MATH. SCAND. 56 (1985), 163–170

HILBERT SCHEMES OF HYPERSURFACES AND NUMERICAL CRITERIONS

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Introduction.

By a n-hypersurface in $\boldsymbol{P}^N$ we will mean a hypersurface in a $(n+1)$-plane in $\boldsymbol{P}^N$. Let $P_{n,d}$ denote the Hilbert polynomial of a n-hypersurface of degree d. In this paper we prove that the Hilbert scheme $\mathrm{Hilb}_{\boldsymbol{P}^N}^{P_{n,d}}$ consists of n-hypersurfaces only. This generalizes a result of P. Ionescu [9].

From this we prove that $P_{n,d}$ is the minimal Hilbert polynomial of subschemes in $\boldsymbol{P}^N$ of dimension n and degree d. Furthermore write the Hilbert polynomial of X

$$HP_X(t) = d\binom{t+n}{n} + a_1\binom{t+n-1}{n-1} + \ldots + a_n.$$

Then we get $a_1 \geqq -\binom{d}{2}$ and if X is of pure dimension and no embedded components, equality implies that X is an n-hypersurface.

Finally we relate this criterion to criterions given by I. Vainsencher [12] and M. Dale [3].

1. Hilbert schemes of hypersurfaces.

Let S be a noetherian scheme, V a locally free O_S-Module of rank $N+1$. By a n-hypersurface of degree d in $\boldsymbol{P}(V)$ we will mean a closed subscheme $Z \subset \boldsymbol{P}(V)$, flat over S, such that for each $s \in S$ the fiber $Z_s \subset \boldsymbol{P}(V(s))$ is a hypersurface of degree d in some $(n+1)$-plane in $\boldsymbol{P}(V(s))$. If $n = N-1$ we will simply say that Z is a hypersurface.

LEMMA 1.1. *Let $Z \subset \boldsymbol{P}(V)$ be a n-hypersurface of degree $d > 1$ and suppose $n > 0$. Then there is a locally free $(n+2)$-quotient W of V, unique up to equivalence of quotients, such that Z is a hypersurface in $\boldsymbol{P}(W)$.*

PROOF. Since this proof is very similar to the proof of Proposition 3.2 in [1] we skip some of the details.

Let W be the direct image of $O_Z(1)$ on S. We have $h^1(Z_s, O_{Z_s}(1)) = 0$ and since $n > 0$ and $d > 1$, $h^0(Z_s, O_{Z_s}(1)) = n+2$. Thus W is locally free of rank $n+2$ and its formation commutes with base change.

Received November 9, 1983.

The adjoint of $V_Z \to O_Z(1)$ gives a surjection $V \to W$ by Nakayama's lemma. Pulling this back to Z we see that $V_Z \to O_Z(1)$ factors through W_Z. Thus the embedding of Z in $\boldsymbol{P}(V)$ factors through $\boldsymbol{P}(W)$.

The uniqueness is clear since $\boldsymbol{P}(W(s))$ is the linear span of Z_s for each $s \in S$.

LEMMA 1.2. *Let $\varphi: T \to S$ be a base change. If $Z \subset \boldsymbol{P}(V)$ is a n-hypersurface, then $Z_T \subset \boldsymbol{P}(V_T)$ is a n-hypersurface. The converse holds if φ is surjective.*

PROOF. It is enough to show that for a field extension $k \subset K, Z \subset \boldsymbol{P}_k^N$ is a n-hypersurface if and only if $Z_K \subset \boldsymbol{P}_K^N$ is a n-hypersurface.

The only if part is trivial. For the converse notice that

$$\dim_k H^0(\boldsymbol{P}_k^N, I_Z(1)) = \dim_K H^0(\boldsymbol{P}_K^N, I_{Z_K}(1)),$$

where I_Z (respectively I_{Z_K}) is the ideal of $Z \subset \boldsymbol{P}_k^N$ (respectively $Z_K \subset \boldsymbol{P}_K^N$). Therefore $Z_K \subset \boldsymbol{P}_K^{n+1} \subset \boldsymbol{P}_K^N$ gives a $\boldsymbol{P}_k^{n+1} \subset \boldsymbol{P}_k^n$ containing Z. Z is of pure dimension n and without embedded components [6, IV. 4.2.7] and therefore Z is a divisor in this $\boldsymbol{P}_k^{n+1}$ [6, IV. 21.7].

This lemma shows that there is a contravariant functor from noetherian S-schemes to sets,

$$T \to \{n\text{-hypersurfaces of degree } d \text{ in } \boldsymbol{P}(V_T)\}$$

Put $G = \mathrm{Grass}_{n+2}(V)$ and let Q denote the universal $(n+2)$-quotient.

LEMMA 1.3. *For $d > 1$ and $n > 0$ the above mentioned functor is represented by $X = \boldsymbol{P}_G(S^d(Q)^\vee)$.*

PROOF. To give a n-hypersurface in $\boldsymbol{P}(V_T)$ is equivalent to give a hypersurface in $\boldsymbol{P}(W)$ for an $(n+2)$-quotient W in V_T by Lemma 1.1.

This quotient corresponds to a S-morphism $T \to G$ and the hypersurface corresponds to a lifting of this morphism to $\boldsymbol{P}(S^d(Q)^\vee)$ by [1, 3.1].

$$\text{Let } P_{n,d} = \binom{t+n+1}{n+1} - \binom{t-d+n+1}{n+1} = \sum_{i=0}^{n} (-1)^i \binom{d+1}{i+1} \binom{t+n-i}{n-i}$$

denote the Hilbert polynomial of a n-hypersurface of degree d.

THEOREM 1.4. *If $d > 1$ and $n > 0$ there is a canonical S-isomorphism*

$$\boldsymbol{P}_G(S^d(Q)^\vee \cong \mathrm{Hilb}^{P_{n,d}}_{\boldsymbol{P}(V)/S}$$

PROOF. Put $X = \boldsymbol{P}_G(S^d(Q)^\vee)$ and $H = \mathrm{Hilb}^{P_{n,d}}_{\boldsymbol{P}(V)/S}$. The n-hypersurface functor is a subfunctor at the Hilbert functor and we get a morphism $\varphi : X \to H$. Let Y denote the schemetheoretic image of X. Since X is proper, $\varphi : X \to Y$ is surjective. By Lemma 1.2, Y consists of all points in H corresponding to n-hypersurfaces. By the universal property of X we see that $\varphi : X \to Y$ is an isomorphism.

Since a n-hypersurface is the same as an embedded flat family of complete intersections of type $(d,1,\dots,1)$ with $N-n-1$ ones, we conclude from [9, Proposition 1] that Y is open in H.

Since Y is flat over S the ideal of Y in H commutes with base change. Thus to show that $Y = H$ we may assume that S is connected and $V \cong O_S^{N+1}$. In this case H is connected [7] and we are finished.

REMARKS. (1) In [9, Proposition 3] Ionescu proves the case $n = N-1$ of this theorem. In an unpublished talk at the "Week of Algebraic Geometry". Bucharest 1980, Ionescu proved, by induction on the codimension, that the Hilbert schemes $\mathrm{Hilb}^{P_{n,d}}_{\boldsymbol{P}^N}$ consist of n-hypersurfaces only.

(2) When $d = 1$,

$$\mathrm{Hilb}^{P_{n,1}}_{\boldsymbol{P}(V)/S} = \mathrm{Grass}_{n+1}(V).$$

This is proved by Altman and Kleiman [1, Proposition 32], see also Remark 3 in Ionescu's article.

2. Numerical criterion.

Let $\boldsymbol{P}^N$ denote projective N-space over a field k. For the sake of simplicity we suppose that k is algebraically closed. For a closed subscheme $X \subset \boldsymbol{P}^N$ we denote its Hilbert polynomial

$$HP_X(t) = d\binom{t+n}{n} + a_1\binom{t+n-1}{n-1} + \dots + a_n.$$

The ring $\mathbb{Q}[t]$ of rational polynomials is ordered by putting $f \geqq g$ if $f(t) \geqq g(t)$ for $t \gg 0$. This is equivalent to the lexicographic order in the coefficients of both the expansion

$$f(t) = p_0 t^n + p_1 t^{n-1} + \dots + p_n$$

and the binomial expansion

$$f(t) = q_0\binom{t+n}{n} + q_i\binom{t+n-1}{n-1} + \ldots + q_n.$$

THEOREM 2.1. *If $X \subset P^N$ is a closed subscheme of dimension n and degree d, then $HP_X \geqq P_{n,d}$. If $n > 0$, equality implies that X is a n-hypersurface.*

PROOF. The last statement is a corollary of 1.4. We prove the first statement by induction. For $n = 0$, $HP_X = d = P_{0,d}$.

If $n > 0$, let $H \subset P^N$ be a hyperplane with $H \cap \operatorname{Ass} X = \varnothing$ and put $X' = X \cap H$. We have

$$\begin{aligned} HP_{X'}(t) &= HP_X(t) - HP_X(t-1) \\ &= d\binom{t+n-1}{n-1} + a_1\binom{t+n-2}{n-2} + \ldots + a_{n-1}. \end{aligned}$$

Suppose $HP_X \leqq P_{n,d}$, this implies

$$a_i \leqq (-1)^i\binom{d}{i+1}, \quad i = 1, \ldots, n.$$

By induction $HP_{X'} \geqq P_{n-1,d}$ and therefore

$$a_i = (-1)^i\binom{d}{i+1} \quad i = 1, \ldots, n-1.$$

Thus, $HP_X = P_{n,d} - m$, where m is a non-negative integer.

Let Y be the disjoint union of X and m points. Then $HP_Y = P_{n,d}$ and Y is a n-hypersurface. This is absurd if m is positive.

REMARK. For $n = 1$, the theorem simply says that

$$p_a \leqq \frac{(d-1)(d-2)}{2}$$

with equality if and only if X is a plane curve.

C. Peskine has shown me a direct proof of this result, based on a careful study of the sum

$$\sum_{i \geqq 1} h^1(X, O_X(i-1)) - h^1(X, O_X((i)).$$

From the theorem we see that $a_1 \geqq -\binom{d}{2}$. We will prove that equality implies that X is a n-hypersurface under mild conditions on X. For this we need some lemmas.

LEMMA 2.2. *Let $X \subset P^N$ be a closed subscheme with $h^0(X, O_X) = 1$ and let*

$H \subset \boldsymbol{P}^N$ be a hyperplane such that $H \cap \operatorname{Ass} X = \varnothing$. Put $X' = X \cap H$. Suppose that X' is contained in a m-plane in H, then X is contained in a $(m+1)$-plane in $\boldsymbol{P}^N$.

PROOF. Let I be the ideal of X in $\boldsymbol{P}^N, I'$ the ideal of X' in H. From the exact sequence

$$0 \to I \to O_{\boldsymbol{P}^N} \to O_X \to 0$$

and the assumption $h^0(X,O_X) = 1$, we get $h^1(\boldsymbol{P}^N,I) = 0$. From the exact sequence

$$0 \to I \to I(1) \to I'(1) \to 0,$$

we then get $h^0(\boldsymbol{P}^N,I(1)) = h^0(H,I'(1))$ and the lemma follows.

LEMMA 2.3. *Let $X \subset \boldsymbol{P}^N$ be a closed subscheme. Assume that there are no closed points in* Ass X, *then $H^0(X,O_X(-q)) = 0$ for $q \gg 0$.*

PROOF. If x is a closed point in X, then depth $O_{X,x} \geqq 1$ by assumption. Then the lemma follows by the same method as the proof of (i) $\Rightarrow$ (ii) in [8, III. 7.66].

LEMMA 2.4. *Let $X \subset \boldsymbol{P}^N$ be a closed subscheme of dimension $n \geqq 2$. Assume $X' = X \cap H$ is a $(n-1)$-hypersurface, where H is a hyperplane such that $H \cap \operatorname{Ass} X = \varnothing$.*

Then X is a n-hypersurface if and only if X is of pure dimension and without embedded components.

PROOF. If X is a n-hypersurface, then X has pure dimension and no embedded components.

Conversely, suppose that Y is of pure dimension and without embedded components. From the exact sequence

$$0 \to O_X \to O_X(1) \to O_{X'}(1) \to 0$$

we get an exact sequence

$$0 \to H^0(X,O_X(-q)) \to H^0(X,O_X(1-q)) \to H^0(X',O_{X'}(1-q)).$$

Since X' is a $(n-1)$-hypersurface, $H^0(X',O_{X'}(1-q)) = 0$ for $q > 1$ and therefore

$$H^0(X,O_X(-q)) \cong H^0(X,O_X(1-q)) \quad \text{for } q > 1.$$

From this and Lemma 2.3 we see that $H^0(X,O_X(-q)) = 0$ for $q \geqq 1$. Putting $q = 1$ in the sequence above we get

$$0 \to H^0(X,O_X) \to H^0(X',O_{X'}) \cong k$$

and therefore $h^0(X,O_X) = 1$. From Lemma 2.2 we conclude that $X \subset \mathbf{P}^{n+1} \subset \mathbf{P}^N$. Since X has pure dimension and no embedded components, X is a hypersurface in $\mathbf{P}^{n+1}$ by [6, IV. 21.7].

THEOREM 2.5. *Let $X \subset \mathbf{P}^N$ be a closed subscheme of pure dimension $n \geq 1$, degree d and no embedded components. Then $a_1 \geqq -\binom{d}{2}$ and equality holds if and only if X is a n-hypersurface.*

PROOF. It only remains to prove that $a_1 = -\binom{d}{2}$ implies that X is a n-hypersurface.

If $n = 1$ this is 2.1. For $n \geqq 2$, let H be a general hyperplane. Then $X' = X \cap H$ is of pure dimension $n-1$ and without embedded components by [6, IV.9.7.6]. By induction X' is a $(n-1)$-hypersurface and we conclude from Lemma 2.4 that X is a n-hypersurface.

We will now study the relation between this criterion and other criterions by Vainsencher and Dale. From now on let $X \subset \mathbf{P}^N$ be a projective variety (i.e. a reduced and irreducible closed subscheme) of dimension n and degree d.

Let

$$\omega_X = \operatorname{Ext}^{N-n}_{O_{\mathbf{P}^N}}(O_X, O_{\mathbf{P}^N}(-N-1))$$

be the dualizing sheaf and write its Hilbert polynomial

$$HP_{\omega_X}(t) = \sum_{i=0}^{n} b_i \binom{t+n-i}{n-i}.$$

Then $b_0 = d$ and we define $\deg_X \omega_X = b_1 - a_1$ (see Vainsencher [12]).

The proof of Vainsencher's criterion essentially proves the following theorem.

THEOREM 2.6. *Let $X \subset \mathbf{P}^N$ be a projective variety of dimension $n \geqq 1$ and degree d, then*

(1) $\qquad \deg_X \omega_X \leqq -2a_1 - (n+1)d$

(2) $\qquad \deg_X \omega_X = -2a_1 - (n+1)d$ *if X is Cohen-Macaulay.*

PROOF. Suppose X is Cohen-Macaulay, then by Serre duality

$$H^i(X,\omega_X(t)) \cong H^{n-i}(X,O_X(-t))^\vee.$$

This implies that $HP_{\omega_X}(t) = (-1)^n HP_X(-t)$ and a short calculation gives $b_1 = a_1 - (n+1)d$ and (2) follows.

For the proof of (1) we use induction after n. If $n = 1$, X is Cohen-Macaulay and we have equality. If $n \geqq 2$, let $H \subset \boldsymbol{P}^N$ be a general hyperplane. Then $X' = X \cap H$ is a variety (see f.ex. Jouanolou [11, Theorem 6.3]). The essence of Vainsencher's proof is the inequality

$$\deg_{X'} \omega_{X'} \geqq \deg_X \omega_X + d.$$

By induction we have

$$\deg_X \omega_X \leqq \deg_{X'} \omega_{X'} - d \leqq (-2a_1 - nd) - d = -2a_1 - (n+1)d.$$

From this inequality and our criterion Vainsencher's criterion follows.

COROLLARY 2.7. *We have* $\deg_X \omega_X \leqq d(d-n-2)$ *and equality holds if and only if X is a n-hypersurface.*

PROOF. From 2.5 and 2.6 we have

$$\deg_X \omega_X \leqq -2a_1 - (n+1)d \leqq d(d-n-2).$$

Equality implies that $a_1 = -\frac{1}{2}d(d-1)$ and X is a n-hypersurface by 2.5.

Now we will study the criterion of M. Dale [3]. Let $X \subset \boldsymbol{P}^N$ be a projective variety, $\dim X = n$, $\deg X = d$. Let $s_1(X) \in A_{n-1}X$ be part of the Segre class of X. See f.ex. Johnson [10] for the definition. By $\deg s_1$ we mean $\deg i_* S_1$, where $i: X \to \boldsymbol{P}^N$ is the given embedding.

We are now able to formulate Dale's criterion.

PROPOSITION 2.8. (Dale [3, 2.11]). *We have* $\deg s_1 \leqq d(d-n-2)$, *and equality implies that X is a n-hypersurface.*

An easy application of the Baum-Fulton-MacPherson-Riemann-Roch theorem [2] gives

$$\deg \tau_1 = a_1 + d\frac{n+1}{2},$$

where $\tau_1 \in A_{n-1}X_{\mathbf{Q}}$ is part of the Todd class introduced there and the degree is taken by the given embedding of X in $\boldsymbol{P}^N$.

The next proposition gives a partial answer to the question of relations between the numerical invariants $\deg s_1$ and a_1.

PROPOSITION 2.9. *If X is regular in codimension one, then* $s_1 = -2\tau_1$ *in* $A_{n-1}X_{\mathbf{Q}}$, *in particular* $\deg s_1 = -2a_1 - d(n+1)$.

PROOF. Suppose first that X is regular. Then

$$\tau_1 = \mathrm{Td}_1(X) \cap [X] = \tfrac{1}{2}c_1 \cap [X] = -\tfrac{1}{2}s_1$$

by part (2) of B. F. M. Riemann-Roch.

If X is regular in codimension one, put $U = X_{\mathrm{reg}}$ and let $j : U \to X$ denote the embedding. By [4, 1.9] we have an isomorphism

$$j^* : A_{n-1}X_{\mathbf{Q}} \cong A_{n-1}U_{\mathbf{Q}}.$$

We have $j^*\tau_1(x) = \tau_1(U)$ by part (3) of B. F. M. Riemann-Roch, and $j^*s_1(X) = s_1(U)$ by [5, Proposition 3.2].

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