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THE GALOIS GROUP OF THE TANGENCY PROBLEM FOR PLANE CURVES

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1. Introduction.

Consider the following enumerative problem: given curves $D_1,\ldots,D_N$ in a projective plane over an algebraically closed field k, find all the reduced curves of a given degree r simultaneously tangent to these curves.

Two natural and important problems arise when the number of solutions is finite.

The first problem is to determine the number of solutions. This is completely solved only when $r \leqq 2$. For $r=3$ the problem was treated by Maillard and Zeuthen, and by Zeuthen for $r=4$, but this work has not yet been verified rigorously. Considerable progress towards the solution of the problem, by means of its reduction to the computation of the characteristic numbers of families, has been made recently by W. Fulton, S. Kleiman and R. MacPherson (cf. [1]).

The second problem is to determine the Galois group of the field extension L/K, where K is the field obtained by adjoining to k the coefficients of the D_i, $i=1,\ldots,N$, taken as indeterminates over k, and L is obtained by adjoining to K the coefficients of all the curves which are solutions of the problem. The hypothesis on the coefficients of the D_i means that $D_1,\ldots,D_N$ are in general position in the plane.

Our subject in this paper is the second problem. When $k=\mathbf{C}$ and all the curves involved are conics (the Steiner 5 conic problem), the problem was solved by J. Harris in [3], where he proves that the Galois group is the full symmetric group on 3264 letters (3264 is the number of solutions, when the five conics are in general position). Our main result is that, at least when $k=\mathbf{C}$ and $\deg D_i \geqq 2$, $i=1,\ldots,N$, the Galois group is always the full symmetric group on s letters where s is the number of solutions of the general configuration. Our method of proof consists in showing that the group is twice transitive and contains a transposition (this is also the method used by Harris). The existence

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of a transposition is established using a multiplicity formula which generalizes to curves of arbitrary degrees a formula obtained in the case of conics by Fulton and MacPherson (cf. [2]).

The problem we solve here was recently presented in a more refined version as an open problem in [1]

In the following we take $k=\mathbf{C}$, because, on the one hand, it is convenient to have $\operatorname{char} k=0$ when dealing with duality and, on the other, we will use a lemma from [3] about the existence of a transposition, which is proved using the topology of $\mathbf{C}$.

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2. Preliminaries.

Let $D\subset \boldsymbol{P}^2$ be a curve of degree d without multiple components. Denote by W_r the complete linear system of curves of degree r in $\boldsymbol{P}^2$. Let $H_D\subset W_r$ denote the subset of the curves which are tangent to D; more precisely,

$$H_D = \{C\in W_r \mid \exists P\in C\cap D \text{ such that } m_P(C.D)\geqq 2\} .$$

H_D is a hypersurface unless D is a line and $r=1$. We will exclude this exception from our considerations. Give H_D a cycle structure as follows:

$$\bar{D}^{\check{}} + \sum_{P\in D} e_P L_P \tag{1}$$

where $\bar{D}^{\check{}}$ denotes the dual variety of $\bar{D}$, the r-fold Veronese embedding of D in $\boldsymbol{P}^N$, $N=r(r+3)/2$ (more precisely, we take its associated cycle), L_P denotes the hyperplane in W_r (we are identifying W_r with $\check{\boldsymbol{P}}^N$) parametrizing the curves of degree r passing through P, and $e_P=e_P(D)$ is the multiplicity of the jacobian ideal of D at the point P. Note that $e_P\neq 0$ if and only if P is a singular point of D, therefore the sum (1) is in fact over the singular points of D. With this structures on H_D, it is easy to check, using formulas (IV, 49) and (IV, 76) in [5], that the degree of H_D is equal to $d(d+2r-3)$.

In what follows the same symbol C will be used to indicate either a plane curve of degree r or the corresponding point in W_r.

The description of the local structure of H_D at an arbitrary point will be a consequence of Proposition 1. To prove this proposition we will need the following lemma:

LEMMA 1. *Let $X \subset \boldsymbol{P}^n$, with $n \geqq 3$, be an irreducible curve and let H be a hyperplane not containing X. If the center of a projection is a general linear subspace of H of dimension m, with $0 \leqq m \leqq n-3$, then the projection gives a birational isomorphism from X* onto its image.

PROOF. Since a linear projection with m-dimensional center can be obtained composing $m+1$ linear projections with zero-dimensional centers, it is enough to prove the lemma in the case $m=0$.

Let

$$S = \{(P,Q;\, R;\, \lambda, \mu) \mid R = \lambda P + \mu Q\} \subset (X \times X \setminus \Delta) \times \boldsymbol{P}^n \times \boldsymbol{P}^1 .$$

S is the secant bundle of X. It is clear that $\dim S = 3$. Let pr: $S \to \boldsymbol{P}^n$ be the projection on the factor $\boldsymbol{P}^n$. Since $X \not\subset H$ it follows that $\operatorname{pr}(S) \not\subset H$. So the points of H which are on infinitely many secant lines, are points over which the fibers of the morphism

$$S \setminus \operatorname{pr}^{-1}(H) \to H$$

$$(P,Q;\, R;\, \lambda, \mu) \mapsto \overline{PQ} \cap H$$

($\overline{PQ}$ denoting the line determined by P and Q) have dimension $\geqq 2$. It then follows that the set of points of H which are on infinitely many secant lines has dimension $\leqq 1$. Hence a central projection from a general point of H gives a birational isomorphism from X onto its image.

PROPOSITION 1. *Let $X \subset \boldsymbol{P}^n$ be an irreducible curve which is not a line, and let H be a hyperplane in $\boldsymbol{P}^n$ not containing X. The tangent cone to $\check{X}$ at $\check{H}$ is given as a divisorial cycle by*

$$TC_{\check{H}}(\check{X}) = \sum_{P \in X \cap H} r_P \check{P} ,$$

where $r_P = m_P(X.\,.H) - m_P(X)$ and $\check{P}$ is the hyperplane corresponding by duality to the point P.

PROOF. We first prove the case $n=2$ and then reduce the general case to this particular one.

CASE $n=2$. Each branch β of $\check{X}$ centered at $\check{H}$ comes by duality from a branch α centered at some point $P \in X \cap H$ and with tangent H. Since in char 0, the dual of β is again α, it follows that the tangent to β is the line $\check{P}$. Hence the contribution of the point $P \in X \cap H$ to the tangent cone of $\check{X}$ at $\check{H}$ is the line $\check{P}$ counted with the multiplicity $r_P = \sum \operatorname{ord} \beta$, this sum is taken over all

β which are images of branches α with center P and tangent to H. But since $\operatorname{ord}\beta = \operatorname{ord}_\alpha H - \operatorname{ord}\alpha$ (cf. [7], p. 153]), it follows that

$$r_P = \sum \operatorname{ord}\beta = \sum_{\substack{\alpha \text{ with center } P \\ \text{and tangent to } H}} [\operatorname{ord}_\alpha H - \operatorname{ord}\alpha]$$

$$= \sum_{\alpha \text{ with center } P} [\operatorname{ord}_\alpha H - \operatorname{ord}\alpha] = m_P(X.H) - m_P(X).$$

CASE $n \geqq 3$. Let P be any point of $X \cap H$, and let $\pi \subset \check{P}^n$ be a general plane through the point $\check{H}$. The dual of π, $\check{\pi} \subset P^n$, is a linear space of dimension $n-3$ contained in the hyperplane H and not containing P.

Let $f\colon P^n \to P^2$ denote a linear projection with center $\check{\pi}$. By the principle of section and projection we have in the duality of P^2 that

$$(2) \qquad (f(X))\check{} = \check{X} \cap \pi \quad \text{and} \quad (f(P))\check{} = \check{P} \cap \pi$$

(cf. [6, Theorem (5.1), (iii)], plus the fact, which is a consequence of Lemma (5.2) there, that the dual hypersurface of a curve is the $(n-2)$th osculating developable of the dual curve. These results as stated in the reference require that X generate P^n, but we can easily reduce our statement to this case).

Now since X is not contained in H and $\check{\pi}$ is general in H, it follows from Lemma 1 that f gives a birational isomorphism from X onto $f(X)$.

From the birationality of f and the fact that no tangent to X at P meets a general $\check{\pi}$, it follows that, for any branch α of X centered at P, we have

$$(3) \qquad \operatorname{ord} f(\alpha) = \operatorname{ord}\alpha.$$

By a projection formula, which can be easily verified directly, we have

$$(4) \qquad \operatorname{ord}_\alpha H = \operatorname{ord}_{f(\alpha)} f(H).$$

From (2), the case $n=2$, and the fact that $\check{\pi}$ do not meet any secant of X lying on H, we have

$$(5) \qquad TC_{\check{H}}(\check{X} \cap \pi) = \sum_{P \in X \cap H} r_P(\check{P} \cap \pi) \left(= \left(\sum_{P \in X \cap H} r_P \check{P} \right) \cap \pi \right)$$

where

$$r_P = \sum_{\alpha \text{ with center } P} [\operatorname{ord}_{f(\alpha)} f(H) - \operatorname{ord} f(\alpha)].$$

From (3) and (4) it follows that

$$(6) \qquad r_P = \sum_{\alpha \text{ with center } P} [\operatorname{ord}_\alpha H - \operatorname{ord}\alpha] = m_P(X.H) - m_P(X).$$

Since π is general we also have as a cycle:

$$TC_{\check{H}}(\check{X} \cap \pi) = TC_{\check{H}}(\check{X}) \cap \pi . \tag{7}$$

From (5) and (7) we get

$$TC_{\check{H}}(\check{X}) \cap \pi = \left(\sum_{P \in X \cap H} r_P \check{P} \right) \cap \pi .$$

Now if one looks at the forms defining $TC_{\check{H}}(\check{X})$ and $\sum_{P \in X \cap H} r_P \check{P}$, it follows easily that

$$TC_{\check{H}}(\check{X}) = \sum_{P \in X \cap H} r_P \check{P} .$$

This equation, together with (6), completes the proof.

Proposition 2. *Let D be a reduced plane curve. Suppose that $r \geqq 2$ or no component of D is a line. If C is a curve of degree r such that C and D have no common components, then the tangent cone of H_D at C is given, as a divisorial cycle, by*

$$TC_C(H_D) = \sum_{P \in C \cap D} (r_P + e_P) L_P ,$$

where $r_P = m_P(D.C) - m_P(D)$ and e_P is the multiplicity of the jacobian ideal of D at P. In particular we have

$$m_C(H_D) = \sum_{P \in C \cap D} (r_P + e_P) .$$

Proof. If we denote by $\tilde{C}$ the hyperplane in $\boldsymbol{P}^N$ corresponding to the curve C, and by $\bar{P}$ (respectively $\bar{D}$) the image of P (respectively D) in $\boldsymbol{P}^N$ under the Vernose embedding, we have

$$m_P(D.C) = m_{\bar{P}}(\bar{D}.\tilde{C}) .$$

Now our proposition is an immediate consequence of (1) and of Proposition 1.

Corollary. *If D and C are as in the proposition and in addition all the points of $C \cap D$ are simple points of D, then*

$$TC_C(H_D) = \sum_{P \in C \cap D} r_P L_P$$

where $r_P = m_P(D.C) - 1$. In particular,

$$m_C(H_D) = \sum_{P \in C \cap D} [m_P(D.C) - 1] = dr - (\# \, D \cap C) .$$

Remark 1. The corollary above was given in the special case where C and D are smooth conics, using an "ad hoc" argument, by Fulton and MacPherson in [2].

Denote by X_D the locus of the curves of degree r which are bitangent to D, and by Y_D the locus of the curves of degree r which have a point of tangency of higher order with D. More precisely,

$$X_D = \{C \in H_D \mid \exists P, Q \in C \cap D,\ C.D \geqq 2P + 2Q\}\ ,$$
$$Y_D = \{C \in H_D \mid \exists P \in C \cap D,\ C.D \geqq 3P\}\ .$$

If D is a smooth curve, it follows from the corollary above that

$$\operatorname{Sing}(H_D) = X_D \cup Y_D\ .$$

3. Some properties of H_D, Y_D and X_D.

In this section we will discuss the irreducibility of the sets Y_D, $H_D \cap H_{D'}$ and $Y_D \cap H_{D'}$. The main results here are Propositions 4 and 5. Although the proofs are elementary in spirit, they require an extensive analysis of the fibers of certain morphisms. We will include only the main features of this analysis.

Let D and D' be two irredicible plane curves, P and P' simple points respectively of D and D'. For non negative integers i and j, consider the set

$$Y(i,j;\ D,P;\ D',P') = \{C \in W_r \mid C.D \geqq iP,\ C.D' \geqq jP'\}\ .$$

We will denote this set by $Y(i,j)$ if no confusion is possible.

We set as always $N = r(r+3)/2$. Let V be the $(N+1)$-dimensional vector space of all homogeneous polynomials of degree r in three indeterminates.

Let G and G' be polynomials in V such that $G(P) \neq 0$ and $G'(P') \neq 0$. If M (respectively M') is the maximal ideal of $\mathcal{O}_{D,P}$ (respectively $\mathcal{O}_{D',P'}$), then we have k-linear homomorphisms

$$\Phi_{i,j} : V \to (\mathcal{O}_{D,P}/M^i) \times (\mathcal{O}_{D',P'}/(M')^j)\ ,$$
$$F \to \left(\left(\frac{\bar{F}}{\bar{G}}\right), \left(\frac{\bar{F}}{\bar{G}'}\right)\right)$$

with the convention that if $D = D'$ and $P = P'$, the set on the right has to be replaced by $\mathcal{O}_{D,P}/M^{i+j}$. In this case we have $\Phi_{i,j} = \Phi_{i+j,0} = \Phi_{0,i+j}$.

If we denote by π the canonical map $V \setminus \{0\} \to \boldsymbol{P}^N = W_r$, then

$$Y(i,j) = \pi(\operatorname{Ker} \Phi_{i,j} \setminus \{0\})\ .$$

Hence these sets are linear subspaces of W_r. Since for $i' \geqq i$ and $j' \geqq j$ we have inclusions $\operatorname{Ker} \Phi_{i',j'} \subset \operatorname{Ker} \Phi_{i,j}$, it follows that

$$Y(i',j') \subset Y(i,j) \quad \text{if } i' \geqq i \text{ and } j' \geqq j\ .$$

In this situation we also have

$$(1) \qquad \operatorname{codim}(Y(i',j'), Y(i,j)) = \operatorname{codim}(\operatorname{Ker}\Phi_{i',j'}, \operatorname{Ker}\Phi_{i,j}).$$

If one restricts the maps $\Phi_{i+1,j}$ and $\Phi_{i,j+1}$ to $\operatorname{Ker}\Phi_{i,j}$, it is easy to verify that

$$(2)\ \operatorname{codim}(\operatorname{Ker}\Phi_{i+1,j}, \operatorname{Ker}\Phi_{i,j}) \leqq 1 \quad \text{and} \quad \operatorname{codim}(\operatorname{Ker}\Phi_{i,j+1}, \operatorname{Ker}\Phi_{i,j}) \leqq 1.$$

We will denote by $t_P(D)$ the tangent line to D at P.

LEMMA 2. *If* $i \leqq r+1$, *then* $\operatorname{codim}(Y(0,i), Y(0,i-1)) = 1$.

PROOF. Let $i \leqq r+1$. Let $C = l_1 l_2 \ldots l_{i-1} l_i \ldots l_r$, where $l_1, \ldots, l_{i-1}$ are lines passing through P' and transversal to D' at P', and $l_i, \ldots, l_r$ are lines not passing through P'. It is clear that $C \in Y(0,i-1) \setminus Y(0,i)$, hence by (1) and (2) we have that $\operatorname{codim}(Y(0,i), Y(0,i-1)) = 1$.

From now on in this section we will assume $r \geqq 2$. Consider the following chain of inclusions

$$(3) \qquad Y(3,2) \subset Y(2,2) \subset Y(1,2) \subset Y(0,2) \subset Y(0,1) \subset \boldsymbol{P}^N.$$

From the lemma above it follows that

$$(4) \qquad \operatorname{codim}(Y(0,1), \boldsymbol{P}^N) = \operatorname{codim}(Y(0,2), Y(0,1)) = 1.$$

LEMMA 3. *If* $\operatorname{codim}(Y(1,2), Y(0,2)) = 0$, *then* $P = P'$ *and* $D \neq D'$.

PROOF. It is enough to show that if $P \neq P'$ or $D = D'$ and $P = P'$, then $\operatorname{codim}(Y(1,2), Y(0,2)) = 1$.

Suppose that $D = D'$ and $P = P'$, then $Y(1,2) = Y(0,3)$, hence by Lemma 2 it follows that $\operatorname{codim}(Y(1,2), Y(0,2)) = \operatorname{codim}(Y(0,3), Y(0,2)) = 1$.

Suppose that $P \neq P'$. Let C_0 be a conic tangent to the line $t_{P'}(D')$ at P' and not passing through P (C_0 trivially exists). Let $l_1, \ldots, l_{r-2}$ be lines which do not pass through the point P. Define $C = C_0 l_1 \ldots l_{r-2}$, so $C \in Y(0,2) \setminus Y(1,2)$, hence $\operatorname{codim}(Y(1,2), Y(0,2)) = 1$.

LEMMA 4. *If* $\operatorname{codim}(Y(2,2), Y(1,2)) = 0$, *then either*

(i) $r=2$, $P \neq P'$ *and* $t_P(D) = t_{P'}(D')$, *or*

(ii) $r=2$, $D = D'$ *and* $P = P'$ *is a hyperflex of* D (i.e., $m_P(D.t_P(D)) > 3$), *or*

(iii) $D \neq D'$, $P = P'$ *and* $t_P(D) = t_{P'}(D)$.

PROOF. It is sufficient to prove that in the cases not listed above the codimension is equal to 1. The way we do this is by making a systematic analysis of the situations of Figure 1:

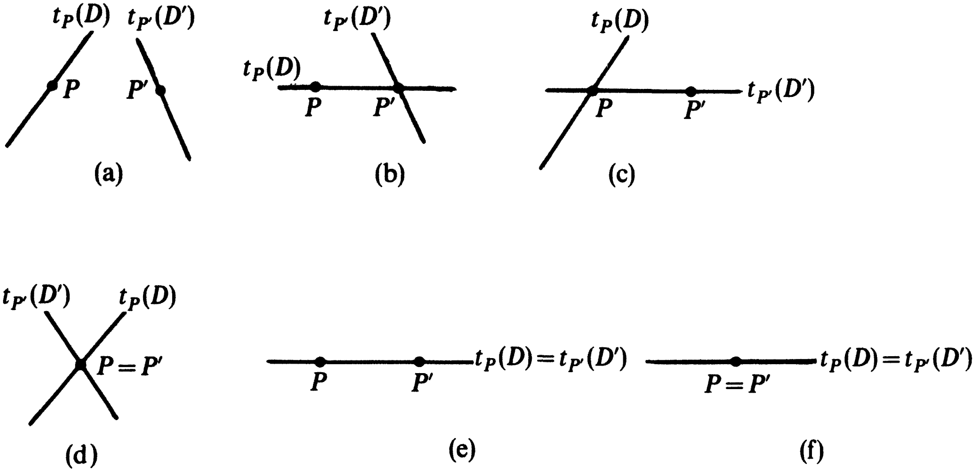

Figure 1.

For example if we are in situation (c), take $C=t_{P'}(D')l_1 \ldots l_{r-1}$, where $l_1,\ldots,l_{r-1}$ are lines not passing through P, then $C \in Y(1,2)\setminus Y(2,2)$, hence codim $(Y(2,2), Y(1,2))=1$.

We do just another case to illustrate the kind of arguments involved.

Suppose that $D=D'$ and $P=P'$ (so $t_P(D)=t_{P'}(D')$), then we have $Y(1,2) = Y(0,3)$ and $Y(2,2)=Y(0,4)$. In order to exclude case (ii) we have to assume either $r \geqq 3$ or P is not a hyperflex of D. Suppose $r \geqq 3$, then by Lemma 2, it follows that codim $(Y(2,2), Y(1,2))=1$. Suppose now that P is not a hyperflex of D. Let $C=t_P(D)l_1 \ldots l_{r-1}$ where $l_2,\ldots,l_{r-1}$ are lines not passing through P and l_1 is a line, not passing through P if P is a flex of D, or transversal to D at P if P is not a flex of D. So

$$C \in Y(0,3)\setminus Y(0,4) = Y(1,2)\setminus Y(2,2)$$

and therefore codim $(Y(2,2), Y(1,2))=1$.

LEMMA 5. *Suppose* $D \neq D'$. *If* codim $(Y(3,2), Y(2,2))=0$, *then either*
(i) $r=2$, $P \neq P'$, *P is a flex of D and* $P' \in t_P(D)$, *or*
(ii) $t_P(D)=t_{P'}(D')$.

PROOF. Similar to the proof of Lemma 4.

It follows from (3) and (4) that codim $(Y(1,2), \mathbf{P}^N) \geqq 2$, therefore

$$\operatorname{codim}(Y(2,2), \mathbf{P}^N) \geqq 2 .$$

By Lemma 2, codim $(Y(3,0), \boldsymbol{P}^N)=3$, and since $Y(3,2)\subset Y(3,0)$, it follows that codim $(Y(3,2), \boldsymbol{P}^N)\geqq 3$. Putting together this and the lemmas above we get the following more precise statements.

PROPOSITION 3. *In parts* (c) *and* (d) *below we assume* $D\neq D'$.

(a) *If* codim $(Y(2,2), \boldsymbol{P}^N)\leqq 3$, *then either*
(i) $P=P'$ *and* $D\neq D'$ *or*
(ii) $r=2$, $P\neq P'$ *and* $t_P(D)=t_{P'}(D')$ *or*
(iii) $r=2$, $D=D'$ *and* $P=P'$ *is a hyperflex of* D.

(b) *If* codim $(Y(2,2), \boldsymbol{P}^N)=2$, *then* $D\neq D'$, $P=P'$ *and* $t_P(D)=t_{P'}(D')$.

(c) *If* codim $(Y(3,2), \boldsymbol{P}^N)\leqq 4$, *then either*
(i) $P=P'$ *or*
(ii) $t_P(D)=t_{P'}(D')$ *or*
(iii) $r=2$, $P\neq P'$, P *is a flex of* D *and* $P'\in t_P(D)$.

(d) *If* codim $(Y(3,2), \boldsymbol{P}^3)=3$, *then either*
(i) $P=P'$ *and* $t_P(D)=t_{P'}(D')$ *or*
(ii) $r=2$, $P\neq P'$, $t_{P'}(D')=t_P(D)$ *and* P *is a flex of* D.

PROOF. Combining Lemmas 2, 3, and 4 with (4) and using some formal logic we get (a), (b) and (c). We now prove (d). Again Lemmas 2, 3, and 4 and (4) give that if codim $(Y(3,2), \boldsymbol{P}^N)=3$, then either
(i) $P=P'$ and $t_P(D)=t_{P'}(D')$ or
(ii)$'$ $r=2$, $P\neq P'$ and $t_P(D)=t_{P'}(D')$.

Now we will show that codim $(Y(3,2), \boldsymbol{P}^N)=3$ plus (ii)$'$ imply that P is a flex of D. Indeed, in this case $N=5$, $\dim Y(3,0)=2$, and if P is not a flex of D, the reducible conics C' such that $m_P(C'.D)\geqq 3$ form a subspace of dimension one of $Y(3,0)$. Take an irreducible conic C such that $m_P(C.D)\geqq 3$, then $P'\notin C$, so $C\in Y(3,0)\setminus Y(3,1)$ and therefore codim $(Y(3,1), \boldsymbol{P}^5)=4$. Since $Y(3,2)\subset Y(3,1)$, it follows that codim $(Y(3,2), \boldsymbol{P}^5)\geqq 4$, contradiction.

COROLLARY. *Let* D *and* D' *be smooth plane curves. In* (c) *and* (d) *below we assume* $D\neq D'$.

(a) *Excepting the case* $r=2$, $D=D'$ *and* $\deg D=1$, *we have*

$$\dim\{(P,P')\in D\times D' \mid \operatorname{codim}(Y(2,2), \boldsymbol{P}^N)\leqq 3\}\leqq 1\,.$$

This dimension is zero if $r\geqq 3$ *or if* D, D' *is not a pair of tangent curves such that one of them is a line.*

(b) *If* $D=D'$ *or* D *is transversal to* D', *then*

$$\{(P,P') \in D \times D' \mid \operatorname{codim}(Y(2,2), \boldsymbol{P}^N) = 2\} = \varnothing .$$

(c) $\dim \{(P,P') \in D \times D' \mid \operatorname{codim}(Y(3,2), \boldsymbol{P}^N) \leqq 4\} \leqq 1$.

This dimension is zero if D, D' is not a pair of tangent curves such that one of them is a line.

(d) *If D is transversal to D' and no inflectional tangent of D is tangent to D', then*

$$\{(P,P') \in D \times D' \mid \operatorname{codim}(Y(3,2), \boldsymbol{P}^N) = 3\} = \varnothing ,$$

PROPOSITION 4. *Let D be a smooth plane curve. Then*
(i) $\dim X_D \leqq N-2$.
(ii) Y_D *is irreducible and if* $r=2$ *and* $\deg D = 1$, $\dim Y_D = N-3$, *otherwise* $\dim Y_D = N-2$.

PROOF. (i) Let

$$I(D) = \{(C;\ P,Q) \in W_r \times D \times D \mid C.D \geqq 2P+2Q\} .$$

Let $p_1\colon I(D) \to W_r$ and $p_2\colon I(D) \to D \times D$ be the projections. It is clear that $p_1(I(D)) = X_D$ and $p_2^{-1}(P,Q) \cong Y(2,2;\ D,P;\ D,Q)$. If $r \geqq 3$, or $D \neq D'$, or $\deg D > 1$, we have from the corollary above, part (a), that $\dim I(D) = N-2$, hence $\dim X_D \leqq N-2$. If $r=2$, $D=D'$, and $\deg D=1$, then X_D is the set of conics having D as a component, this set has dimension $2(=N-3)$.

(ii) If $r=2$ and $\deg D=1$, then Y_D is the set of conics having D as a component. In this case Y_D is irreducible and $\dim Y_D = 2 = N-3$. Suppose now that $r>2$ or $\deg D>1$. Let

$$I'(D) = \{(C;\ P) \in W_r \times D \mid m_P(C.D) \geqq 3\} \subset W_r \times D .$$

Let $p_1\colon I'(D) \to W_r$ and $p_2\colon I'(D) \to D$ be the projections. We have that $Y_D = p_1(I'(D))$ and $p_2^{-1}(P) = Y(0,3;\ D,P;\ D,P)$. From Lemma 2 it follows that every fiber of p_2 is a $\boldsymbol{P}^{N-3}$, therefore $I'(D)$ is irreducible of dimension $N-2$, so Y_D is irreducible. Since the fibers of p_1 are generally finite, it follows that Y_D has dimension $N-2$.

PROPOSITION 5. *Let D, D' be plane curves such that D is smooth and not a component of D'. We have*
(i) *The set $H_D \cap H_{D'}$ is pure $(N-2)$-dimensional. If D' is smooth and transversal to D, then $H_D \cap H_{D'}$ is irreducible.*
(ii) *If* $r>2$ *or* $\deg D > 1$, *then $Y_D \cap H_{D'}$ is pure $(N-3)$-dimensional. If moreover D' is smooth, transversal to D and no inflectional tangent of D is tangent to D', then $Y_D \cap H_{D'}$ is irreducible.*

PROOF. (i) Since D is not a component of D', $H_D \not\subset \bar{D}'^{\vee}$, and since H_D is not a hyperplane, $H_D \not\subset L_P$ for all P. It follows that $H_D \not\subset H_{D'}$, and consequently $H_D \cap H_{D'}$ is pure $(N-2)$-dimensional.

Assume now that D' is smooth and transveral to D. Let

$$I(D,D') = \{(C;\, P,P') \mid C.D \geqq 2P,\ C.D' \geqq 2P'\} \subset W_r \times D \times D' .$$

Let $p_1\colon I(D,D') \to W_r$ and $p_2\colon I(D,D') \to D \times D'$ be the projections. We have that $H_D \cap H_{D'} = p_1(I(D,D'))$ and $p_2^{-1}(P,P') = Y(2,2;\, D,P;\, D',P')$. From the corollary, parts (a) and (b), it follows that the fibers of p_2 are generally isomorphic to $\boldsymbol{P}^{N-4}$, that they are not isomorphic to $\boldsymbol{P}^{N-4}$ only for finitely many pairs $(P,P') \in D \times D'$, and that in this case they are isomorphic to $\boldsymbol{P}^{N-3}$. Therefore $I(D,D')$ has dimension $N-2$ and a unique irreducible component of that dimension. Since $H_D \cap H_{D'}$ is pure $(N-2)$-dimensional and it is the image of $I(D,D')$, it follows that it is irreducible.

(ii) It is clear that $Y_D \not\subset \bar{D}'^{\vee}$ and that $Y_D \not\subset L_P$ for all P, therefore $Y_D \not\subset H_{D'}$. It follows that $Y_D \cap H_{D'}$ is pure $(N-3)$-dimensional.

Suppose now that D' is smooth, transversal to D, and no inflectional tangent of D is tangent to D'. Let

$$I'(D,D') = \{(C;\, P,P') \mid C.D \geqq 3P,\ C.D' \geqq 2P'\} \subset W_r \times D \times D' .$$

If $p_1\colon I'(D,D') \to W_r$ and $p_2\colon I'(D,D') \to D \times D'$ are the projections, then

$$Y_D \cap H_{D'} = p_1(I'(D,D')) \quad \text{and} \quad p_2^{-1}(P,P') = Y(3,2;\, D,P;\, D',P') .$$

From the corollary, parts (c) and (d), it follows that, on the complement of a finite set, the fibers of p_2 are $\boldsymbol{P}^{N-5}$ and, on this finite set, the fibers are $\boldsymbol{P}^{N-4}$. It follows that $I'(D,D')$ has dimension $N-3$ and has a unique irreducible component of that dimension. Since $Y_D \cap H_{D'}$ is pure $(N-3)$-dimensional and it is the image of $I'(D,D')$, it follows that it is irreducible.

4. The Galois group.

In this section we compute the Galois group of the tangency problem for plane curves, in the case in which the degrees d_i of the curves D_i are all $\geqq 2$, and either $r \geqq 2$ or for some i, $d_i > 2$. The case $r=1$ and $d_1 = d_2 = 2$ will be treated in the next section. To carry out the computation we first identify the Galois group of the problem with a monodromy group associated to a map of finite degree. Then we show that the action of that monodromy group on the general fiber is twice transistive. Finally we establish the existence of a transposition.

Let $\tilde{W}_r$ be the open subset of W_r consisting of points which represent reduced curves. Let W'_r be the closed subset of points of $\tilde{W}_r$ which represent the singular curves.

Let, as always, $N=r(r+3)/2$. We are assuming $d_i \geqq 2$ for $i=1,\ldots,N$. Let

$$X = W_{d_1} \times \ldots \times W_{d_N} .$$

Let

$$Y = \left\{ (D_1,\ldots,D_N; C) \in X \times \tilde{W}_r \;\middle|\; C \in \bigcap_{i=1}^{N} H_{D_i} \right\} .$$

Consider the projections

$$\begin{array}{ccc} & Y & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & & \tilde{W}_r \end{array}$$

p_2 is clearly proper since it is obtained by base extension of a projective morphism.

LEMMA 6. *Y has a unique component $\bar{Y}$ of maximal dimension* ($=\dim X$). *Moreover $\bar{Y}$ is the closure of $p_2^{-1}(\tilde{W}_r \setminus W_r')$ in $X \times \tilde{W}_r$.*

PROOF. Notice that for $C \in \tilde{W}_r$ we have

$$p_2^{-1}(C) \cong H_C \times \ldots \times H_C ,$$

where each H_C is sitting in the appropriate W_{d_i}, $i=1,\ldots,N$. Since all the fibers of p_2 have dimension $\dim X - N$ and $\tilde{W}_r$ has dimension N, it follows that $\dim Y = \dim X$. On the other hand, $p_2^{-1}(C)$ is irreducible for C in the open set $\tilde{W}_r \setminus W_r'$ and, since p_2 is proper, we have that $p_2^{-1}(\tilde{W}_r \setminus W_r')$ is irreducible. It is clear now that the closure of $p_2^{-1}(\tilde{W}_r \setminus W_r')$ in $X \times \tilde{W}_r$ is the unique component of Y of maximal dimension.

REMARK 2. The proof above works also if $r \geqq 2$ and for some i, $d_i=1$. So the only case excluded is when $r=1$ and some $d_i=1$, but in this case the enumerative problem has no solution.

To give content to our results, we will assume that the restriction of p_1 to $\bar{Y}$ is dominating. This fact is a corollary of the main result in [1].

Let G be the Galois group or the monodromy group of $p_1 \mid \bar{Y}$ (cf. [3]). This is the Galois group we are interested in determining.

Since $\bar{Y}$ is irreducible, G acts transtively on the general fiber of $p_1 \mid \bar{Y}$. In fact the transitivity of the group action is equivalent to the irreducibility of $\bar{Y}$.

LEMMA 7. *The general fiber of p_1 consists of points which represent non singular curves. In particular, the general fiber of p_1 is contained in $\bar{Y}$.*

PROOF. In fact, let

$$Y' = \{(D_1,\ldots,D_N;\, C) \in Y \mid C \text{ is singular}\} = p_2^{-1}(W'_r)\,.$$

By lemma 6 we have that $\bar{Y} \not\subset Y'$, so $\dim Y' < \dim \bar{Y}$ $(= \dim Y = \dim X)$, hence $\dim p_1(Y') < \dim X$.

Let $U = X \setminus p_1(Y')$, where $Y' = p_2^{-1}(W'_r)$ as above.

LEMMA 8. *$p_1^{-1}(U) \times_U p_1^{-1}(U) \setminus \Delta$, where Δ is the diagonal, has a unique irreducible component of maximal dimension* $(= \dim X)$.

PROOF. The fiber of $p_2\colon p_1^{-1}(U) \times_U p_1^{-1}(U) \setminus \Delta \to (\bar{W}_r \setminus W'_r)^2$ over the point (C, C') with $C \neq C'$ is isomorphic to

$$(H_C \cap H_{C'}) \times \ldots \times (H_C \cap H_{C'})\,.$$

From Proposition 5 (note that here the d_i's play the role of r there) these fibers are $(\dim X - 2N)$-dimensional and generally are irreducible. From this it follows that $p_1^{-1}(U) \times_U p_1^{-1}(U) \setminus \Delta$ has the same dimension as X and it has a unique component of that dimension.

PROPOSITION 6. *G acts doubly transitively on the general fiber of $p_1 \mid \bar{Y}$.*

PROOF. Consider the diagram

$$\begin{tikzcd}
p_1^{-1}(U) \times_U p_1^{-1}(U) \setminus \Delta \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, "p"] & p_1^{-1}(U) \arrow[d, "p_1"] \\
p_1^{-1}(U) \arrow[r, "p_1"] & U \subset X\,.
\end{tikzcd}$$

From Lemma 8 it follows that the Galois group G' associated to the map p acts transitively on its general fiber. Let (y_1, y_2), (y'_1, y'_2) be two points in a general fiber of p (y_1, y_2, y'_1, y'_2 are then points on a general fiber of p_1). From the transitivity of the action of G', there exists a path γ in $p_1^{-1}(U) \times_U p_1^{-1}(U) \setminus \Delta$ such that

$$\gamma(0) = (y_1, y_2) \quad \text{and} \quad \gamma(1) = (y'_1, y'_2)\,.$$

Let α be the path in U defined by $\alpha = p \circ \gamma$. Then $\pi_1 \circ \gamma$ and $\pi_2 \circ \gamma$ are the liftings of α in $p_1^{-1}(U)$ with initial points respectively y_1 and y_2. Since

$$\pi_1 \circ \gamma(1) = y'_1 \quad \text{and} \quad \pi_2 \circ \gamma(1) = y'_2\,,$$

it follows that if $g \in G =$ Galois group associated to $p_1\colon p_1^{-1}(U) \to U$ (=Galois group associated to $p_1\colon \bar{Y} \to X$) is the element associated to the path α, then we have

$$g(y_1) = y_1' \quad \text{and} \quad g(y_2) = y_2' .$$

This proves the double transitivity of the action of G.

Let $D_1, \ldots, D_N$ be any curves and $C \in H_{D_1} \cap \ldots \cap H_{D_N}$. We are interested in computing the intersection multiplicity $m_C(H_{D_1} \ldots H_{D_N})$ of the hypersurfaces $H_{D_1}, \ldots, H_{D_N}$ at the point C. This computation can be done very easily when the tangent cones of $H_{D_1}, \ldots, H_{D_N}$ at C intersect only in $\{C\}$. In fact, this condition is equivalent to the equality

$$(1) \qquad m_C(H_{D_1} \ldots H_{D_N}) = m_C(H_{D_1}) \ldots m_C(H_{D_N}) .$$

In the above situation we say that $H_{D_1}, \ldots, H_{D_N}$ are transverse at C.

From the description of the tangent cones to the H_{D_i} we gave in Proposition 2, it follows that $H_{D_1}, \ldots, H_{D_N}$ are not transverse at C if and only if $(D_1, \ldots, D_N; C) \in \Gamma$, where

$$\Gamma = \Big\{(D_1, \ldots, D_N; C) \in Y \;\Big|\; C \in \bigcap_{i=1}^{N} H_{D_i} \quad \text{and} \quad \exists\, C' \in W_r,\ C' \neq C \text{ such that}$$

$$\forall i = 1, \ldots, N,\ \exists\, P_i \in D_i \cap C' \text{ with } m_{P_i}(D_i . C) - m_{P_i}(D_i) + e_{P_i}(D_i) > 0\Big\} .$$

Note that the last inequality above is equivalent to $m_{P_i}(D_i . C) \geqq 2$.

REMARK 3. There exists an open set U in X such that, for every $(D_1, \ldots, D_N) \in U$ and every $C \in H_{D_1} \cap \ldots \cap H_{D_N} \cap \tilde{W}_r$, the curves $D_1, \ldots, D_N$ and C are smooth and the H_{D_i} are transverse at C. In particular for such D_i and C we have

$$m_C(H_{D_1} \ldots H_{D_N}) = \prod_{i=1}^{N} \Bigg(\sum_{P \in C \cap D_i} [m_P(D_i . C) - 1] \Bigg) .$$

PROOF. The assertion about the smoothness of C follows from Lemma 7. The existence of U on which the transversality assertion is valid is insured by the fact that the closure of Γ is a proper closed subset of Y not containing $\bar{Y}$. Indeed, the fiber over $C \in \tilde{W}_r \setminus W_r'$ of $p_2\colon \Gamma \to \tilde{W}_r$ is given by

$$p_2^{-1}(C) \cong \bigcup H_{C,P_1} \times H_{C,P_2} \times \ldots \times H_{C,P_N} ,$$

where the union is taken over all points $P_1, \ldots, P_N$ such that there exists $C' \in W_r$, $C' \neq C$, such that $C' . C \geqq P_1 + \ldots + P_N$, and where H_{C,P_i} is the set parametrizing the curves of degree d_i tangent to C at the point P_i.

We have that

$$\dim p_2^{-1}(C) \leqq \sum_{i=1}^{N} (\dim W_{d_i}-2)+\dim H^0(C,\mathcal{O}_C(r)) ,$$

and since $\dim H^0(C,\mathcal{O}_C(r))=N-1$, we have that

$$\dim p_2^{-1}(C) \leqq \dim X-N-1 ,$$

therefore $\dim \Gamma \leqq \dim X-1=\dim Y-1$. The formula is now a consequence of the corollary after Proposition 2.

Our task in the rest of the section is to show the existence of a fiber of $p_1 \mid \bar{Y}$ with a point of simple ramification, around which $\bar{Y} \setminus$ ramification locus is connected (in the $\mathbf{C}$ topology), and no other ramification occurs. From this fact, it follows that G contains a simple transposition (cf. [3], Lemma on page 698]). The presence of the transposition and the double transitivity of G imply that G is the whole symmetric group.

Let C be a fixed smooth curve of degree r. Set

$$Z = H_C \times \ldots \times H_C \times Y_C \subset X .$$

Since C is smooth we have that $Z \times \{C\} \subset p_2^{-1}(C) \subset \bar{Y}$.

We say that an N-tuple $(P_1,\ldots,P_N)$ is in general position on C if the P_i's are general points on C (i.e. points of C in the complement of some finite subset. This finite subset will be made precise in the course of the proofs of Lemmas 9 and 11) and there is no curve of degree r other than C passing through these points. It is clear from the definition of Γ that if for each i, D_i touches C in a unique point P_i, and if $(P_1,\ldots,P_N)$ is in general position on C, then $(D_1,\ldots,D_N;\, C) \notin \Gamma$.

LEMMA 9. *If either $r \geqq 2$ or for some i, $d_i > 2$, then the set*

$$Z' = \{(D_1,\ldots,D_N) \in Z \mid m_C(H_{D_1} \ldots H_{D_N})=2\}$$

is dense in Z.

PROOF. Consider the set

$$Z'' = \bigcup H'_{C,P_1} \times \ldots \times H'_{C,P_{N-1}} \times Y'_{C,P_N} ,$$

where the union is taken over all N-tuples $(P_1,\ldots,P_N)$ in general position on C, and where H'_{C,P_i} is the open subset of

$$H_{C,P_i} = \{D \in H_C \mid m_{P_i}(C.D) \geqq 2\} ,$$

defined by $\{D \in H_{C,P_i} | m_C(H_D)=1\}$ and Y'_{C,P_N} is the open subset of

$$Y_{C,P_N} = \{D \in Y_C \mid m_{P_N}(C.D) \geqq 3\}$$

defined by $\{D \in Y_{C,P_N} \mid m_C(H_D)=2\}$. Note that if $r=1$ and $d_1=d_2=2$, then $Y'_{C,P_i}=\varnothing$ for $i=1,2$.

Each

$$H'_{C,P_1} \times \ldots \times H'_{C,P_{N-1}} \times Y'_{C,P_N},$$

if not empty, is open and dense in

$$H_{C,P_1} \times \ldots \times H_{C,P_{N-1}} \times Y_{C,P_N}.$$

$Z'' \subset Z'$ because if $(D_1,\ldots,D_N) \in Z''$, D_i determines a unique point P_i in which D_i touches C, so by the note just before the lemma and formuls (1), it follows that $(D_1,\ldots,D_N) \in Z'$. To complete the proof of the lemma it is enough to show that for general points $P_1,\ldots,P_N$ on C one has

$$H'_{C,P_1} \times \ldots \times H'_{C,P_{N-1}} \times Y'_{C,P_N} \neq \varnothing.$$

Indeed, let P_i, for some $i=1,\ldots,N-1$ be a general point of C. If $r>1$, P_i is neither a flex of C nor a point on a bitangent of C. Form the curve D_i with the tangent line to C at P_i as a component and d_i-1 other lines transversal to C and such that no two of these d_i lines intersect on C. From Proposition 2 it follows that $D_i \in H'_{C,P_i}$, so $H'_{C,P_i} \neq \varnothing$. If $r=1$, then $N=2$, and it is enough to take a curve D_1 of degree d_1 such that C is neither an inflectional tangent of D_1 at P_1 nor C is a bitangent of D_1 (This is always possible because $d_1 \geqq 2$). $D_1 \in H'_{C,P_1}$, hence $H'_{C,P_1} \neq \varnothing$.

Now let P_N be a general point of C. If $r>1$, take an irreducible conic D_0 such that $m_{P_N}(C,D_0)=3$ and D_0 is not bitangent to C. This is possible. For suppose $r=2$ for a moment (not the same r as above), the subset $Y_{C,P_N} \setminus X_{C,P_N}$ of $\mathbf{P}^5$ is non empty because the curve $t_{P_N}(C).l$, where l is a line transversal to C and passing through P_N, belongs to it (X_{C,P_N} stands for $X_C \cap H_{C,P_N}$). So $Y_{C,P_N} \setminus X_{C,P_N}$ has dimension 2 (recall that $Y_{C,P_N} = Y(0,3;\, C,P_N;\, C,P_N)$), but the set of reducible conics D such that $m_{P_N}(C.D)=3$ is one dimensional (this because P_N is not a flex of C), hence D_0 exists. Now let D_N be the curve with components D_0 and d_N-2 lines transversal to C and such that no two of these d_N-1 curves meet on C. It is clear that $D_N \in Y'_{C,P_N}$, so $Y'_{C,P_N} \neq \varnothing$. If $r=1$ and $d_2>2$ (we are assuming without loss of generality that $d_1 \leqq d_2$), then take a smooth curve D_2 of degree d_2 such that $m_{P_2}(C.D_2)=3$ but C is not a bitangent of D_2 (D_2 obviously exists), clearly $D_2 \in Y'_{C,P_2}$, so $Y'_{C,P_2} \neq \varnothing$.

Notice that if $(D_1,\ldots,D_N) \in Z'$, then C is not bitangent to any of the D_i's, because otherwise

$$m_C(H_{D_1} \ldots H_{D_N}) \geqq m_C(H_{D_1}) \ldots m_C(H_{D_N}) > 2;$$

and the points $P_1, \ldots, P_N$ such that $m_{P_i}(C.D_i) \geqq 2$ uniquely determine a curve of degree r which is C, because otherwise $(D_1, \ldots, D_N; C) \in \Gamma$, so

$$m_C(H_{D_1} \ldots H_{D_N}) > m_C(H_{D_1}) \ldots m_C(H_{D_N}) \geqq 2 .$$

Define

$$K = p_1^{-1}(Z) \setminus p_2^{-1}(C) = \left\{ (D_1, \ldots, D_N; C') \;\middle|\; C' \in \bigcap_{i=1}^{N} H_{D_i}, \right.$$

$$\left. C \in H_{D_1} \cap \ldots \cap H_{D_{N-1}} \cap Y_{D_N}, C \neq C' \right\} .$$

Lemma 10. *K has a unique irreducible component $\bar{K}$ which dominates Z. Moreover for every $C' \in \tilde{W}_r \setminus W'_r$, $p_2^{-1}(C') \cap K = p_2^{-1}(C') \cap \bar{K}$ and $\bar{K} \subset \bar{Y}$.*

Proof. Let $C' \in \tilde{W}_r \setminus \{C\}$. We have

$$p_2^{-1}(C') \cap K \cong (H_C \cap H_{C'}) \times \ldots \times (H_C \cap H_{C'}) \times (Y_C \cap H_{C'}) .$$

Recall that for C' non singular we have $p_2^{-1}(C') \subset \bar{Y}$, so generally $p_2^{-1}(C') \cap K \subset \bar{Y}$. From Proposition 5 we have that $p_2^{-1}(C') \cap K$ generally is irreducible and always of dimension $\dim Z - N$. It follows that $\dim K = \dim Z$ and K has a unique irreducible component $\bar{K}$ of that dimension. Moreover $\bar{K}$ has the specified property.

Let K' be the closed subset of $\bar{K}$ defined by

$$K' = \{(D_1, \ldots, D_N; C') \in \bar{K} \mid m_{C'}(H_{D_1} \ldots H_{D_N}) \geqq 2\} .$$

Lemma 11. *If $r \geqq 2$ or for some i, $d_i > 2$, then* $\dim K' < \dim \bar{K}$.

Proof. We have only to exhibit a point in $\bar{K}$ which is not in K'.

If $r \geqq 2$, choose a smooth curve C' transversal to C and let $P_1, \ldots, P_N$ be points on C' in general position. For each $i = 1, \ldots, N-1$ construct a curve D_i in the following way: D_i has $t_{P_i}(C')$ as a component and as other components $d_i - 1$ lines transversal to C', one of them tangent to C and such that no two of the D_i lines meet on C'. Now since P_N is in general position on C', the tangent line to C' at P_N meets C at a point P such that $P \notin C'$ and the tangent line to C at P is transversal to C'. Let D_N be a curve having $t_{P_N}(C')$ and $t_P(C)$ as components and as other components $d_N - 2$ lines transversal to C' and such that no two of these d_N lines meet on C'. From this construction of $C', D_1, \ldots, D_N$, and by Lemma 10 it is clear that $(D_1, \ldots, D_N; C') \in \bar{K} \setminus K'$.

If $r = 1$ and $d_2 > 2$, take a smooth cubic D_0 with C as an inflectional tangent, take for C' a tangent line to D_0 at a point P_2 which is not a flex of D_0, take a conic C_0 tangent to C and tangent to C' at a point $P_1 \neq P_2$. It is clear how to complete C_0 and D_0 with lines in order to get curves D_1 and D_2 with the desired degrees and such that $(D_1, D_2; C') \in \bar{K} \setminus K'$.

PROPOSITION 7. *If $r \geqq 2$ or if for some i, $d_i > 2$, then G contains a simple transposition.*

PROOF. Take a point $(D_1, \ldots, D_N) \in Z' \setminus p_1(K')$. Such a point exists by Lemmas 9, 10, and 11. Then we have $m_C(H_{D_1} \ldots H_{D_N}) = 2$, and for every $C' \neq C$ such that

$$(D_1, \ldots, D_N; C') \in p_1^{-1}(D_1, \ldots, D_N) \cap \bar{Y},$$

we have $m_{C'}(H_{D_1} \ldots H_{D_N}) = 1$. It follows that the fiber of $p_1 \mid \bar{Y}$ over the point $(D_1, \ldots, D_N)$ has a point of simple ramification and no other ramification occurs. To complete the proof we have only to show that there exists an arbitrary small neighborhood (in the $\mathbf{C}$ topology) of $(D_1, \ldots, D_N; C)$ in $\bar{Y}$ such that the complement of the ramification locus in it is connected.

From the note just after Lemma 9 we have that C is not bitangent to any D_i and the points P_i, $i = 1, \ldots, N$, with $m_{P_i}(C . D_i) \geqq 2$ determine a unique curve of degree r which is C. Consider now the set

$$I = \{(D'_1, \ldots, D'_N; C'; P'_1, \ldots, P'_N) \in X \times \tilde{W}_r \times (\mathbf{P}^2)^N \mid m_{P'_i}(C' . D'_i) \geqq 2\} .$$

It is clear that $\pi_1(I) = Y$, where π_1 is the projection $I \to X \times \tilde{W}_r$. By considering the projection $\pi_2 \colon I \to (\mathbf{P}^2)^N$, we see that

$$\pi_2^{-1}(U) \cong \mathbf{P}^{N_1 - 2} \times \ldots \times \mathbf{P}^{N_N - 2} \times U ,$$

where U is some open neighborhood of $(P_1, \ldots, P_N)$ in $(\mathbf{P}^2)^N$, so $\pi_2^{-1}(U)$ is smooth and contains $(D_1, \ldots, D_N; C; P_1, \ldots, P_N)$. On the other hand, since no C' close to C is bitangent to any D'_i close to D_i, it is possible to find arbitrary small neighborhoods V' and V respectively of $(D_1, \ldots, D_N; C; P_1, \ldots, P_N)$ in I and of $(D_1, \ldots, D_N; C)$ in $\bar{Y}$ such that $\pi_1 \mid V' \colon V' \to V$ is bijective. By taking V' compact, it follows that V' and V are homeomorphic. Since the complement of a codimension one subset of a smooth variety over $\mathbf{C}$ is connected and since V' is smooth and homeomorphic to V, it is clear that V has the desired property.

5. The case $r = 1$, $d_1 = d_2 = 2$.

The problem we have to solve now is the determination of the Galois group of the four tangents to two smooth conics in general position. This is equivalent, by duality, to find the Galois group of the four intersections of two smooth conics in general position. We will show that the Galois group of the problem is the symmetric group on four elements, by solving a more general problem.

Let $C \subset \boldsymbol{P}^N$ be an irreducible curve of degree $n > 1$. Let B be defined by

$$B = \{(H,P) \mid P \in H \cap C\} \subset \check{\boldsymbol{P}}^N \times C .$$

Consider the projections

$$\begin{tikzcd} & B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \check{\boldsymbol{P}}^N & & C \end{tikzcd}$$

It is clear that B is an irreducible N-dimensional variety and p_1 is a map of degree n. If we denote by G the Galois group associated to the map p_1, it follows that the action of G is transitive on the general fiber of p_1.

The following proposition was proved by J. Harris in [4, (Lemma on page 38)]. We will give a different proof of it.

Proposition 8. *G is the symmetric group on n letters.*

Proof. Consider the morphism induced by p_2:

$$B \times_{\check{\boldsymbol{P}}^N} B \setminus \Delta \to C \times C .$$

Over a point $(P,Q) \in C \times C$ with $P \neq Q$, the fiber is the set of hyperplanes containing P and Q, so it is a $\boldsymbol{P}^{N-2}$, therefore $B \times_{\check{\boldsymbol{P}}^N} B \setminus \Delta$ is irreducible, hence the same argument we used in the proof of Proposition 6, shows that G acts doubly transitively on the general fiber of p_1.

To produce the special fiber, and hence the transposition, take a general point H in $\check{C}$. From Proposition 1, it follows that for some smooth point P_0 of C we have that $m_{P_0}(C.H)=2$ and $m_P(C.H)=1$ for all the other points $P \in C \cap H$. So the fiber of p_1 over H has a unique ramification point (H,P_0) which is a simple ramification. Since B is smooth at (H,P_0), it follows that there are arbitrary small neighborhoods of (H,P_0) in B such that the complement of the ramification locus in each of these neighborhoods is connected. From this it follows that G contains a transposition and hence it is equal to S_n.

This proposition gives us the Galois group of the Bézout problem:

Corollary 1. *The Galois group of the $m.n$ intersection points of a fixed curve of degree m with a general curve of degree n is the symmetric group S_{nm}.*

PROOF. Apply the proposition to the n-fold Veronese embedding of the curve of degree m.

COROLLARY 2. *The Galois group of the tangency problem for plane curves in the case* $r=1$, $d_1=d_2=2$ *is* S_4.

REFERENCES

1. W. Fulton, S. Kleiman, and R. MacPherson, *About the enumeration of contacts* in *Algebraic geometry-open problems* (Proc., Ravello, 1982), eds. C. Ciliberto, F. Ghione, F. Orecchia. (Lecture Notes in Math. 997), pp. 156–196. Springer-Verlag, Berlin - Heidelberg - New York, 1983.
2. W. Fulton and R. MacPherson, *Defining algebraic intersection*, in *Algebraic geometry* (Proc. Sympos. Univ. Tromsø, Norway, 1977), ed. L. D. Olson. (Lecture Notes in Math. 687), pp. 1–30. Springer-Verlag, Berlin - Heidelberg - New York, 1978.
3. J. Harris, *Galois groups of enumerative problems*, Duke Math. J. 46 (1979), 685–724.
4. J. Harris, *A bound on the geometric genus of projective varieties*, Ann. Scuola Norm. Sup. Pisa (4) 8 (1981), 35–68.
5. S. L. Kleiman, *The enumerative theory of singularities*, in *Real and complex singularities* (Proc. Sympos., Oslo, 1976), ed. P. Holm, pp. 297–396. Sijthoff and Noordhoff, Alphen aan den Rijn 1977.
6. R. Piene, *Numerical characters of a curve in projective n-space*, in *Real and complex singularities* (Proc. Sympos., Oslo, 1976), ed. P. Holm, pp. 475–495. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
7. R. J. Walker, *Algebraic curves*, (Princeton Math. Series 13), Princeton Univ. Press, Princeton, N.J. 1950.

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