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NONUNIQUENESS OF IMMEDIATE MAXIMAL EXTENSIONS OF A VALUATION*

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An extension w of a valuation v is *immediate* if the residue field and value group of w are the same as those of v, that is, if $e(w/v)=f(w/v)=1$ where $e(w/v)$ is the ramification index, $f(w/v)$ the residue class degree of w over v. A valuation is *maximal* if it admits no proper immediate extensions. A maximal valuation of K is henselian [5, p. 231], that is, v admits only one extension to any algebraic extension of K (we adopt the convention that an extension of a valuation v of K to an algebraic extension takes its values in the divisible group generated by the value group of v).

Krull [3, § 13] established that every valuation has an immediate maximal extension, and in [2] Kaplansky obtained conditions insuring the uniqueness of an immediate maximal extension. (An immediate maximal extension w to M of a valuation v of K is *unique* if for every immediate maximal extension w' to M' of v, there is a K-isomorphism σ from M to M' satisfying $w' \circ \sigma = w$.)

If w is a valuation of a field M whose residue field k has prime characteristic p, we shall say that w is a *Kaplansky valuation* if w is maximal, the value group G of w satisfies $p.G=G$, and k satisfies the following condition:

(K) For any $\beta, \beta_0, \beta_1, \ldots, \beta_{n-1} \in k$, the polynomial $X^{p^n} + \beta_{n-1}X^{p^{n-1}} + \ldots + \beta_1 X^p + \beta_0 X + \beta$ has a root in k.

If w is a maximal valuation of M that is an immediate extension of valuation v of K, we shall say that w satisfies the *Uniqueness Condition* relative to v if for every subfield L of M containing K, w is a unique immediate maximal extension of its restriction to L.

Kaplansky [2, Theorem 5] proved that an immediate maximal extension w of a valuation v is unique if either the residue field k of w has characteristic zero or k has prime characteristic and w is a Kaplansky valuation. Since these conditions pertain only to w, we conclude:

THEOREM A (Kaplansky). *Let w be an immediate maximal extension of a*

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valuation v. If the residue field k of w has characteristic zero, or if k has prime characteristic and w is a Kaplansky valuation, then w satisfies the Uniqueness Condition relative to v.

Kaplansky showed by an example [2, pp. 318–9] that an immediate maximal extension did not always satisfy the Uniqueness Condition. Our principal purpose here is to show that if the residue field of a valuation v of K has prime characteristic, and if the value group of v is archimedean in case K has zero characteristic, then an immediate maximal extension w of v to M satisfying the Uniqueness Condition is necessarily a Kaplansky valuation unless M is a rather small extension of K, more precisely, unless the algebraic closure of K in M is dense. For this, we need to recall some concepts and theorems concerning maximal valuation:

An *Ostrowski net* $(a_\beta)_{\beta\in B}$ for a valuation v of K is a family of elements of K indexed by a totally ordered set B having no largest element such that $v(a_\lambda - a_\mu) < v(a_\mu - a_\nu)$ whenever $\lambda<\mu<\nu$. The *gauge* of an Ostrowski net $(a_\beta)_{\beta\in B}$ is the family $(\gamma_\beta)_{\beta\in B}$ of elements of the value group G of v such that $\gamma_\beta = v(a_\beta - a_\lambda)$ for all $\lambda>\beta$; it is a strictly increasing family. Obviously, an Ostrowski net for a valuation v is also one for any extension of v. Committing an abuse of language, we shall say that an element $c \in K$ is *adherent* to an Ostrowski net $(a_\beta)_{\beta\in B}$ *in* K (or *for* v) if c is adherent to the filter base $(a_\beta + M_{\gamma_\beta})_{\beta\in B}$ on K, where $(\gamma_\beta)_{\beta\in B}$ is the gauge of $(a_\beta)_{\beta\in B}$ and

$$M_{\gamma_\beta} = \{x \in K : v(x) \geqq \gamma_\beta\} .$$

As each M_{γ_β} is closed, c is adherent to $(a_\beta)_{\beta\in B}$ if and only if

$$c \in \bigcap_{\beta\in B} (a_\beta + M_{\gamma_\beta}) ,$$

or equivalently, if and only if $v(c-a_\beta) = \gamma_\beta$ for all $\beta \in B$. We note that the associated filter base $(a_\beta + M_{\gamma_\beta})_{\beta\in B}$ is a Cauchy filter base if and only if the gauge $(\gamma_\beta)_{\beta\in B}$ is cofinal in G, or equivalently, if and only if $(\gamma_\beta)_{\beta\in B}$ is unbounded above in G. Kaplansky [2, Theorem 4] proved that a valuation v of K is maximal if and only if each Ostrowski net for v has an adherent point in K.

Among the theorems we shall use are the following:

THEOREM B [2, Theorem 1]. *If a valuation w of a field L is an immediate extension of a valuation v of K, then for any $c \in L \setminus K$ there is an Ostrowski net $(a_\beta)_{\beta\in B}$ for v such that the points adherent to $(a_\beta)_{\beta\in B}$ for w include c but no point of K.*

THEOREM C [2, p. 306; 5, pp. 94–5]. *If $(a_\beta)_{\beta \in B}$ is an Ostrowski net for a valuation v of K and if f is a nonzero polynomial over K, then $(v(f(a_\beta)))_{\beta \in B}$ is either eventually strictly increasing or eventually stationary, and the former case occurs if and only if some root of f in the algebraic closure Ω of K is adherent to $(a_\beta)_{\beta \in B}$ for some extension of v to Ω.*

If there is a nonzero polynomial $f \in K[X]$ such that $(v(f(a_\beta)))_{\beta \in B}$ is eventually strictly increasing, $(a_\beta)_{\beta \in B}$ is called *algebraic*, and any monic polynomial q of least degree such that $(v(q(a_\beta)))_{\beta \in B}$ is eventually strictly increasing is called a *minimal* polynomial of $(a_\beta)_{\beta \in B}$. If $(a_\beta)_{\beta \in B}$ is not algebraic, it is called *transcendental*. The following theorem was proved in a different way by Kaplansky [2, pp. 307–8]:

THEOREM D. *Let $(a_\beta)_{\beta \in B}$ be an Ostrowski net for a valuation v of K, and let w be an extension of v to L. If $c \in L$ is adherent to $(a_\beta)_{\beta \in B}$ for w, then for any nonzero polynomial $f \in K[X]$ such that $(v(f(a_\beta)))_{\beta \in B}$ is eventually stationary, $w(f(c)) = v(f(a_\mu))$ for all sufficiently large $\mu \in B$.*

PROOF. There is a polynomial $h \in L[X]$ such that

$$f(c) - f(X) = (c - X)h(X) .$$

As $(w(c - a_\beta))_{\beta \in B}$ is strictly increasing and as $(w(h(a_\beta)))_{\beta \in B}$ is either eventually stationary or eventually strictly increasing, $(w(f(c) - f(a_\beta)))_{\beta \in B}$ is eventually strictly increasing. Therefore for sufficiently large λ, if $\mu > \lambda$, then

$$\begin{aligned} w(f(c) - f(a_\mu)) > w(f(c) - f(a_\lambda)) &\geqq \min\{w(f(c)), w(f(a_\lambda))\} \\ &= \min\{w(f(c)), w(f(a_\mu))\} . \end{aligned}$$

Consequently, either $w(f(c)) < w(f(c) - f(a_\mu))$ or $w(f(a_\mu)) < w(f(c) - f(a_\mu))$, and in either case

$$w(f(c)) = \min\{w(f(c) - f(a_\mu)), w(f(a_\mu))\} = w(f(a_\mu)) .$$

This theorem is used to establish, in particular, the following two theorems:

THEOREM E [2, Theorem 2]. *If $(a_\beta)_{\beta \in B}$ is a transcendental Ostrowski net for a valuation v of K and if c is adherent to $(a_\beta)_{\beta \in B}$ for an extension w of v to L, then c is transcendental over K, for any polynomial $f \in K[X]$,*

$$w(f(c)) = w(f(a_\mu)) \quad \textit{for all sufficiently large } \mu \in B ,$$

and the restriction of w to $K(c)$ is an immediate extension of v.

THEOREM F [2, Theorem 3]. *If q is a minimal polynomial of an algebraic Ostrowski net $(a_\beta)_{\beta \in B}$ for a valuation v of K, then q is a prime polynomial, and there is an immediate extension of v to a stem field of q.*

Theorems E and F are used to establish the following theorem, which is implicit in [2]:

THEOREM G. *Let v be a valuation of K. An immediate maximal extension of v satisfies the Uniqueness Condition relative to v if and only if for every immediate maximal extension w of v to a field M and for each subfield L of M containing K, every minimal polynomial of every algebraic Ostrowski net on L has a root in M.*

We begin with some results concerning henselian valuations:

THEOREM 1. *The restriction of a henselian valuation w of L to a subfield K of L that is algebraically closed in L is henselian.*

PROOF. A valuation u of a field F is henselian if and only if every polynomial over F of the form

$$X^n + a_{n-1}X^{n-1} + \ldots + a_1 X + a_0 ,$$

where $u(a_i) \geqq 0$ for all $i \in [0, n-1]$, $u(a_1) = 0$, $u(a_0) > 0$, has a root c in F satisfying $u(c) > 0$ [4, pp. 94-5]. Let f be such a polynomial over K for v. Then f has a root $c \in L$ satisfying $w(c) > 0$. But as K is algebraically closed in L, $c \in K$. Therefore v is henselian.

Theorem 1 provides an easy proof of the fact that a henselization of a valuation [5, pp. 175-8] is an immediate extension:

THEOREM 2 [5, p. 184]. *A henselization of a valuation v of K is an immediate extension of v.*

PROOF. By Krull's theorem, v has an immediate maximal extension w to a field M. As w is henselian, its restriction to the algebraic closure H of K in M is henselian by Theorem 1. As H is an algebraic extension of K, H contains a henselization of K, which therefore is an immediate extension of K. Since any two henselizations of v are equivalent, the assertion follows.

The same proof establishes the following:

THEOREM 3. *If w is a henselian valuation of a field L, every subfield of L has a henselization contained in L.*

We may now identify those immediate maximal extensions of a valuation that are so small their uniqueness is assured without any further hypothesis:

THEOREM 4. *Let w be an immediate maximal extension to M of a valuation v of K. If $M=\hat{K_h}$, the completion of a henselization K_h of K, then w satisfies the Uniqueness Condition relative to v.*

PROOF. We first observe that if K_r is another henselization of K contained in M, then $M=\hat{K_r}$. Indeed, there is a K-isomorphism from K_h to K_r satisfying $(w\circ\sigma)(x)=w(x)$ for all $x \in K_h$, and σ extends by continuity to a K-isomorphism $\sigma\hat{}$ from $M=\hat{K_h}$ to $\hat{K_r}$ satisfying $w\circ\sigma\hat{}=w$. Therefore $\hat{K_r}$ is a maximal field. As M is a immediate extension of $\hat{K_r}$, $M=\hat{K_r}$.

Now let L be any subfield of M containing K. By Theorem 3, M contains a henselization L_h of L. Again by Theorem 3, L_h contains a henselization K_r of K. As M is complete, $M\supseteqq\hat{L_h}\supseteqq\hat{K_r}=M$. Let w' be an immediate maximal extension to M' of the restriction of w to L. By Theorem 3, M' contains a henselization L'_h of L. Thus there is an L-isomorphism τ from L_h to L'_h satisfying $(w'\circ\tau)(x)=w(x)$ for all $x \in L_h$, and τ extends by continuity to an L-monomorphism $\tau\hat{}$ from $M=\hat{L_h}$ into M' satisfying $w'\circ\tau\hat{}=w$. Again, M' is an immediate maximal extension of the maximal field $\tau\hat{}(M)$, so $M'=\tau\hat{}(M)$.

COROLLARY. *If v is a valuation of a field K and if its completion $v\hat{}$ on $K\hat{}$ is a maximal valuation, then $v\hat{}$ satisfies the Uniqueness Condition relative to v.*

Our principal theorem is the following:

THEOREM 5. *Let v be a valuation of a field K whose residue field k has prime characteristic p, let w be an immediate maximal extension of v to M, let L be the algebraic closure of K in M, and let G be the value group of v and w. If w satisfies the Uniqueness Condition relative to v and if $M\neq L\hat{}$, then k satisfies* (K); *in addition, if either K has characteristic p or G is archimedean, $p.G=G$.*

PROOF. Let $c \in M\setminus L\hat{}$, and let v' be the restriction of w to L. By Theorem B there is an Ostrowski net $(a_\beta)_{\beta\in B}$ for v' whose adherent points in M include c but no point of L. As $c \notin L\hat{}$, the filter base corresponding to $(a_\beta)_{\beta\in B}$ is not a Cauchy filter base, so the gauge $(\gamma_\beta)_{\beta\in B}$ of $(a_\beta)_{\beta\in B}$ is bounded above by some $\gamma \in G$. Let $a \in L$ satisfy $v'(a)=\gamma$. Then $(a^{-1}a_\beta)_{\beta\in B}$ is an Ostrowski net for v' whose gauge is bounded above by zero and whose adherent points in M include $a^{-1}c$ but no points of $a^{-1}L=L$. Therefore we may assume that $(\gamma_\beta)_{\beta\in B}$ is bounded above by zero.

Suppose $(a_\beta)_{\beta\in B}$ were an algebraic Ostrowski net, and let $q \in L[X]$ be a minimal polynomial. By Theorem G, M contains a root d of q, and $d \in L$ as L is algebraically closed in M. By Theorem C, some conjugate d' of d in an algebraic closure Ω of L is adherent to $(a_\beta)_{\beta \in B}$ for an extension v'' of v' to Ω. As w is henselian, so is v' by Theorem 1. If σ is an L-automorphism of Ω such that $\sigma(d)=d'$, then $v''\circ\sigma=v''$ as v' is henselian, so

$$\gamma_\beta = v''(d'-a_\beta) = v''(\sigma(d)-\sigma(a_\beta)) = (v''\circ\sigma)(d-a_\beta) = v'(d-a_\beta),$$

and hence $d \in L$ is adherent to $(a_\beta)_{\beta\in B}$, a contradiction. Thus $(a_\beta)_{\beta\in B}$ is a transcendental Ostrowski net. (If K has characteristic p, we are now in the situation of Kaplansky's counter-example [2, pp. 318-9].)

We shall show that if $b, b_0, b_1, \ldots, b_{n-1}$ belong to the valuation ring $A_{v'}$ of v' and if

$$g(X) = X^{p^n}+b_{n-1}X^{p^{n-1}}+\ldots+b_1X^p+b_0X,$$

then there exists $c_1 \in M$ such that $g(c)-g(c_1)=-b$. To do so, we shall first establish that for some subset C of B of the form $B\setminus I$, where $I=\varnothing$ or I is an initial segment of B (a set of the form $\{\beta \in B\colon \beta\leqq\delta\}$ for some $\delta \in B$), $(g(a_\beta))_{\beta\in C}$ is an Ostrowski net for v' whose gauge is bounded above by zero, and $g(c)$ is adherent to $(g(a_\beta))_{\beta\in C}$ for w.

CASE 1. K has characteristic p. We take $C=B$. For any $x \in M$ such that $w(x)<0$, clearly $w(g(x))=p^n w(x)$. Since $g(y)-g(z)=g(y-z)$ for all $y,z \in M$, it follows readily that $(g(a_\beta))_{\beta\in B}$ is an Ostrowski net whose gauge is $(p^n\gamma_\beta)_{\beta\in B}$, and that $g(c)$ is adherent to $(g(a_\beta))_{\beta\in B}$ for w.

CASE 2. K has zero characteristic. We first observe that if $x, b \in M$ satisfy $w(b)<w(x-b)<0$ and if $w(p)\neq(p-1)(w(x-b)-w(b))$, then

$$w(x^p-b^p) = \min\{pw(x-b), w(p)+(p-1)w(b)-w(x-b)\}. \tag{1}$$

Indeed, let $f(X)=X^p-b^p$. Expanding f in Taylor series about b, we obtain

$$f(X) = \sum_{j=1}^{p}\binom{p}{j}b^{p-j}(X-b)^j.$$

If $1\leqq j<p$, p divides $\binom{p}{j}$ in the valuation A_w of w but p^2 does not, so $w\left(\binom{p}{j}\right)=w(p)$. Consequently, if $1<j<p$, then as $w(x-b)>w(b)$,

$$\begin{aligned} w\left(\binom{p}{j}b^{p-j}(x-b)^j\right) &= w(p)+(p-j)w(b)+jw(x-b) \\ &> w(p)+(p-1)w(b)+w(x-b) = w(pb^{p-1}(x-b)). \end{aligned}$$

Thus

$$w\left(\sum_{j=1}^{p-1}\binom{p}{j}b^{p-j}(x-b)^j\right) = w(p)+(p-1)w(b)+w(x-b)\,.$$

Also

$$w((x-b)^p) = pw(x-b) \neq w(p)+(p-1)w(b)+w(x-b)$$

by hypothesis. Hence (1) holds.

As $(a_\beta)_{\beta\in B}$ is transcendental, $(v'(a_\beta))_{\beta\in B}$ is eventually stationary, so there exists $\beta_0 \in B$ and $\gamma \in G$ such that $v'(a_\beta)=\gamma$ for all $\beta \geqq \beta_0$.

Let

$$B_0 = \{\beta \in B : \beta > \beta_0\}\,.$$

For any $\beta \in B_0$,

$$\gamma_\beta > \gamma_{\beta_0} = v'(a_\beta - a_{\beta_0}) \geqq \min\{v'(a_\beta), v'(a_{\beta_0})\} = \gamma\,.$$

Thus $(a_\beta)_{\beta\in B_0}$ is an Ostrowski net for v' whose gauge $(\gamma_{0,\beta})_{\beta\in B_0}$ satisfies

$$v'(a_\beta) = \gamma < \gamma_{0,\beta} \qquad \text{for all } \beta \in B_0\,.$$

As $(\gamma_{0,\beta})_{\beta\in B_0}$ is strictly increasing, there is at most one $\tau_0 \in B$ such that $(p-1)(\gamma_{0,\tau_0}-\gamma)=v'(p)$; let

$$B_1 = \{\beta \in B_0 : \beta > \tau_0\}$$

if there is such a τ_0, $B_1 = B_0$ otherwise. By (1), for any $\lambda, \mu, \nu \in B_1$ such that $\lambda < \mu < \nu$,

$$\begin{aligned} v'(a_\lambda^p - a_\mu^p) &= \min\{p\gamma_{0,\lambda}, v'(p)+(p-1)\gamma+\gamma_{0,\lambda}\} \\ &< \min\{p\gamma_{0,\mu}, v'(p)+(p-1)\gamma+\gamma_{0,\mu}\} = v'(a_\mu^p - a_\nu^p)\,. \end{aligned}$$

Thus $(a_\beta^p)_{\beta\in B_1}$ is an Ostrowski net whose gauge $(\gamma_{1,\beta})_{\beta\in B_1}$ satisfies

$$\gamma_{1,\beta} = \min\{p\gamma_{0,\beta}, v'(p)+(p-1)\gamma+\gamma_{0,\beta}\}\,.$$

In particular,

$$\gamma_{1,\beta} \leqq p\gamma_{0,\beta} < \gamma_{0,\beta} < 0 \qquad \text{for all } \beta \in B_1\,.$$

Moreover, $v'(a_\beta^p)=p\gamma<\gamma_{1,\beta}$ for all $\beta \in B_1$. Furthermore, by (1)

$$\begin{aligned} w(c^p - a_\beta^p) &= \min\{pw(c-a_\beta), w(p)+(p-1)w(a_\beta)+w(c-a_\beta)\} \\ &= \min\{p\gamma_{0,\beta}, v'(p)+(p-1)\gamma+\gamma_{0,\beta}\} = \gamma_{1,\beta} \end{aligned}$$

for all $\beta \in B_1$, so c^p is adherent to $(a_\beta^p)_{\beta\in B_1}$. Applying this result successively to

$$(a^p), (a^{p^2}), \ldots, (a^{p^{n-1}})\,,$$

we obtain a decreasing sequence $B_0, B_1, \ldots, B_n$ of subsets of B, each either B or the complement of an initial segment of B, such that

$$(a_\beta)_{\beta \in B_0}, (a_\beta^p)_{\beta \in B_1}, \ldots, (a_\beta^{p^n})_{\beta \in B_n}$$

are Ostrowski nets whose respective gauges

$$(\gamma_{0,\beta})_{\beta \in B_0}, (\gamma_{1,\beta})_{\beta \in B_1}, \ldots, (\gamma_{n,\beta})_{\beta \in B_n}$$

satisfy $\gamma_{n,\beta} < \ldots \gamma_{1,\beta} < \gamma_{0,\beta} < 0$ for all $\beta \in B_n$ and $v'(a^{p^j}) = p^j\gamma < \gamma_{j,\beta}$ for all $j \in [0, n]$ and all $\beta \in B_j$, and whose adherent points for w respectively include $c, c^p, \ldots, c^{p^n}$. Let $C = B_n$. If $j \in [0, n-1]$, then $v'(b_j) \geqq 0$, so if $\lambda, \mu \in C$ and $\lambda < \mu$,

$$v'(b_j a_\lambda^{p^j} - b_j a_\mu^{p^j}) \geqq v'(a_\lambda^{p^j} - a_\mu^{p^j}) = \gamma_{j,\lambda} > \gamma_{n,\lambda} = v(a_\lambda^{p^n} - a_\mu^{p^n}),$$

whence

$$v'(g(a_\lambda) - g(a_\mu)) = v'\left((a_\lambda^{p^n} - a_\mu^{p^n}) + \sum_{j=0}^{n-1} b_j(a_\lambda^{p^j} - a_\mu^{p^j})\right) = \gamma_{n,\lambda}.$$

As $(\gamma_{n,\beta})_{\beta \in C}$ is the gauge of the Ostrowski net $(a_\beta^{p^n})_{\beta \in C}$, therefore, $(g(a_\beta))_{\beta \varepsilon C}$ is an Ostrowski net whose gauge $(\gamma_{n,\beta})_{\beta \in C}$ is bounded above by zero. By the same reasoning,

$$w(g(c) - g(a_\beta)) = \gamma_{n,\beta} \quad \text{for all } \beta \in C,$$

so $g(c)$ is adherent in M to $(g(a_\beta))_{\beta \in C}$.

Thus in both cases $(g(a_\beta))_{\beta \in C}$ is an Ostrowski net for v' whose gauge is bounded above by zero, and $g(c)$ is adherent in M to $(g(a_\beta))_{\beta \in C}$. As $v'(b) \geqq 0$, therefore, $g(c) + b$ is also adherent in M to $(g(a_\beta))_{\beta \in C}$. Moreover, $(g(\alpha_\beta))_{\beta \in C}$ is clearly transcendental as $(a_\beta)_{\beta \in C}$ is. Therefore by Theorem E, $g(c)$ and $g(c) + b$ are transcendental over L, and for any $f \in L[X]$,

$$w(f(g(c) + b))) = w(f(g(a_\mu))) = w(f(g(c)))$$

for all sufficiently large $\mu \in C$. Let $N = L(g(c)) = L(g(c) + b)$, and let τ be the L-automorphism of N satisfying $\tau(g(c)) = g(c) + b$. Then $(w \circ \tau)(z) = w(z)$ for all $z \in N$.

Let $\bar{\tau}$ be the automorphism of $N[X]$ induced by τ. In $N(c)$, c is a root of $g(X) - g(c)$. Let r be the minimal polynomial of c over N and let c' be a root of $\bar{\tau}(r)$ in a stem field $N(c')$ of $\bar{\tau}(r)$. Then c' is also a root of

$$\bar{\tau}(g(X) - g(c)) = g(X) - \tau(g(c)) = g(X) - (g(c) + b),$$

that is, $g(c') = g(c) + b$. Let τ' be the unique isomorphism from $N(c)$ to $N(c')$ extending τ such that $\tau'(c) = c'$, and let w' be an immediate maximal extension to M' of the valuation $w \circ \tau'^{-1}$ of $N(c')$. Both w and w' are then immediate maximal extensions of the restriction of w to N, as $(w \circ \tau)(z) = w(z)$ for all $z \in N$.

By hypothesis, there is an N-isomorphism σ from M to M'. As $c' \in M'$ is a root of $g(X)-g(c)-b \in N[X]$, $c_1=\sigma^{-1}(c')$ is a root of $g(X)-g(c)-b$ in M. Thus $g(c)-g(c_1)=-b$.

CASE 1. K has characteristic p. Then $g(c-c_1)=g(c)-g(c_1)=-b$, so

$$X^{p^n}+b_{n-1}X^{p^{n-1}}+\ldots+b_1X^p+b_0X+b$$

has the root $c-c_1$ in M, and clearly $w(c-c_1)\geqq 0$. Consequently, k satisfies (K). Moreover, $p.G=G$, for if $\gamma\geqq 0$ and if $b \in L$ satisfies $v'(b)=\gamma$, then X^p-b has a root x in M, so $w(x) \in G$ and $p.w(x)=\gamma$.

CASE 2. K has characteristic zero. As p divides $g(c-c_1)-(g(c)-g(c_1))$ in the valuation ring A_w of w,

$$g(c-c_1) \equiv g(c)-g(c_1) \pmod{pA_w},$$

that is $g(c-c_1)\equiv -b \pmod{pA_w}$. In particular, as p belongs to the maximal ideal M_w of A_w, $g(c-c_1)\equiv -b \pmod{M_w}$. Thus the polynomial in $k[X]$ corresponding to

$$X^{p^n}+b_{n-1}X^{p^{n-1}}+\ldots+b_1X^p+b_0X+b$$

has a root in k. Hence k satisfies (K). We shall next show that the isolated subgroup H of G generated by $v(p)$ satisfies $p.H=H$. First, suppose $\gamma \in G$ satisfies $0\leqq\gamma<v(p)$. Applying the preceding to X^p-b, where $b \in L$ satisfies $v'(b)=\gamma$, we conclude there exists $x \in A_w$ such that $x^p\equiv b \pmod{pA_w}$, that is, $w(x^p-b)\geqq w(p)$. Hence as $w(b)<w(p)$, $w(x^p)=w(b)$, so $w(x) \in G$ and $p.w(x)=\gamma$; in particular, $0\leqq w(x)\leqq\gamma$, so $w(x) \in H$. Suppose next that $p^{-1}.\sigma \in G$ whenever $0\leqq\sigma<m.v(p)$, where $m\geqq 1$, and let $\gamma \in G$ satisfy $m.v(p)\leqq\gamma<(m+1).v(p)$. Then $0\leqq\gamma-m.v(p)<v(p)$. By hypothesis, $M\neq \hat{K}$, so v is not discrete. Consequently, there exists $\tau \in G$ such that $\gamma-m.v(p)<\tau<v(p)\leqq\gamma$, whence $0<\gamma-\tau<m.v(p)$, so both $p^{-1}.\tau$, $p^{-1}.(\gamma-\tau) \in G$, and therefore

$$p^{-1}.\gamma = p^{-1}.\tau+p^{-1}.(\gamma-\tau) \in G.$$

As $0\leqq p^{-1}.\gamma\leqq\gamma$, $p^{-1}.\gamma \in H$. Thus $p.H=H$. In particular, if G is archimedean, then $H=G$, so $p.G=G$.

COROLLARY. *Let v be a valuation of a field K whose residue field has prime characteristic p, and let w be an immediate maximal extension of v to M. If w satisfies the Uniqueness Condition relative to v, if the algebraic closure of K in M is not dense in M, and if either K has characteristic p or the value group of w is archimedean, then w is a Kaplansky valuation.*

With the notation of Theorem 5, assume K has prime characteristic or that the value group is archimedean, and let K_h be a henselization of K in M, whence $K_h \subseteq L$. If $M = \hat{K_h}$, w satisfies the Uniqueness Condition relative to v without any further restrictions by Theorem 4, but if $M \supset \hat{L}$, w satisfies the Uniqueness Condition relative to v if and only if w is a Kaplansky valuation by Theorems A and 5. Whether w must be a Kaplansky valuation for it to satisfy the Uniqueness Condition remains an open question for the case $\hat{K_h} \subset \hat{L} = M$.

To demonstrate the ubiquity of nonunique immediate maximal extensions, we shall apply Theorem 5 to the classical example of an immediate maximal extension introduced by Krull. Let k be a field, G a totally ordered abelian group, and let $S(k, G)$ be the set of all functions from G to k whose support is a well-ordered subset of G (the support, Supp f, of $f \in k^G$ is defined to be $\{x \in G: f(x) \neq 0\}$). Under the usual addition and convolution (defined by $(fg)(z) = \sum f(x)g(y)$, the sum over all $(x, y) \in G \times G$ such that $x + y = z$), $S(k, G)$ is a field; we equip $S(k, G)$ with the valuation w satisfying $w(f) =$ the smallest element in Supp f for every nonzero $f \in S(k, G)$. (See [1, Exercise 2, § 3, Ch. 6, pp. 173–4].) We denote by $F[k, G]$ the subring of all functions with finite support, and by $F(k, G)$ its field of quotients in $S(k, G)$. Krull proved that w is a maximal valuation, and hence that $S(k, G)$ is an immediate maximal extension of $F(k, G)$ [3, Satz 26]. (We shall apply the terminology for valuations to the fields on which they are defined, as we are henceforth interested only in the restrictions of w to subfields of $S(k, G)$.) For each $x \in G$, we denote by δ_x the member of $F[k, G]$ defined by $\delta_x(x) = 1$, $\delta_x(y) = 0$ if $y \neq x$. Clearly $(\delta_x f)(y) = f(y - x)$ for all $y \in G$.

Let G be a cyclic group. There is a topological isomorphism from $S(k, G)$ to $k((X))$, equipped with the X-adic valuation, that takes $F(k, G)$ to $k(X)$. Consequently $S(k, G)$ is the completion of $F(k, G)$, so by the Corollary of Theorem 4, $S(k, G)$ satisfies the Uniqueness Condition relative to $F(k, G)$. This establishes the sufficiency of the condition in the following theorem:

THEOREM 6. *Let k be a field of prime characterstic p, and let G be a subgroup of the additive group $\mathbf{Q}$ of rational numbers such that $p.G \neq G$. Then $S(k, G)$ satisfies the Uniqueness Condition relative to $F(k, G)$ if and only if G is cyclic.*

We note that each element of $F[k, G]$ is contained in $F[k, a\mathbf{Z}]$ for some rational $a > 0$, and consequently each element of $F(k, G)$ is contained in $F(k, b\mathbf{Z})$ for some rational $b > 0$. To establish the necessity of the condition, we need two lemmas:

LEMMA 1. *Let L be a subfield of $S(k, \mathbf{Q})$ containing $F(k, ea\mathbf{Z})$, where a is a positive rational, e a positive integer. If the value group of L is $a\mathbf{Z}$ and if $p \nmid e$, then $L \subseteq S(k, a\mathbf{Z})$.*

PROOF. We shall first prove that any uniformizer u of L belongs to $S(k,a\mathbf{Z})$. As $w(u)=a$, $u(a)\neq 0$; multiplying u by $u(a)^{-1}\in k$, we may assume $u(a)=1$. Suppose $u\notin S(k,a\mathbf{Z})$. Then there is a smallest $t\in \operatorname{Supp} u\setminus a\mathbf{Z}$. Let $n\geqq 1$ be the largest integer such that $na<t$. We define recursively $u_1,\ldots,u_n\in L$ by $u_1=u$, $u_{r+1}=u_r-u_r((r+1)a)u_r^{r+1}$ for all $r\in[1,n-1]$. Straightforward calculations establish that for each $r\in[1,n]$, $u_r(a)=1$, $u_r(t)=u(t)\neq 0$, and the numbers $<t$ in $\operatorname{Supp} u_r$ are among a, $(r+1)a, (r+2)a,\ldots,na$. Let $z=u_n\in L$. Then $z(a)=1$, $z(t)\neq 0$, and a, t are the smallest numbers in $\operatorname{Supp} z$. Easy calculations establish that for each integer $q\geqq 0$, $z^{q+1}((q+1)a)=1$, $z^{q+1}(qa+t)=(q+1)z(t)$, and $z^{q+1}(x)=0$ for all $x<qa+t$ other than $(q+1)a$. Consequently,

$$(z^{q+1}-\delta_{qa}z)(qa+t) = qz(t) \quad\text{and}\quad (z^{q+1}-\delta_{qa}z)(x) = 0$$

for all $x<qa+t$. In particular, let $q=e$; then $\delta_{ea}\in F(k,ea\mathbf{Z})\subseteqq L$, so $z^{e+1}-\delta_{ea}z\in L$,

$$(z^{e+1}-\delta_{ea}z)(ea+t) = ez(t) \neq 0 \quad \text{as } p\nmid e,$$

and consequently $w(z^{e+1}-\delta_{ea}z)=ea+t\notin \mathbf{Z}a$, a contradiction. Therefore, $u\in S(k,a\mathbf{Z})$.

Next, we shall show that a unit y of the valuation ring of L belongs to $S(k,a\mathbf{Z})$. As before, we may assume $y(0)=1$. Then $w(y-\delta_0)>0$, so $w(y-\delta_0)\geqq a$. If $w(y-\delta_0)=a$, then by the preceding $y-\delta_0\in S(k,a\mathbf{Z})$, whence $y\in S(k,a\mathbf{Z})$. If $w(y-\delta_0)>a$, let u be a uniformizer of L; then $w(y-\delta_0+u)=a$, so $y-\delta_0+u\in S(k,a\mathbf{Z})$, whence again $y\in S(k,a\mathbf{Z})$. Therefore $L\subseteqq S(k,a\mathbf{Z})$ as each nonzero element of L is a product of a unit and a power of a uniformizer.

LEMMA 2. *Let $(G_n)_{n\geqq 1}$ be an increasing sequence of cyclic groups such that $G=\bigcup_{n=1}^{\infty} G_n$. The algebraic closure $A(k,G)$ of $F(k,G)$ in $S(k,G)$ is contained in $\bigcup_{n=1}^{\infty} S(k,G_n)$.*

PROOF. Since $p.G\neq G$, there is a largest integer s such that $p^{-s}\in G$; we may assume $p^{-s}\in G_1$. Let $z\in A(k,G)$. There exists $m\geqq 1$ such that each coefficient of the minimal polynomial of z belongs to $F(k,G_m)$, so $z\in A(k,G_m)$. Let $G_m=b\mathbf{Z}$, where $b>0$, and let e be the ramification index of $F(k,b\mathbf{Z})[z]$ over $F(k,b\mathbf{Z})$. The value group of $F(k,b\mathbf{Z})[z]$ is then $a\mathbf{Z}$ where $b=ea$. As $z\in S(k,G)$ and $b\mathbf{Z}\subseteqq G$, $F(k,b\mathbf{Z})[z]\subseteqq S(k,G)$, and therefore $a\mathbf{Z}\subseteqq G$. As $p^{-s}\in G_1\subseteqq b\mathbf{Z}$, $p^{-s}=bc$ for some integer c. If $e=pd$ for some integer d, then $p^{-(s+1)}=acd\in a\mathbf{Z}\subseteqq G$, a contradiction. Thus $p\nmid e$, so by Lemma 1, $F(k,b\mathbf{Z})[z]\subseteqq S(k,a\mathbf{Z})$. As $a\mathbf{Z}\subseteqq G$, $a\mathbf{Z}\subseteqq G_n$ for some $n\geqq 1$, whence $z\in S(k,G_n)$.

To complete the proof of Theorem 6, it suffices by Theorem 5 and Lemma 2 to show that if G is not cyclic, then $\bigcup_{n=1}^{\infty} S(k,G_n)$ is not dense in $S(k,G)$, where $(G_n)_{n\geqq 1}$ is an increasing sequence of cyclic groups whose union is G. As G is not cyclic, G contains a strictly increasing sequence $(r_n)_{n \geqq 1}$ of rationals in $(0, 1)$ such that $\sup r_n = 1$. Let $g \in S(k, G)$ be the characteristic function of $\{r_n : n \geqq 1\}$. Suppose $\bigcup_{n=1}^{\infty} S(k, G_n)$ were dense in $S(k, G)$. Then for some $m \geqq 1$ and some $f \in S(k, G_m)$, $w(g - f) > 1$, whence $f(r_n) = g(r_n) = 1$ for all $n \geqq 1$. But as G_m is cyclic, some $r_n \notin G_m$, whence $f \notin S(k, G_m)$, a contradiction.

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