.1.

$$\varinjlim_{\Lambda_+} F = (\Lambda_+)\cdot A = \mathfrak{m}\,.$$

PROOF. For every $\lambda \in \Lambda_+$, consider the morphism $\eta_\lambda\colon F(\lambda) \to A$, the multiplication by λ. This defines a morphism

$$\eta\colon \varinjlim_{\Lambda_+} F \to \mathfrak{m}\,.$$

Given an element $\alpha \in \mathfrak{m}$, there is a unique representation $\alpha = \sum_{i=1}^N \alpha_i \cdot \lambda_i$; $\alpha_i \in k$, $\lambda_i \in \Lambda_+$. Consider α_i as an element of $F(\lambda_i)$ and let $\bar{\alpha}_i$ be the image of α_i in $\varinjlim_{\Lambda_+} F$. Define $\mu\colon \mathfrak{m} \to \varinjlim_{\Lambda_+} F$ by $\mu(\alpha) = \sum_{i=1}^N \bar{\alpha}_i$. Then μ is an inverse of η.

LEMMA 1.2.

$$\varinjlim_{\Lambda_+}{}_{(n)} F = 0 \quad \text{for } n \geqq 1\,.$$

PROOF. By [1, (1, 1.4)] it is enough to show that F is coflabby (coflasque). Let $\lambda \in \Lambda_+$ and put

$$\bar{\lambda} = \{\lambda' \in \Lambda_+ \mid \lambda \leqq \lambda'\}\,.$$

Suppose $\Lambda_1 \subseteq \bar{\lambda}$ is such that if $\lambda' \in \Lambda_1$ and $\lambda' \leqq \lambda''$, then $\lambda'' \in \Lambda_1$. F is coflabby if in this situation

$$\varinjlim_{\Lambda_1} F \to \varinjlim_{\bar{\lambda}} F = F(\lambda) = A$$

is an injection.

However, the proof of Lemma 1.1 applies to show that

$$\varinjlim_{\Lambda_1} F = \{\lambda'/\lambda \mid \lambda' \in \Lambda_1\}\cdot A$$

and that the morphism

$$\varinjlim_{\Lambda_1} F \to \varinjlim_{\bar{\lambda}} F = A$$

is the obvious inclusion. Therefore we are done.

Consider the resolving complex $C.(\Lambda_+; -)$ for $\varinjlim_{\Lambda_+}$, see [1, (1.2)]. By Lemma 1.2, $C.(\Lambda_+; F)$ is an A-free resolution of the maximal ideal $\mathfrak{m}$ of A. Therefore

$$\operatorname{Tor}_i^A(k,k) \cong \begin{cases} k & i=0 \\ H_{i-1}(C.(\Lambda_+; F)\otimes_A k) & i \geqq 1 . \end{cases}$$

Now $C.(\Lambda_+; F)\otimes_A k = C.(\Lambda_+; F\otimes_A k)$, therefore

$$H_{i-1}(C.(\Lambda_+; F)\otimes_A k) = \varinjlim_{\Lambda_+}{}_{(i-1)} (F\otimes_A k) .$$

Observe that the projective system $F\otimes_A k$ is isomorphic to $\coprod_{\lambda\in\Lambda_+} k(\lambda)$, where $k(\lambda)$ is the projective system defined by

$$k(\lambda)(\lambda') = \begin{cases} 0 & \text{if } \lambda' \neq \lambda \\ k & \text{if } \lambda' = \lambda . \end{cases}$$

Put for any $\lambda \in \Lambda_+$,

$$\begin{aligned} \hat{\lambda} &= \{\lambda' \in \Lambda_+ \mid \lambda' \leqq \lambda\} \\ L(\lambda) &= \{\lambda' \in \Lambda_+ \mid \lambda' \leqq \lambda, \lambda' \neq \lambda\} = \hat{\lambda} - \{\lambda\} . \end{aligned}$$

It is easy to see that there are isomorphisms

$$\varinjlim_{\Lambda_+}{}_{(n)} k(\lambda) \cong \varinjlim_{\hat{\lambda}}{}_{(n)} k(\lambda) \qquad \text{for } n \geqq 0 .$$

In fact this follows from the existence of a $\coprod$-projective resolution of $k(\lambda)$, trivial outside of $\hat{\lambda}$, see [1, (1.2)]. Let $\boldsymbol{k}_\lambda$ be the constant projective system on $\hat{\lambda}$ defined by $\boldsymbol{k}_\lambda(\lambda') = k$, and let $\boldsymbol{k}'_\lambda$ be the subprojective system of $\boldsymbol{k}_\lambda$ defined by $\boldsymbol{k}'_\lambda(\lambda') = 0$ if $\lambda' = \lambda$ and $\boldsymbol{k}'_\lambda(\lambda') = k$ if $\lambda' \neq \lambda$. Then there is an exact sequence of projective systems on $\hat{\lambda}$

$$0 \to \boldsymbol{k}'_\lambda \to \boldsymbol{k}_\lambda \to k(\lambda) \to 0 .$$

As

$$\varinjlim_{\hat{\lambda}}{}_{(n)} \boldsymbol{k}_\lambda = \begin{cases} k & \text{for } n=0 \\ 0 & \text{for } n \geqq 1 \end{cases}$$

and since

$$\varinjlim_{\hat{\lambda}}{}_{(n)} \boldsymbol{k}'_\lambda \cong \varinjlim_{L(\lambda)}{}_{(n)} \boldsymbol{k} \cong H_n(E(\lambda); k) \qquad n \geqq 0$$

where $\boldsymbol{k}$ is the constant projective system k on $L(\lambda)$, and where we denote by $E(\lambda)$ the simplicial set defined by the ordered set $L(\lambda)$, see [1, (1.1)], we obtain an exact sequence

$$0 \to \varinjlim_{\lambda}{}_{(1)} k(\lambda) \to \varinjlim_{\lambda} k'_\lambda \to k \to \varinjlim_{\lambda} k(\lambda) \to 0$$

and isomorphisms

$$\varinjlim_{\lambda}{}_{(n)} k(\lambda) \cong H_{n-1}(E(\lambda); k) \qquad n \geqq 2\,.$$

Notice that $\varinjlim_{\lambda} k(\lambda)=0$ unless λ is minimal in Λ_+, in which case $\varinjlim_{\lambda} k(\lambda)\cong k$, and $\varinjlim_{\lambda}{}_{(1)} k(\lambda)=0$.

If λ is not minimal, then

$$\varinjlim_{\lambda}{}_{(1)} k(\lambda) \cong \tilde{H}_0(E(\lambda); k)$$

where $\tilde{H}_.$ is the augmented homology.

Summing up we have proved the following

PROPOSITION 1.3.

$$\mathrm{Tor}_n^A(k,k) \cong \begin{cases} k & n=0 \\ k^\varrho & n=1 \\ \coprod_{\lambda\in\Lambda_+} \tilde{H}_{n-2}(E(\lambda); k) & n\geqq 2 \end{cases}$$

where ϱ is the number of minimal elements of Λ_+.

2. Application to 2-dimensional Torus embeddings.

Let Λ' be a submonoid of $\mathbf{Z}_+^2$, satisfying the following condition:
There exist $(m_i, n_i) \in \Lambda'$, $i=1,2$ with m_i and n_i relatively prime such that

(i) $\Lambda' = \{(m_0, n_0) \in \mathbf{Z}_+^2 \mid \exists\, t_j \in \mathbf{Z}_+,\ j=0,1,2 \text{ such that } t_0(m_0,n_0)=t_1(m_1,n_1)+t_2(m_2,n_2)\}$

(ii) $m_1 \cdot n_2 - m_2 n_1 = p > 0.$

Any such monoid will be referred to as a saturated rational (sub)monoid (of $\mathbf{Z}_+^2$), see Fig. 1.

There is a one-to-one correspondence between saturated rational submonoids Λ' of $\mathbf{Z}_+^2$ and affine 2-dimensional normal torus imbeddings A, see [3], such that $A=k(\Lambda')$.

By (1.3) we know that

$$\mathrm{Tor}_n^{k(\Lambda')}(k,k) \cong \coprod_{\lambda\in\Lambda'_+} \tilde{H}_{n-2}(E(\lambda), k), \qquad n\geqq 2\,.$$

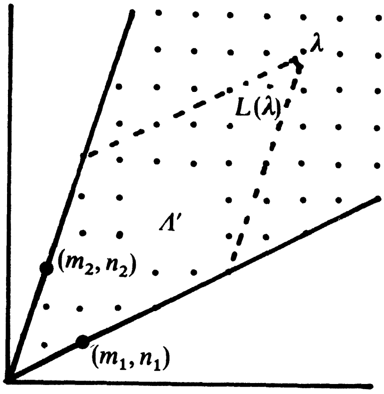

Fig. 1. $\Lambda', L(\lambda)$

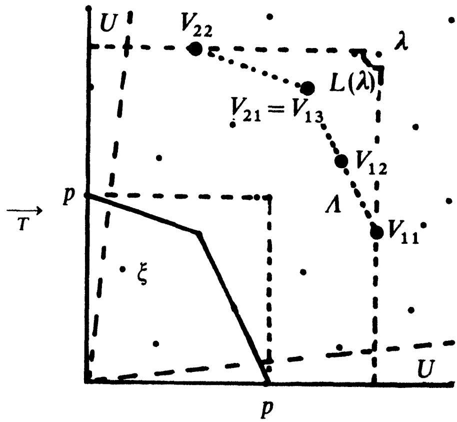

Fig. 2. Λ

The purpose of the rest of this paper is to establish a recursion formula for computing $\tilde{H}_r(E(\lambda), k)$, $\lambda \in \Lambda'_+$, $r \geqq 0$, see (2.17), from which we easily deduce the rationality of the Betti series

$$B(t) = \sum_{n=0}^{\infty} \beta_n t^n$$

where

$$\beta_n = \dim_k \operatorname{Tor}_n^{k(\Lambda')}(k, k) .$$

We are therefore interested in the simplicial structure of $E(\lambda)$, $\lambda \in \Lambda'_+$, which is determined by the structure of the ordered set $L(\lambda)$.

Consequently we shall have to study the ordered sets $L(\lambda)$ for arbitrary $\lambda \in \Lambda'_+$, see Fig. 1 and 2.

Consider first the unique linear transformation T: $\mathbf{Z}^2 \to \mathbf{Z}^2$ mapping (m_1, n_1) to $(p, 0)$ and (m_2, n_2) to $(0, p)$. T is represented by the 2×2 matrix

$$\begin{bmatrix} n_2 & -m_2 \\ -n_1 & m_1 \end{bmatrix}$$

Put $\Lambda = T(\Lambda')$. Notice that Λ is submonoid of $\mathbf{Z}_+^2$ but no longer a saturated rational submonoid, see Fig. 2. Nevertheless T defines an isomorphism $\Lambda' \cong \Lambda$, and we shall, from now on, find it more convenient to work with Λ. We may assume $p \geqq 2$, since otherwise $A = k(\Lambda)$ is a polynomial algebra in 2 variables, for which the Betti series is well known.

Let, for $n \geqq 1$, $\Lambda_n = \{(n, r) \in \Lambda \mid r \in \mathbf{Z}_+\}$. Then the following lemma holds:

LEMMA 2.1. *There exists a unique* $\xi \in \mathbf{Z}_+$, *with* $0<\xi<p$, such that

$$\Lambda_1 = \{(1,\xi+\eta\cdot p) \mid \eta \in \mathbf{Z}_+\}$$
$$\Lambda_n = \{(n,n\cdot\xi+\eta\cdot p \mid \eta \in \mathbf{Z}, n\cdot\xi+\eta\cdot p \geqq 0\}\,.$$

PROOF. Since $(m_2,n_2)=1$, there exists an integer pair $(x_0,y_0) \in \mathbf{Z}^2$ such that

$$T(x_0,y_0) = (n_2x_0-m_2y_0, -n_1x_0+m_1y_0) \in (\{1\}\times \mathbf{Z})\,.$$

The set $\{(m_1,n_1),(m_2,n_2)\}$ forms a basis for $\mathbf{Q}^2$, and there exist $\alpha,\beta \in \mathbf{Q}$ such that

$$(*)\qquad (x_0,y_0) = \alpha(m_1,n_1)+\beta_0(m_2,n_2)\,.$$

But T is a linear map so we have

$$\begin{aligned} T(x_0,y_0) &= \alpha\cdot T(m_1,n_1)+\beta_0\cdot T(m_2,n_2)\\ &= \alpha\cdot(p,0)+\beta_0\cdot(0,p) \in (\{1\}\times\mathbf{Z})\,. \end{aligned}$$

This implies $\alpha=1/p$ and from equation (*) and the fact $(m_1,n_1)=1$ we deduce that $\beta_0 \notin \mathbf{Z}$. So there exists an integer $\mu \in \mathbf{Z}$ such that $0<\beta_0+\mu<1$ and

$$T((x_0,y_0)+\mu(m_2,n_2)) = \alpha\cdot(p,0)+(\beta_0+\mu)(0,p) \in (\{1\}\times[0,p])\,.$$

Put $\beta=\beta_0+\mu$ and $(x,y)=(x_0,y_0)+\mu(m_2,n_2) \in \mathbf{Z}_+^2$, and let γ be the product of the denominators of α and β. The numbers $\gamma\cdot\alpha$, $\gamma\cdot\beta$ are integers, and

$$\gamma\cdot(x,y) \in \Lambda'\,.$$

Since the monoid Λ' is saturated, it follows that $(x,y) \in \Lambda'$. Let $\xi=\beta\cdot p$. Then $T(n\cdot(x,y))=(n,n\cdot\xi)$. Now consider the equivalence

$$\begin{aligned} n\cdot\xi+\eta\cdot p &= n\cdot\beta\cdot p+\eta\cdot p\\ &= (n\cdot\beta+\eta)\cdot p \geqq 0\\ \Leftrightarrow\quad & n\cdot\beta+\eta \geqq 0\,. \end{aligned}$$

If $n\cdot\xi+\eta\cdot p\geqq 0$ then we have

$$\begin{aligned} (n,n\cdot\xi+\eta\cdot p) &= T(n(x,y)+\eta(m_2,n_2))\\ &= T(n\cdot\alpha(m_1,n_1)+(n\cdot\beta+\eta)(m_2,n_2)) \end{aligned}$$

and $(n,n\cdot\xi+\eta\cdot p) \in \Lambda$. This follows from the fact that an integer pair, positively generated by (m_1,n_1) and (m_2,n_2) is element of Λ'.

Suppose $(x,y),(x',y') \in \Lambda'$ satisfy $T(x,y) \in \Lambda_a$, $T(x',y') \in \Lambda_a$ for some $a \in \mathbf{Z}_+$. Then we have

$$n_2 \cdot x - m_2 \cdot y = n_2 \cdot x' - m_2 \cdot y'$$

or equivalently

$$n_2(x-x') = m_2(y-y') .$$

Since $(m_2, n_2) = 1$ this is equivalent to

$$x - x' = c \cdot m_2, \qquad y - y' = c \cdot n_2$$

for some $c \in \mathbf{Z}$. But then we have

$$\begin{aligned} -n_1 \cdot x + m_1 \cdot y &= -n_1(c \cdot m_2 + x') + m_1(y' + c \cdot n_2) \\ &= -n_1 \cdot x' + m_1 \cdot y' - c(n_1 \cdot m_2 - m_1 \cdot n_2) \\ &= -n_1 \cdot x' + m_1 \cdot y' + c \cdot p \end{aligned}$$

It is easy to see that this proves the lemma.

Thus we have a complete description of Λ given by

$$\Lambda = \{(a,b) \in \mathbf{Z}_+^2 \mid a \cdot \xi \equiv b (\operatorname{mod} p)\} .$$

If we interchange (m_1, n_1) and (m_2, n_2) and apply the proof of Lemma 2.1 we get a number $\eta \in \mathbf{Z}_+$ satisfying

i) $0 < \eta < p$

ii) $\eta \cdot \xi \equiv 1 \pmod p$

The use of this will appear later.

REMARK 2.2. One of the advantages with this description of Λ is the following property of Λ: If $\lambda = (a,b)$, $\lambda' = (a',b') \in \Lambda$ and if $\lambda' - \lambda = (a'-a, b'-b) \in \mathbf{Z}_+^2$, then $\lambda' - \lambda \in \Lambda$.

In fact since for (a,b), $(a',b') \in \Lambda$; $b \equiv a \cdot \xi (\operatorname{mod} p)$, $b' \equiv a' \cdot \xi (\operatorname{mod} p)$, and $a' - a \geqq 0$, $b' - b \geqq 0$, we find $b' - b = (a'-a) \cdot \xi (\operatorname{mod} p)$ therefore $(a'-a, b'-b) \in \Lambda$. Notice that this implies that the order relation on Λ (see section 1) induced by the order relation on Λ' is the restriction of the ordinary order relation on $\mathbf{Z}_+^2$.

DEFINITION 2.3. Let $P \in \mathbf{Z}_+^2$. Define the ordered set $\hat{P}$ associated with P by $\hat{P} = \{\lambda \in \Lambda \mid \lambda \leqq P\} \subseteqq \Lambda$. The associated simplicial set will also be denoted by $\hat{P}$.

Correspondingly we shall let $L(P) = \{\lambda \in \Lambda \mid \lambda \lneqq P\}$ also denote the associated simplicial set. (When $P \in \Lambda$, this is precisely the set $E(P)$ of section 1, see Fig. 2, 3, and 4.)

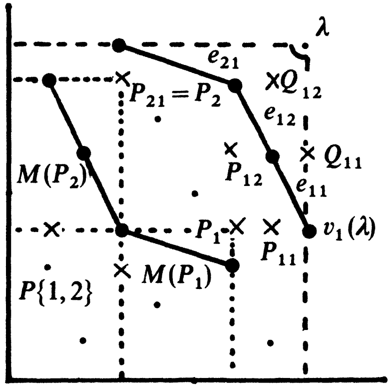

Fig. 3. $L(\lambda), M(\lambda)$

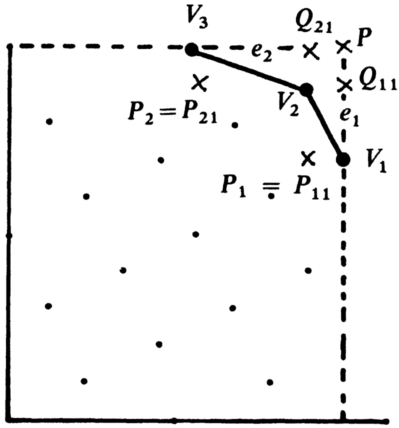

Fig. 4. $\hat{P}, M(P)$

REMARK 2.4. Notice that for $P \in \mathbf{Z}_+^2 - \Lambda$ we have $L(P)=\hat{P}$.

LEMMA 2.5. *Let ξ and η be defined as above. Let $U \subseteqq \mathbf{Z}_+^2$ be the set defined by*

$$U = \{(a,b) \in \mathbf{Z}_+^2 \mid b > p + a\cdot\xi \text{ or } a > p + b\cdot\eta\}\,.$$

Then for any $P \in U$

$$\tilde{H}_n(L(P)) = 0 \qquad n \geqq 0\,.$$

PROOF. It is obviously sufficient to prove the lemma in the case where $P=(a,b)$ satisfies the condition $b>p+a\cdot\xi$. Given $P=(a,b)\in\mathbf{Z}_+^2$, and suppose $b>p+a\cdot\xi$. Then there exist integers $\alpha,\beta\in\mathbf{Z}$ such that

$$b-a\cdot\xi = \alpha\cdot p+\beta$$

with $0<\beta\leqq p$ and $\alpha\geqq 1$. We shall prove the lemma by induction on the integer a.

Suppose $a=0$. Then $L(P)$ has a final object and the homology vanishes.

Suppose $a>0$. Let $P=(a,b)\in U$, and suppose the formula is valid for all $(m,c)\in U$ with $m<a$. Notice that Lemma 2.1 implies $(a,b-\beta)=(a,a\cdot\xi+\alpha\cdot p)\in\Lambda$.

Now it is easy to see that

i) $L(P) = (a-1,b)\hat{} \cup (a,b-\beta)\hat{}$

ii) $(a-1,b-\beta)\hat{} = (a-1,b)\hat{} \cap (a,b-\beta)\hat{}$.

Apply the Mayer-Vietoris sequence and obtain the long exact sequence

$$\ldots \to \tilde{H}_n(a-1,b-\beta) \to \tilde{H}_n(a,b-\beta) \oplus \tilde{H}_n(a-1,b) \to$$
$$\to \tilde{H}_n(L(P)) \to \tilde{H}_{n-1}(a-1,b-\beta) \to \ldots$$

where $\tilde{H}.(P)$ is the homology of the ordered set associated with P. But now we have $b > p + a \cdot \xi > p + (a-1) \cdot \xi$ and $b - \beta = \alpha \cdot p + a \cdot \xi \geqq p + a \cdot \xi > p + (a-1) \cdot \xi$, so $(a-1,b) \in U$ and $(a-1,b-\beta) \in U$. The induction hypothesis implies

$$\tilde{H}_n(a-1,b-\beta) = \tilde{H}_n(a-1,b) = 0 \quad \forall n \geqq 0 .$$

$(a,b-\beta) \in \Lambda$ and $(a,b-\beta)\hat{}$ has a final object; therefore

$$\tilde{H}_n(a,b-\beta) = 0 \quad \forall n \geqq 0 .$$

Thus, using the exactness of the above sequence, we get

$$\tilde{H}_n(P) = 0 \quad \forall n \geqq 0$$

which proves the lemma.

Definition 2.6. Let $P \in \mathbf{Z}_+^2$. The maximal polygon associated with $P, M(P)$ is the set of maximal elements of the convex hull of $L(P)$ in $\mathbf{R}_+^2$, see Fig. 3 and 4.

Put $M_0(P) = L(P) \cap M(P)$. Then the following lemma holds.

Lemma 2.7. *$M_0(P)$ is the set of maximal elements of $L(P)$.*

Proof. Let $\max L(P)$ be the set of maximal elements of $L(P)$. Obviously $M_0(P) \subseteqq \max L(P)$. Assume $\lambda \in \max L(P)$ and $\lambda \notin M_0(P)$. $M(P)$ is a convex polygon and λ has to sit strictly below some edge e. Pick vertices of e, $\mu, \mu' \in M_0(P)$, $\mu \neq \mu'$, and consider the element $\eta = \mu + \mu' - \lambda$. Since $\eta \in \mathbf{Z}_+^2$ we have seen (Remark 2.2) that $\eta \in \Lambda$. An easy argument then shows that $\eta \in L(P)$ and that η is above the edge e, a contradiction.

It is easily seen that $M(P)$ must lie inside a square, $p \times p$, with P as the maximal point.

Lemma 2.8. *For every $P \in \mathbf{Z}_+^2$ with $P \geqq (p,p)$, and every $\lambda \in \Lambda$*

$$M(P+\lambda) = M(P) + \lambda .$$

Proof. It is enough to show the equality $M_0(P+\lambda) = M_0(P) + \lambda$. So let $\mu \in M_0(P)$. Then $\lambda \leqq \mu + \lambda < P + \lambda$. Now choose $\eta \in M_0(P+\lambda)$ such that $\lambda \leqq \mu + \lambda \leqq \eta < P + \lambda$. Then we have $\mu \leqq \eta - \lambda < P$. Since $\mu, \eta, \lambda \in \Lambda$, the remark (2.2) implies $\eta - \lambda \in \Lambda$, thus we get $\mu = \eta - \lambda$ or $\eta = \mu + \lambda$. Consequently $\mu + \lambda \in$

$M_0(P+\lambda)$ and $M_0(P)+\lambda \subseteqq M_0(P+\lambda)$. To prove the inverse inclusion, we first notice that if $\mu \in M_0(P+\lambda)$, then $\mu \geqq \lambda$. This follows from the fact that $P \geqq (p, p)$ and that $M_0(P+\lambda)$ sits inside a square $p \times p$ with $P+\lambda$ as the maximal point.

So let $\mu \in M_0(P+\lambda)$. Then $\mu < P+\lambda$ or $\mu - \lambda < P$. Choose $\eta \in M_0(P)$ such that $\mu - \lambda \leqq \eta < P$. This implies $\mu \leqq \eta + \lambda < P + \lambda$. But $\mu \in M_0(P+\lambda)$ so the last equation implies $\mu = \eta + \lambda$, which proves the lemma.

DEFINITION 2.9. Let $P \in \mathbf{Z}_+^2$ and denote by

$$\{V_{i,j}(P) \mid i=1,2,\ldots,n;\ j=1,2,\ldots,m_i\}$$

the lattice points on $M(P)$ where i is the number of the edge counted from right, and j is the number of the lattice point on the edge, also counted from right. (See Fig. 2, 3, and 4.)

Put $V_i = V_{i,1}$ for $i=1,2,\ldots,n$ and $V_{n+1} = V_{n,m}$. Notice that for $i=1,2,\ldots,n$ we have $m_i \geqq 2$ and $V_{i,m_i} = V_{i+1}$.

Denote by

$$\{e_{i,j}(P) \mid i=1,2,\ldots,n;\ j=1,\ldots,m_i\}$$

the edges between $V_{i,j}(P)$ and $V_{i,j+1}(P)$. For $i=1,\ldots,n$,

$$e_i(P) = \bigcup_{j=1}^{m_i-1} e_{i,j}(P)$$

are then the edges of $M(P)$.

Let $\{S_i(P)\}_{i=1,\ldots,n}$ be the absolute values of the slopes of the $e_i(P)$'s and let finally

$$\{X_i(P)\}_{i=1,\ldots,n}, \qquad X_i = X(V_{i,2}) - X(V_{i,1})$$

and

$$\{Y_i(P)\}_{i=1,\ldots,n}, \qquad Y_i = Y(V_{i,2}) - Y(V_{i,1})$$

be the differences in the values of the coordinates of $V_{i,1}(P)$ and $V_{i,2}(P)$.

It is clear that $M(P)$ is determined by these families of numbers. Moreover, we deduce the following

$$Y_i(P) = S_i(P) \cdot X_i(P) \qquad i=1,\ldots,n\,.$$

Put, as a shorthand, $\alpha_i(P) = X(P) - X(V_i(P))$ and $\beta_i(P) = Y(P) - Y(V_i(P))$, and notice that $\alpha_{i+1}(P) > \alpha_i(P)$, $\beta_{i+1}(P) < \beta_i(P)$.

For every pair (i,j), $i=1,\ldots,n$, $j=1,\ldots,m_i$ the proof of Lemma 2.7 gives the existence of unique points

$$Q_{i,j}(P) = \big(X(V_{i,j}(P)),\, Y(V_{i,j+1}(P))\big)$$

and

$$P_{i,j}(P) = (X(V_{i,j+1}(P)), Y(V_{i,j}(P)))$$

with the properties

$$L(Q_{i,j}(P)) = V_{i,j}(P)\hat{} \cap V_{i,j+1}(P)\hat{}$$
$$P_{i,j}(P)\hat{} = V_{i,j}(P)\hat{} \cap V_{i,j+1}(P)\hat{} .$$

(See Fig. 3 and 4.)

DEFINITION 2.10. Denote by P_i the unique element of $\mathbf{Z}_+^2$ such that $P_i\hat{} = \bigcap_{j=1}^{m_i} P_{i,j}\hat{}$.

Let $\lambda \in \Lambda$ and let n be the number of edges of $M(\lambda)$. The next lemma will show that $M(P_i(\lambda))$ is congruent to the polygon $M(\lambda)$ with the ith edge removed. We shall therefore index the vertices and the edges etc. of $M(P_i(\lambda))$ by restricting the corresponding indexing of $M(\lambda)$. Thus $e_i(P_i(\lambda))$ does not exist and, modulo translation, $e_j(P_i(\lambda))$ is congruent to $e_j(\lambda)$ whenever $i \neq j$. Likewise $V_i(P_i(\lambda))$ does not exist and $V_{i-1,m_{i-1}}(P_i(\lambda)) = V_{i+1}(P_i(\lambda))$. Notice that the intersection points $P_j(P_i(\lambda))$ and $P_i(P_j(\lambda))$ are, in general, different when $i \neq j$. Let $P_{\{i,j\}}(\lambda)$ denote their intersection, i.e. the unique element of $\mathbf{Z}_+^2$ such that

$$P_{\{i,j\}}(\lambda)\hat{} = P_i(P_j(\lambda))\hat{} \cap P_j(P_i(\lambda))\hat{}, \quad \text{see Fig. 3.}.$$

In general we make the following definition ($\lambda \gg 0$ means $X(\lambda), Y(\lambda) \gg 0$).

DEFINITION 2.11. Let $\lambda \in \Lambda$ and $M(\lambda)$ as above, $\lambda \gg 0$. Let $I \subseteqq \{1,2,\ldots,n\}$ be a set of integers different from the empty set. Define $P_I(\lambda)$ recursively via the intersection property

$$P_I(\lambda)\hat{} = \bigcap_{i \in I} P_i(P_{I-\{i\}}(\lambda))\hat{}$$

where $P_{\varnothing}(\lambda) = \lambda$.

Lemma 2.12 will show that $M(P_{\{i,j\}}(\lambda))$ is congruent to $M(\lambda)$ with the ith and the jth edge removed, and that in general $M(P_I(\lambda))$ is congruent to $M(\lambda)$ with the ith edge removed for every $i \in I \subseteqq \{1,2,\ldots,n\}$.

LEMMA 2.12. *Let $\lambda, M(\lambda)$ be as above and let $I \subseteqq \{1,2,\ldots,n\}$ be a set of integers, the empty set included.*

i) *The maximal polygon $M(P_I(\lambda))$ of the set $P_I(\lambda)\hat{}$ is congruent to the maximal polygon $M(\lambda)$ of $\lambda\hat{}$ with the ith edge removed for every $i \in I$.*

ii) *Let for* $i=1,2,\ldots,n$, $r_i=(\alpha_i,\beta_i)$. *Then for every* $j\notin I$

$$P_j(P_I(\lambda)) = \lambda-\sum_{i\in I} r_i-r_{n+1}-\sum_{\substack{h\notin I\\ h\geqq j}} e_h-(\alpha_{j+1}-\alpha_j,0)\,,$$

where e_h *is the vector* $\overrightarrow{V_hV_{h+1}}$ *associated to the edge* $e_h(\lambda)$, *and* $\alpha_i=\alpha_i(\lambda)$, $\beta_i=\beta_i(\lambda)$.

Proof. We shall prove the lemma by induction on the number of elements of I, $\#I=k$.

The case $k=0$ is vacuous; just notice that $e_h=r_h-r_{h+1}$ so

$$\lambda-r_{n+1}-\sum_{\substack{h\notin I\\ h\geqq j}} e_h = \lambda-r_j\,.$$

Suppose the lemma holds for $\#I=k-1$, $0<k\leqq n$, and let $I\subseteq\{1,.,\ldots,n\}$ with $\#I=k$. To simplify notation, write for every $i\in I$; $P_{I,i}(\lambda)=P_i(P_{I-\{i\}}(\lambda))$. Obviously

$$P_I(\lambda)\hat{} = \bigcap_{i\in I} P_{I,i}(\lambda)\hat{} = \left(\min_{i\in I} X(P_{I,i}(\lambda)), \min_{i\in I} Y(P_{I,i}(\lambda))\right)\hat{}$$

so we have to study the relation between the intersection points $P_{I,i}(\lambda)$. The induction hypothesis gives

$$(**)\qquad P_{I,j}(\lambda) = \lambda-\sum_{i\in I-\{j\}} r_i-r_{n+1}-\sum_{\substack{h\notin I-\{j\}\\ h\geqq j}} e_h-(\alpha_{j+1}-\alpha_j,0)$$

$$= \lambda-\sum_{i\in I} r_i+\sum_{\substack{h\in I\\ h>j}} e_h-(\alpha_{j+1}-\alpha_j,0)\,.$$

Consider the last part of the above sum, $\sum_{h\in I,h>j}e_h+(X(e_j),0)$. The fact that $\alpha_{j+1}>\alpha_j$ and $\beta_{j+1}<\beta_j$ shows that the X-value of this vector increases and the Y-value decreases with increasing $j\in I$. So it follows that

$$P_I(\lambda)\hat{} = P_{I,i_1}(\lambda)\hat{}\cap P_{I,i_k}(\lambda)\hat{}$$
$$= (X(P_{I,i_1}(\lambda)), Y(P_{I,i_k}(\lambda)))\hat{}$$

where $I=\{i_1<i_2<\ldots<i_k\}$. From (**) we deduce that $X(P_I(\lambda))=X(P_{I,i_1}(\lambda))=X(\lambda-\sum_{i\in I}r_{i+1})$ and $Y(P_I(\lambda))=Y(P_{I,i_k}(\lambda))=Y(\lambda-\sum_{i\in I}r_i)$. In addition we get the two inequalities

$$P_{I,j_1}(\lambda) < -\sum_{i\in I} r_{i+1}$$

$$P_{I,i_k}(\lambda) < \lambda-\sum_{i\in I} r_i\,.$$

Obviously $\lambda-\sum_{i\in I} r_i \geqq \lambda-\sum_{i\in I} r_{i+1}-r_1$ and $\lambda-\sum_{i\in I} r_{i+1} \geqq \lambda-\sum_{i\in I} r_i-r_{n+1}$ and therefore

$$\lambda-\sum_{i\in I} r_{i+1}-r_1 < P_I(\lambda) \quad \text{and} \quad \lambda-\sum_{i\in I} r_i-r_{n+1} < P_I(\lambda).$$

Thus $\lambda-\sum_{i\in I} r_{i+1}-r_1$ and $\lambda-\sum_{i\in I} r_i-r_{n+1}$ are the "endpoints" of the maximal polygon of $P_I(\lambda)$.

Using the fact that $\sum_{h=1}^{n} e_h = r_1-r_{n+1}$ we have the equalities

$$\begin{aligned}
\lambda-\sum_{i\in I} r_{i+1}-r_1 &= \lambda-\sum_{i\in I} r_{i+1}-r_{n+1}-\sum_{h=1}^{n} e_h \\
&= \lambda-\sum_{i\in I} (r_{i+1}-r_i)-\sum_{i\in I} r_i-r_{n+1}-\sum_{h=1}^{n} e_h \\
&= \lambda-\sum_{i\in I} r_i-r_{n+1}-\sum_{h\notin I} e_h
\end{aligned}$$

This proves part i).

To prove ii) observe that i) implies

$$\begin{aligned}
X(P_j(P_I(\lambda))) &= X\Bigg(\lambda-\sum_{i\in I} r_{i+1}-r_1+\sum_{\substack{h\notin I\\ h\leqq j}} e_h\Bigg) \\
&= X\Bigg(\lambda-\sum_{i\in I} r_i-r_{n+1}-\sum_{\substack{h\notin I\\ h\geqq j}} e_h+e_j\Bigg).
\end{aligned}$$

We already know

$$Y(P_j(P_I(\lambda))) = Y\Bigg(\lambda-\sum_{i\in I} r_i-r_{n+1}-\sum_{\substack{h\notin I\\ h\geqq j}} e_h\Bigg)$$

and therefore

$$P_j(P_I(\lambda)) = \lambda-\sum_{i\in I} r_i-r_{n+1}-\sum_{\substack{h\notin I\\ h\geqq j}} e_h+(X(e_j),0),$$

which is the claimed equation for $P_j(P_I(\lambda))$, $\# I=k$.

Corollary 2.13. *$P_I(\lambda)\in \Lambda$ if and only if $I=\{1,2,\ldots,n\}$ or $I=\varnothing$.*

Proof. $0\leqq\sum_{i\in I}\alpha_{i+1}-\alpha_i\leqq p$ with equality on the left or right if and only if $I=\varnothing$, respectively $I=\{1,2,\ldots,n\}$.

In the next few lemmas we shall relate the homology of $L(P)$ to the homology of ordered sets connected with $M(P)$. Let $P \in \mathbf{Z}_+^2$ and assume $P \gg 0$. Put $M = M(P)$, $V_i = V_i(P)$, etc.

LEMMA 2.14. *In the situation above we have an isomorphism for every* $r \geqq 0$

$$\bigoplus_{j=1}^{m_i-1} \tilde{H}_r(P_{i,j}) \cong \bigoplus_{j=2}^{m_i-1} \tilde{H}_r(L(V_{i,j})) \oplus \tilde{H}_r(P_i).$$

PROOF. Define $V = V_i - r_{i+1} \in \Lambda$. Then for $j = 1, 2, \ldots, m_i$

$$\begin{aligned} \hat{P_{i,j}} &= (X(P_{i,j}), Y(V))\hat{} \cup (X(V), Y(P_{i,j}))\hat{} \\ \hat{V} &= (X(P_{i,j}), Y(V))\hat{} \cap (X(V), Y(P_{i,j}))\hat{}. \end{aligned}$$

The proof of this is left to the reader; an argument analogous to the proof of Lemma 2.7 will give the result.

Applying the reduced Mayer-Vietoris sequence, and using the fact that $\hat{V}$ has a final object, we get an isomorphism for $j = 1, 2, \ldots, m_i - 1$ and $r \geqq 0$

$$(***) \qquad \tilde{H}_r(P_{i,j}) \cong \tilde{H}_r(X(P_{i,j}), Y(V)) \oplus \tilde{H}_r(X(V), Y(P_{i,j})).$$

But we also have for $j = 2, 3, \ldots, m_i - 1$

$$\begin{aligned} L(V_{i,j}) &= (X(P_{i,j-1}), Y(V))\hat{} \cup (X(V), Y(P_{i,j}))\hat{} \\ \hat{V} &= (X(P_{i,j-1}), Y(V))\hat{} \cap (X(V), Y(P_{i,j}))\hat{}. \end{aligned}$$

So for every $r \geqq 0$

$$(****) \qquad \tilde{H}_r(L(V_{i,j})) \cong \tilde{H}_r(X(P_{i,j-1}), Y(V)) \oplus \tilde{H}_r(X(V), Y(P_{i,j})).$$

Summing over $j = 1, 2, \ldots, m_i - 1$ the isomorphisms (***), changing parentheses, and using (****) we get

$$\begin{aligned} \bigoplus_{j=1}^{m_i-1} \tilde{H}_r(P_{i,j}) &\cong \bigoplus_{j=2}^{m_i-1} \tilde{H}_r((V_{i,j})) \oplus \tilde{H}_r(X(V), Y(P_{j,1})) \oplus \tilde{H}_r(X(P_{i,m_i-1}), Y(V)) \\ &\cong \bigoplus_{j=2}^{m_i-1} \tilde{H}(L(V_{i,j})) \oplus \tilde{H}_r(P_i) \qquad \forall r \geqq 0 \end{aligned}$$

The next lemma gives the relation between the homology of $L(P)$ and the homology of the intersection points P_i.

LEMMA 2.15. *Let the symbols* $P, M, V_{i,j}$ *be as above; n is the number of edges of* M. *There is an isomorphism for every* $r > 0$

$$\tilde{H}_r(L(P)) \cong \left[\bigoplus_{i=1}^{n} \bigoplus_{j=2}^{m_i-1} \tilde{H}_{r-1}(L(V_{i,j})) \right] \oplus \left[\bigoplus_{i=1}^{n} \tilde{H}_{r-1}(P_i) \right].$$

PROOF. As a consequence of Lemma 2.7 we have

$$L(P) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i-1} \hat{Q_{i,j}}$$

where $Q_{i,j}=Q_{i,j}(P)$ and the intersections $\hat{Q_{i,j}} \cap \hat{Q_{i,j+1}}$ and $\hat{Q_{i,m_i-1}} \cap \hat{Q_{i+1,1}}$ always are ordered sets with $V_{i,j+1}$, respectively $V_{i+1,1}$, as final elements. Using the Mayer-Vietoris sequence repeatedly we find

$$\tilde{H}_r(L(P)) \cong \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i-1} \tilde{H}_r(Q_{i,j})\,.$$

Apply the Mayer-Vietoris sequence once more to the system $(\hat{Q_{i,j}}, \hat{V_{i,j}}, \hat{V_{i,j+1}}, \hat{P_{i,j}})$. Since $\hat{V_{i,j}}$ has a final element we obtain an isomorphism for every $r>0$

$$\tilde{H}_r(Q_{i,j}) \cong \tilde{H}_{r-1}(P_{i,j})$$

where $i=1,\ldots,n$, $j=1,\ldots,m_i-1$. Using Lemma 2.14 the lemma follows immediately.

LEMMA 2.16. *Let $\lambda \in \Lambda$ and let $I \subseteqq \{1,2,\ldots,n\}$. Suppose $2 \leqq \# I = k \leqq n$. Let $P_I = P_I(\lambda)$ and $P_{I,i} = P_i(P_{I-\{i\}}(\lambda))$. Then for every $r \geqq 0$ we have an isomorphism*

$$\bigoplus_{i\in I} \tilde{H}_r(P_{I,i}) \cong \bigoplus_{\substack{i\in I\\ i\neq i_k}} \tilde{H}_r(L(V_i(P_{I-\{i\}}))) \oplus \tilde{H}_r(P_I)\,,$$

where $I=\{i_1<\ldots<i_k\}$.

PROOF. Define $P_{I,i,j}$ via the intersection property

$$\hat{P_{I,i,j}} = \hat{P_{I,i}} \cap \hat{P_{I,j}}$$

for every pair $i,j \in I$. From the proof of Lemma 2.12 we deduce $\hat{P_{I,i_1,i_j}} = \hat{P_{I,i_1,i_{j-1}}} \cap P_{I,i_j}$ for every $j=2,\ldots,k$. For $j=1,\ldots,k-1$ we have the inequalities

$$\hat{P_{I,i_1,i_j}} < P_{I,i_j} < V_{i_j}(P_{I-\{i_j\}})$$

and from Lemma 2.12 the equality

$$(*****) \qquad P_{I,i_{j+1}} = V_{ij}(P_{I-\{i_j\}}) - (0, \beta_{i_{j+1}} - \beta_{i_{j+1}+1})\,.$$

Thus $P_{I,i_{j+1}} < V_{i_j}(P_{I-\{i_j\}})$. In addition we have the inequality $V_{i_j}(P_{I-\{i_j\}}) - r_{i_j+1} < P_{I,i_1,i_{j+1}}$. The last statement is an immediate consequence of the two relations

$$V_{i_j}(P_{I-\{i_j\}}) - r_{i_j+1} < P_{I,i_{j+1}}, \quad V_{i_j}(P_{I-\{i_j\}}) - r_{i_j+1} < P_{I,i_1}\,.$$

The first follows from equation (*****), the other is easily deduced from Lemma 2.12 using the analytic formula for P_{I,i_1}. Thus we have

i) $V_{i_j}(P_{I-\{i_j\}})-r_{i_j+1} \leqq P_{I,i_1,i_j} < V_{i_j}(P_{I-\{i_j\}})$

ii) $V_{i_j}(P_{I-\{i_j\}})-r_{i_j+1} \leqq P_{I,i_{j+1}} < V_{i_j}(P_{I-\{i_j\}})$

iii) $X(P_{I,i_{j+1}}) = X(V_{i_j}(P_{I-\{i_j\}}))$

iv) $Y(P_{I,i_1,i_j}) = Y(V_{i_j}(P_{I-\{i_j\}}))$.

Applying the Mayer-Vietoris sequence three times we obtain for every $r \geqq 0$ an isomorphism

$$\tilde{H}_r(P_{I,i_1,i_j}) \oplus \tilde{H}_r(P_{I,i_{j+1}}) \cong \tilde{H}_r(L(V_{i_j}(P_{I-\{i_j\}}))) \oplus \tilde{H}_r(P_{I,i_1,j_{j+1}}) .$$

But $P_{I,i_1,j_k} = P_I$ so an iterated use of the described process will give the lemma.

We are now in position to state and prove the main result of this paragraph.

THEOREM 2.17. *Let $\lambda \in \Lambda$, $\lambda \gg 0$ and $P_I = P_I(\lambda)$, as above. Let n be the number of edges of $M(\lambda)$. Then for every integer $r \geqq n$ there is an isomorphism*

$$\tilde{H}_r(L(\lambda)) = \left[\bigoplus_{k=1}^{n} \bigoplus_{\#I=k-1} \bigoplus_{i \notin I} \bigoplus_{j=2}^{m_i-1} \tilde{H}_{r-k}(V_{i,j}(P_I)) \right] \oplus$$

$$\oplus \left[\bigoplus_{k=2}^{n} \bigoplus_{\#I=k} \bigoplus_{\substack{i \in I \\ i \neq i_k}} \tilde{H}_{r-k}(L(V_i(P_{I-\{i\}}))) \right]$$

where $P_\varnothing = \lambda$ and $I = \{i_1 < \ldots < i_k\}$.

PROOF. This is just an iterated use of Lemma 2.15 and Lemma 2.16, where we for each step increase the order of I. Remember that if $I \neq \varnothing$, $P_I \in \Lambda$ if and only if $I = \{1, \ldots, n\}$. Therefore the process stops when $\#I = n$. Moreover, for $\#I < n$ we have $L(P_I) = \hat{P_I}$.

Now go back to the calculation of the right-hand side of the equation in Proposition 1.3. In Theorem 2.17 we made the assumption $\lambda \gg 0$. In fact it suffices to know that $\lambda > \sum_{i=1}^{n+1} r_i$. This is to ensure that all the points needed in Lemma 2.16 really are elements of Λ.

Put

$$Z = \left\{ \lambda \in \Lambda \;\middle|\; \lambda > \sum_{i=1}^{n+1} r_i \right\}$$

and recall the definition of

$$U = \{(a,b) \in \mathbf{Z}_+^2 \mid b > p + a \cdot \xi \text{ or } a > p + b \cdot \xi\} ,$$

see (2.5). Put

$$W = (\Lambda - Z) \cap (\Lambda - U).$$

W is a finite set containing all $\lambda \in \Lambda - Z$ with the property $\tilde{H}_.(\lambda) \neq 0$. Since for each $\lambda \in \Lambda$, $L(\lambda)$ is a finite ordered set, there exist N' such that $\tilde{H}_m(L(\lambda)) = 0$ for all $m \geqq N'$. Since W is finite we may choose N' such that $\tilde{H}_m(L(\lambda)) = 0$ for all $m \geqq N'$ and all $\lambda \in W$. Putting $h_m(L(\lambda)) = \dim_k \tilde{H}(L(\lambda))$ we have thus proved

$$\sum_{\lambda \in Z} h_m(L(\lambda)) = \sum_{\lambda \in \Lambda} h_m(L(\lambda))$$

for every $m \geqq N'$. Going back to Theorem 2.17 we see that the problem is to calculate the number $\sum_{\lambda \in Z} h_{m-k}\big(L\big(V_{i,j}(P_I(\lambda))\big)\big)$. So we need a lemma.

LEMMA 2.18. *Let $Z \subseteqq \Lambda$ and N' be defined as above. Let $N = N' + n$. Pick $m \geqq N$ and let (k, I, i, j) be a quadruple which occurs in Theorem 2.17. Then we have the equality*

$$\sum_{\lambda \in Z} h_{m-k}\big(L\big(V_{i,j}(P_I(\lambda))\big)\big) = \sum_{\lambda \in Z} h_{m-k}(L(\lambda)).$$

PROOF. The map $\lambda \mapsto V_{i,j}(P_I(\lambda))$ from Z into Λ, is obviously a rigid translation. Of course we have $\lambda \geqq V_{i,j}(P_I(\lambda))$ so

$$Z \subseteqq \{\lambda \in \Lambda \mid \exists \lambda' \in Z \text{ with } \lambda = V_{i,j}(P_I(\lambda'))\}.$$

Let $\lambda' \in Z$ with $V_{i,j}(P_I(\lambda')) \notin Z$. We have $m - k \geqq N - k \geqq N'$ and by definition of N'; $h_{m-k}\big(L\big(V_{i,j}(P_I(\lambda'))\big)\big) = 0$. Since

$$\sum_{\lambda \in Z} h_{m-k}\big(L\big(V_{i,j}(P_I(\lambda))\big)\big) = \sum_{\lambda \in Z} h_{m-k}(L(\lambda)) + \sum_{\lambda' \in Z'} h_{m-k}\big(L\big(V_{i,j}(P_I(\lambda'))\big)\big)$$

where $Z' \equiv \{\lambda' \in Z \mid V_{i,j}(P_I(\lambda')) \notin Z\}$, we have proved the lemma.

THEOREM 2.19. *Let the number N be as above. Let for every $m \geqq N, \gamma_m = \sum_{\lambda \in \Lambda} h_m L(\lambda)$. Then there exists a recursion in the γ's: $\gamma_m = \sum_{k=1}^{h} R_k \cdot \gamma_{m-k}$ given by*

$$R_k = \binom{n-1}{k-1} \cdot S + \binom{n}{k}(k-1) \qquad k = 1, 2, \ldots, n,$$

where n is the number of edges of the maximal polygon $M(\lambda)$ of λ, $\lambda \gg 0$, and $S = \sum_{i=1}^{n} (m_i - 2)$, where m_i is the number of lattice points on the ith *edge of $M(\lambda)$.*

PROOF. Due to Lemma 2.18 and Theorem 2.17 the only problem is to calculate the sums $(I = \{i_1 < \ldots < i_k\})$

$$S_1 = \sum_{\#I=k-1} \sum_{i \notin I} \sum_{j=1}^{m_i-1} \gamma_{m-k}$$

$$S_2 = \sum_{\#I=k} \sum_{\substack{i \in I \\ i \neq i_k}} \gamma_{m-k} \,.$$

This is a purely combinatorial problem and it is easy to show that

$$S_1 = \binom{n-1}{k-1} \cdot S \cdot \gamma_{m-k}$$

$$S_2 = \binom{n}{k} \cdot (k-1) \cdot \gamma_{m-k}$$

which proves the theorem.

COROLLARY 2.20. *Let $\Lambda' \subseteqq \mathbf{Z}_+^2$ be a saturated rational monoid, and let $k[\Lambda']$ be the associated monoid algebra. Consider the corresponding isolated singularity of the affine scheme $X = \operatorname{Spec} k[\Lambda']$. The Betti series $B(t) = \sum_{n \geqq 0} \beta_m t^m$ of the local ring of this singularity is rational with denominator*

$$-1 + \sum_{k=1}^{n} \left[\binom{n-1}{k-1} \cdot S + \binom{n}{k} (k-1) \right] t^k \,.$$

PROOF. Follows immediately from Theorem 2.17 and the formula of Proposition 1.3 implying $\beta_m = \gamma_{m-2}$ for $m \gg 0$.

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