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RESIDUAL FIELDS IN VALUATION THEORY*

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Our purpose here is to suggest that the residual fields of a valuation, which are defined in section 1, may provide a suitable framework for the formulation and solution of certain problems in valuation theory. In section 2 we shall recast the Krull-Ribenboim theory of valuations complete by stages in the language of residual fields and give short proofs of several of their theorems concerning that concept. Also, by use of residual fields, we shall extend in section 3 a classical theorem of Ostrowski asserting the finite dimensionality of a complete, separable algebraic extension of a rank one henselian valuation. For example, "complete" and "rank one" in Ostrowski's theorem may be replaced by "complete by stages." In section 4 we investigate nonhenselian valuations having a finite-dimensional henselian extension.

1. Residual fields.

Let v be a valuation of a field K with valuation ring A, maximal ideal M, and value group G. Let P be a nonmaximal prime ideal of A. Then P is the maximal ideal of the valuation subring A_P of K. If Q is any prime ideal strictly containing P, then A_Q/P is a proper valuation subring of the field A_P/P. If R is any other prime ideal strictly containing P, then A_Q/P and A_R/P are comparable since Q and R are, and therefore they define the same nondiscrete topology on A_P/P. We therefore define the *residual topology* on A_P/P to be the common topology defined by the valuation rings A_Q/P where Q is any prime ideal strictly containing P. Depending on the context, we shall the field A_P/P or the topological field A_P/P, furnished with its residual topology, the *residual field of v determined by P*. If B is any subring of K properly containing A, then $B = A_P$ for some nonmaximal prime ideal P of A, and P is thus the maximal ideal of B; we shall also say that B/P is the *residual field of v determined by B*.

The valuation subring of A_P/P determined by M is, of course, A/P; we shall denote by $\bar{v}_P$ the valuation on A_P/P determined by v; the valuation ring of $\bar{v}_P$ is A/P, its maximal ideal is M/P, and its residue field is canonically isomorphic

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with the residue field A/M of v; finally, its value group is the image H_P under v of the group of invertible elements of A_P, and

$$\bar{v}_P(x+P) = v(x)$$

for all $x \in A_P \setminus P$. In particular, $\bar{v}_P$ defines the residual topology of A_P/P. In the sequel we shall also denote by v_P the valuation of K determined by v and P; the valuation ring of v_P is A_P, its residue field is the residual field A_P/P of v, its value group is G/H_P, and

$$v_P(x) = v(x)+H_P$$

for all nonzero $x \in K$.

Identifying $K/(0)$ with K, we see that K itself is the residual field of v determined by the zero ideal, and its residual topology is the given topology defined by v.

Let K' be an algebraic extension of K, A' the valuation ring of an extension v' of v to K', G' its value group. Then $P' \mapsto P' \cap A$ is a bijection from the set of all prime ideals of A' to the set of all prime ideals of A. Let P' be a prime nonmaximal ideal of A', $P = P' \cap A$ the corresponding prime nonmaximal ideal of A. Under the standard identification of A_P/P with the subfield $(A_P + P')/P'$ of $A'_{P'}/P'$, the residual (topological) field determined by P is a subfield of the residual (topological) field determined by P'. Moreover, if $[K' : K] < +\infty$, then

$$[A'_{P'}/P' : A_P/P] = f(v'_P/v_P) \leqq [K' : K] < +\infty .$$

We shall adopt the convention that extensions of v to an algebraic extension of K take their values in the divisible group generated by G. Under this convention, distinct extensions of v to an algebraic extension are inequivalent.

THEOREM 1. *Let v be a valuation of a field K, and let P be the intersection of a family $(P_\lambda)_{\lambda \in L}$ of prime ideals of the valuation ring A of v, each strictly containing P. Then $(P_\lambda/P)_{\lambda \in L}$ is a fundamental system of neighborhoods of zero for the residual topology of A_P/P.*

PROOF. P is, of course, a nonmaximal prime ideal of A. The residual topology $\mathcal{T}_P$ of A_P/P is defined by the valuation ring A/P. For each $\lambda \in L$, P_λ/P is a nonzero prime ideal of A/P and hence is a neighborhood of zero for $\mathcal{T}_P$. Since the ideals of A/P are totally ordered and the intersection of $(P_\lambda/P)_{\lambda \in L}$ is the zero ideal of A/P, every nonzero ideal of A/P contains some P_λ/P. Therefore $(P_\lambda/P)_{\lambda \in L}$ is a fundamental system of neighborhoods of zero for $\mathcal{T}_P$.

2. Valuations complete by stages.

In 1932, Krull [8, p. 177] introduced "perfekt" valuations, which were called "complete" by Schilling [20, p. 45] and Ribenboim [15, p. 472; 16, p. 12; 17, p. 454], later "complet par etages" by Ribenboim [18, p. 72]. A valuation v of a field K is *complete by stages* if the following two conditions hold:

1°. If P is a nonmaximal prime ideal of the valuation ring A of v and if there is a smallest prime ideal Q strictly containing P, then A_P/P is complete for the topology defined by the valuation ring A_Q/P.

2°. If $(P_\lambda)_{\lambda\in L}$ is a decreasing, well-ordered family of prime ideals of A and if $(a_\lambda)_{\lambda\in L}$ is a family of elements of A satisfying $a_\lambda \equiv a_\mu \pmod{P_\lambda}$ whenever $P_\mu \subset P_\lambda$, then there exists $x \in A$ such that $x \equiv a_\lambda \pmod{P_\lambda}$ for all $\lambda \in L$.

We may characterize valuations complete by stages as follows:

THEOREM 2. *A valuation v of a field K is complete by stages if and only if all its residual fields are complete.*

PROOF. Let P be a nonmaximal prime ideal of the valuation ring A of v. If there is a smallest prime ideal strictly containing P, then 1° is the assertion that the residual topology of A_P/P is complete.

In the contrary case, P is the intersection of the prime ideals strictly containing it. Let $(P_\lambda)_{\lambda\in L}$ be a well-ordered, cofinal subset (for the inclusion ordering) of those prime ideals Then $\bigcap_{\lambda\in L} P_\lambda = P$, so $(P_\lambda/P)_{\lambda\in L}$ is a fundamental system of neighborhoods of zero for the residual topology $\mathscr{T}_P$ by Theorem 1.

Assume 2°. To show that $\mathscr{T}_P$ is complete, it suffices to show that the neighborhood A/P of zero in A_P/P is complete for its induced topology. Let φ be the canonical epimorphism from A to A/P. Let $\mathscr{F}$ be a Cauchy filter on A/P, and for each $\lambda \in L$ let $a_\lambda \in A$ be such that $\varphi(a_\lambda)$ belongs to a (P_λ/P)-small subset F_λ of $\mathscr{F}$. Then $F_\lambda \subseteq \varphi(a_\lambda) + P_\lambda/P$, and therefore $\varphi^{-1}(F_\lambda) \subseteq a_\lambda + P_\lambda$. As $\varphi^{-1}(\mathscr{F})$ is a filter base on A,

$$(a_\lambda + P_\lambda) \cap (a_\mu + P_\mu) \neq \emptyset$$

whenever $P_\mu \subset P_\lambda$, and hence $a_\lambda \equiv a_\mu \pmod{P_\lambda}$. By 2° there exists $x \in A$ such that $x \equiv a_\lambda \pmod{P_\lambda}$ for all $\lambda \in L$. Consequently $(\varphi(a_\lambda))_{\lambda\in L}$ converges to $\varphi(x)$ in A/P for $\mathscr{T}_P$, and therefore $\mathscr{F}$ does also.

Conversely, if $\mathscr{T}_P$ is complete and if $(P_\lambda)_{\lambda\in L}$ and $(a_\lambda)_{\lambda\in L}$ satisfy the hypotheses of 2°, then the family $\mathscr{F}$ of all the subsets $(\varphi(a_\lambda) + (P_\lambda/P))_{\lambda\in L}$ is a

Cauchy filter base on A/P, as $(P_\lambda/P)_{\lambda\in L}$ is a fundamental system of neighborhoods of zero for $\mathscr{T}_P$. Consequently there exists $x \in A_P$ such that $\mathscr{F}$ converges to $\varphi(x)$, so as each member of $\mathscr{F}$ is a coset of an open and hence closed ideal,

$$\varphi(x) \in \bigcap_{\lambda\in L} (\varphi(a_\lambda) + (P_\lambda/P)) .$$

Therefore $x \in \bigcap_{\lambda\in L}(a_\lambda + P_\lambda)$, that is, $x \equiv a_\lambda \pmod{P_\lambda}$ for all $\lambda \in L$. Moreover, $x \in A$ since $a_\lambda + P_\lambda \subseteqq A$ for all $\lambda \in L$.

This characterization permits short proofs of theorems of Krull and Ribenboim concerning valuations complete by stages:

THEOREM 3. *Let v be a valuation of a field K, and let A be its valuation ring.*

1° [15, Lemmas 11–13]. *If Q is a nonmaximal prime ideal of A, then v is complete by stages if and only if v_Q and $\bar{v}_Q$ are complete by stages.*

2° [8, Satz 27]. *If v is a maximal valuation, then v is complete by stages.*

3° [8, Satz 12]. *If v is complete by stages, then v is henselian.*

4° [17, Théorème 1]. *If L is a finite-dimensional extension of K and if v is complete by stages, then the unique extension v' of v to L is complete by stages.*

PROOF. 1°. The residual fields of v_Q are precisely the residual fields of v determined by the prime ideals of A strictly contained in Q, for if P is such an ideal, $(A_Q)_P = A_P$, and A_Q/P is a valuation subring of A_P/P determining the residual topologies defined by both v and v_Q. The nonmaximal prime ideals of the valuation ring A/Q of $\bar{v}_Q$ are precisely the ideals P/Q where P is a nonmaximal prime ideal of A containing Q. For each such ideal P,

$$\frac{x+Q}{y+Q} \mapsto \frac{x}{y} + Q, \qquad x \in A,\ y \in A \setminus P$$

is an isomorphism from $(A/Q)_{P/Q}$ to A_P/Q and is the identity map on A/Q and hence also on P/Q. Consequently, it induces an isomorphism φ from $(A/Q)_{P/Q}/(P/Q)$ to $(A_P/Q)/(P/Q)$ that is the identity map on $(A/Q)/(P/Q)$. The canonical isomorphism ψ from $(A_P/Q)/(P/Q)$ to A_P/P takes the valuation ring $(A/Q)/(P/Q)$ to the valuation ring A/P. Thus $\psi \circ \varphi$ is a topological isomorphism from the residual field $(A/Q)_{P/Q}/(P/Q)$ of $\bar{v}_Q$ to the residual field A_P/P of v. In particular, the residual fields of v are complete if and only if the residual fields of v_Q and $\bar{v}_Q$ are complete.

2°. The valuation v is maximal if and only if its valuation ring A is linearly compact, that is, every filter base on A consisting of cosets of ideals of A has a nonempty intersection [21, Theorem 4]. This is easily seen to be equivalent to the assertion that every (not necessarily Hausdorff) topology on A for which the open ideals form a fundamental system of neighborhoods of zero is complete. In particular, for any nonmaximal prime ideal P of A, the topology on A for which all ideals of A properly containing P form a fundamental system of neighborhoods of zero is complete, or equivalently, the topology on the subring A/P of A_P/P determined by the valuation $\bar{v}_P$ is complete. As $\bar{v}_P$ defines the residual topology of A_P/P and as A/P is a neighborhood of zero for that topology, it also is complete.

3°. To show that v is henselian, it suffices to show that if L is a finite extension of K, then v has only one extension to L. Suppose that u and w are distinct extensions of v to L. Let B' be the subring of L generated by the union of their valuation rings, let Q' be its maximal ideal, and let $B = K \cap B'$, $Q = K \cap Q'$. Then $\bar{u}_{Q'}$ and $\bar{w}_{Q'}$ are independent valuations of B'/Q', as the union of their valuation rings generates all of B'/Q'. Since B' properly contains the valuation ring of u, whose intersection with K is A, and since L is an algebraic extension of K, B properly contains A, so Q is a nonmaximal prime ideal of A, and Q is the maximal ideal of B. As B' is the valuation ring of a valuation of L extending the valuation v_Q of K, B'/Q' is a finite-dimensional extension of B/Q, as noted in section 1. By hypothesis, B/Q is complete for its residual topology, which is defined by $\bar{v}_Q$. Therefore B'/Q' has only one Hausdorff topology making it a topological vector space over B/Q. That topology is defined by $\bar{u}_{Q'}$ and $\bar{w}_{Q'}$, as they are extensions of $\bar{v}_Q$; this contradicts the independence of $\bar{u}_{Q'}$ and $\bar{w}_{Q'}$.

4°. Let Q' be a nonmaximal prime ideal of the valuation ring A' of the unique extension v' of v to L, and let $Q = Q' \cap A$. As the topological residual field $A'_{Q'}/Q'$ of v' is a finite-dimensional extension of the topological residual field A_Q/Q of v, which is complete by hypothesis, $A'_{Q'}/Q'$ is also complete. Thus v' is complete by stages.

3. Extensions of a theorem of Ostrowski.

In 1913, Ostrowski [12, pp. 276–280] proved that if v is a rank one henselian valuation of a field K and if L is a separable algebraic extension of K that is complete for the topology defined by the unique extension of v to L, then $[L: K] < +\infty$.

THEOREM 4. *Let v be a henselian valuation of a field K, and let v' be the unique extension of v to a separable algebraic extension L of K. If L is a Baire space for the topology defined by v', then $[L: K] < +\infty$ and K is closed in L.*

PROOF. By Krasner's Lemma [7] (which was originally discovered by Ostrowski [13, Hilfssatz, p. 197]), for each $x \in L$, the set V_x, defined by

$$V_x = \{y \in L : K[x] \subseteq K[y]\},$$

is a neighborhood of x. For each integer $n \geqq 1$, let

$$P_n = \{y \in L : [K[y]: K] \leqq n\}.$$

Then P_n is closed, for if $x \in L \setminus P_n$, clearly $V_x \subseteq L \setminus P_n$. In particular, $K = P_1$ is closed. As $\bigcup_{n=1}^{\infty} P_n = L$, there exists $m \geqq 1$ such that P_m has an interior point c. Consequently, the value group G' of v' contains an element γ such that $x \in P_m$ whenever $v'(x-c) > \gamma$. Thus if $v'(y) > \gamma$, then $y+c \in P_m$, so as $K[y] \subseteq K[c][y+c]$, $[K[y]: K] \leqq m^2$. For each $z \in L$ there exists a nonzero $t \in K$ such that $v'(tz) > \gamma$ as the value group of v is cofinal in G', and therefore

$$[K[z]: K] = [K[tz]: K] \leqq m^2.$$

Hence by the theorem of the primitive element, $[L: K] \leqq m^2$.

Since complete metrizable spaces are Baire spaces, we recover the following generalization of Ostrowski's theorem, due to Kaplansky [5, Theorem 9]:

THEOREM 5. *Let v be a henselian valuation defining a metrizable topology on a field K, and let L be a separable algebraic extension of K. Then L is complete for the topology defined by the unique extension v' of v to L if and only if $[L: K] < +\infty$ and K is complete.*

PROOF. The topology defined by v' is also metrizable, as the value group of v is cofinal in that of v'. The assertion therefore follows from Theorem 4.

The hypothesis concerning separability in Theorem 4 cannot be removed: Nagata [10, p. 56] (see also [2, Exercise 14 c), p. 193]) has given an example of a complete discrete valuation of a field L of prime characteristic p such that L is a purely inseparable extension of degree p of a dense subfield E, and L is an infinite-dimensional algebraic extension of its closed subfield L^p.

An element x of a topological field is *topologically nilpotent* if

$$\lim_{n \to \infty} x^n = 0.$$

If the topology of a field K is defined by a valuation, then it is also defined by a rank one valuation if and only if K possesses a nonzero topological nilpotent.

THEOREM 6. *Let v be a proper valuation of a field K. If K possesses no nonzero* *topological nilpotent, and if those residual fields of v that are metrizable but possess no nonzero topological nilpotent are complete, then K is a Baire space.*

PROOF. It suffices to prove that the valuation ring A of v is a Baire space. For if so, then for each $a \in K$, $a+A$ is an open neighborhood of a that is a Baire space, so K is a Baire space. Let $(G_n)_{n \geqq 1}$ be a sequence of open dense subsets of A. By hypothesis and Theorem 1, the nonzero prime ideals of A form a fundamental system of neighborhoods of zero, so it suffices to show that if P is a nonzero prime ideal and if $a \in A$, then $a+P$ intersects $\bigcap_{n=1}^{\infty} G_n$. For this, we shall construct a strictly decreasing sequence $(P_n)_{n \geqq 0}$ of nonzero prime ideals and a sequence $(a_n)_{n \geqq 0}$ of elements of A such that $P_0 = P$, $a_0 = a$, and

$$a_n + P_n \subseteqq \bigcap_{k=0}^{n-1} [(a_k + P_k) \cap G_{k+1}] .$$

Indeed, if a_n and P_n are defined, then as G_{n+1} is dense, there exists

$$a_{n+1} \in (a_n + P_n) \cap G_{n+1} .$$

As $(a_n + P_n) \cap G_{n+1}$ is open, there exists a nonzero prime ideal P_{n+1} properly contained in P_n such that

$$a_{n+1} + P_{n+1} \subseteqq (a_n + P_n) \cap G_{n+1} = \bigcap_{k=0}^{n} [(a_k + P_k) \cap G_{k+1}] .$$

Let $Q = \bigcap_{n=0}^{\infty} P_n$, and let φ be the canonical epimorphism from A_Q to A_Q/Q. By Theorem 1, $(P_n/Q)_{n \geqq 0}$ is a fundamental system of neighborhoods of zero for the residual topology of A_Q/Q. Consequently, $(\varphi(a_n) + (P_n/Q))_{n \geqq 0}$ is a decreasing Cauchy filter base consisting of cosets of open (and hence closed) ideals. By hypothesis, the residual field A_Q/Q is complete, so there exists $b \in A_Q$ such that

$$\varphi(b) \in \bigcap_{n=0}^{\infty} [\varphi(a_n) + (P_n/Q)] .$$

Consequently,

$$b \in \bigcap_{n=0}^{\infty} (a_n + P_n) \subseteqq \bigcap_{n=1}^{\infty} G_n ,$$

and also $b \in a_0 + P_0 = a + P \subseteqq A$.

From Theorems 2 and 6 we obtain the following corollary:

COROLLARY. *If v is a valuation of a field K that is complete by stages, then K is a Baire space for the topology defined by v.*

THEOREM 7. *Let v be a proper henselian valuation of a field K, let L be a separable algebraic extension of K, and let v' be the unique extension of v to L. If*

v' is complete by stages, then $[L: K] < +\infty$, and K is closed in L.

The assertion is a consequence of Theorem 4 and the Corollary of Theorem 6.

4. Nonhenselian valuation having finite-dimensional henselian extensions.

We shall call a total ordering of a field K *compatible* if it makes K into an ordered field.

Let K be an ordered field. For each subfield L of K, let

$$V_K(L) = \{x \in K : \text{ there exists } a \in L \text{ such that } -a \leqq x \leqq a\}$$

and

$$P_K(L) = \{x \in K : \text{ for all strictly positive } b \in L,\ -b \leqq x \leqq b\}.$$

Baer [1, p. 7] showed that $V_K(L)$ is a valuation subring of K whose maximal ideal is $P_K(L)$. Clearly $V_K(L)$ is a proper subring of K if and only if L is not a cofinal subset of K. In particular, $V_K(Q)$, where Q is the prime subfield of K, is a proper subring if and only if the ordering of K is nonarchimedean.

Wright [6, p. 314] established the equivalence of the following assertions concerning a valuation v of K with valuation ring V, maximal ideal M, residue field k, and value group G:

1°. $V \supseteqq V_K(Q)$.
2°. M contains no element c satisfying $c > 1$.
3°. V is an isolated subgroup of the additive group K.
4°. $V = V_K(L)$ for some subfield L of K.
5°. The restriction of v to the multiplicative group K^*_+ of all strictly positive elements of K is a decreasing epimorphism from K^*_+ to G.
6°. k admits a compatible total ordering such that the canonical epimorphism from V to k is increasing.

Earlier, though not using the language of valuation theory, Baer [1, § 4] had established the equivalence of 1°, 4°, and 6°. If these properties hold, we shall say that v is *compatible* (with the ordering of K). Thus there exist proper compatible valuations of K if and only if K is nonarchimedean. All proper compatible valuations are dependent, since their valuation rings all contain the valuation ring $V_K(Q)$. Consequently, they all define the same topology, which is the *interval* topology of K, generated by all the open intervals (a, b), where $a < b$. Indeed, if V is the valuation ring of a proper compatible valuation, then for any strictly positive $a \in V$, $(-a, a) \subseteqq Va$, and if b is a nonzero element of V such that $a \notin Vb$, then $Vb \subseteqq (-a, a)$.

Knebush and Wright established the following theorem [6, Satz 2.2]:

THEOREM 8. *If v is a valuation of a real-closed field K, then v is henselian if and only if v is compatible with the (unique) ordering of K.*

An elementary proof may be based on the following simple lemma:

LEMMA. *If v is a valuation of a field K whose characteristic is not 2 and if X^2-c is an irreducible polynomial over K, then v has distinct extensions to $K(c^{\frac{1}{2}})$ if and only if there exists $a \in K$ such that $v(a^2-c)>v(4c)$.*

PROOF. Let v' be an extension of v to $K(c^{\frac{1}{2}})$. As $K(c^{\frac{1}{2}})$ is a normal extension of degree 2 and as the only K-automorphism σ of $K(c^{\frac{1}{2}})$ other than the identity automorphism satisfies

$$\sigma(x+yc^{\frac{1}{2}}) = x-yc^{\frac{1}{2}} \quad \text{for all } x, y \in K,$$

v' is not the only extension of v if and only if there exists $a \in K$ such that

$$v'(a-c^{\frac{1}{2}}) < v'(a+c^{\frac{1}{2}}) .$$

From this follows $v'(2c^{\frac{1}{2}})=v'(a-c^{\frac{1}{2}})$, so

$$v(4c) = 2v'(a-c^{\frac{1}{2}}) < v'(a-c^{\frac{1}{2}})+v'(a+c^{\frac{1}{2}}) = v(a^2-c) .$$

Conversely, if for every $a \in K$, $v'(a-c^{\frac{1}{2}})=v'(a+c^{\frac{1}{2}})$, then for every $a \in K$,

$$v'(2c^{\frac{1}{2}}) \geqq v'(a+c^{\frac{1}{2}}) = v'(a-c^{\frac{1}{2}}) ,$$

so

$$v(4c) \geqq v'(a+c^{\frac{1}{2}})+v'(a-c^{\frac{1}{2}}) = v(a^2-c) .$$

PROOF OF THEOREM 8. We merely note that as $K(i)$ is algebraically closed, v is not henselian if and only if it has distinct extensions to $K(i)$, or equivalently by the Lemma, if and only if there exists $a \in K$ such that $v(a^2+1)>v(-4)=0$. As $a^2+1>1$, condition 2° above does not hold. Conversely, if $v(c)>0$ for some $c>1$, then as K is real-closed, $c=a^2+1$ for some $a \in K$, whence $v(a^2+1)>0$ $=v(-4)$, and therefore v has distinct extensions to $K(i)$.

One consequence of Theorem 8 is that a proper valuation v of the field $\mathbf{R}$ of real numbers is not henselian, and so has precisely two extensions to its algebraic closure $\mathbf{C}$, each of which is trivially henselian. This suggests the problem of characterizing valuations v that are not henselian but have a henselian finite-dimensional extension. The problem has been discussed by Kaplansky and Schilling [4, Theorem 4], Bourbaki [2, Exercise 17, § 8, Ch. 6, p. 194], Ribenboim [19, C], and Endler [3]. Endler's theorem on nonhenselian valuations of rank one that have finite-dimensional henselian extensions has generalized results proved earlier concerning this problem.

THEOREM 9. *Let v be a nonhenselian valuation of a field K, v' a henselian valuation extending v to an algebraic extension K' of K such that $[K': K]_s < +\infty$.*

1°. *K' contains a root i of X^2+1, $i \notin K$, and the restriction of v' to $K(i)$ is henselian (and thus is a henselization of v).*

2°. *There exist a compatible total ordering of K and a (possibly improper) henselian valuation u of K such that*

(a) *$u<v$,*
(b) *u is compatible with the ordering,*
(c) *the residue field of u is a real-closed field, and*
(d) *the residue field of the unique extension u' of u to K' is an algebraically closed field of characteristic zero, and $u'<v'$.*

PROOF. Let K'_s be the separable closure of K in K', and let v'_s be the restriction of v' to K'_s. By hypothesis,

$$[K'_s:K] = [K':K]_s < +\infty .$$

As K' is a purely inseparable extension of K'_s and as v' is henselian, clearly v'_s is henselian. Also K is a proper subfield of K'_s since v'_s is henselian but v is not. If the conclusions of the theorem hold when v' is replaced by v'_s, then K has characteristic zero, so $v'_s = v'$, and hence the conclusions hold also for v'. Therefore we may assume, by replacing v' with v'_s if necessary, that $1<[K': K] < +\infty$.

By hypothesis, there is a finite-dimensional extension N_1 of K on which there are two distinct valuations extending v. As $[K'(N_1): K] < +\infty$, there is a finite-dimensional normal extension N of K that contains $K'(N_1)$. Let v'' be an extension of v' to N. Each of the two extensions of v to N_1 has an extension to N, and therefore one of them, w'', is distinct from v''. As N is a normal extension of K, $w''=v''\circ\sigma$ for some K-automorphism σ of N. As v'' is an extension of v', v'' is henselian, so w'' is also henselian. Let V'' and W'' be respectively the valuation rings of v'' and w'', let B'' be the subring of N generated by $V'' \cup W''$, and let P'' be its maximal ideal. Let $B'=B''\cap K'$, $B=B''\cap K$, $P'=P''\cap B'$, $P=P''\cap B$. Then P' is the maximal ideal of the valuation ring B' of K', and P is the maximal ideal of the valuation ring B of K. As B''/P'' and B/P are respectively the residue fields of $v''_{P''}$ and v_P,

$$[B''/P'': B/P] = f(v''_{P''}/v_P) \leqq [N: K] < +\infty .$$

Also, $\bar{v}'_{P''}$ and $\bar{w}'_{P''}$ are independent, for the subring generated by the union, $(V''/P'') \cup (W''/P'')$, of their valuation rings is all of B''/P''. In addition, $\bar{v}'_{P''}$ and $\bar{w}'_{P''}$ are henselian [10, Theorem 14; 18, Proposition 9, p. 210]. Consequently by [2, Ch. 6, Exercise 15 c), p. 193] (see also [4]), B''/P'' is separably algebraically closed.

As $\bar{v}'_{P''}$ and $\bar{w}'_{P''}$ are independent extensions of $\bar{v}_P$ to B''/P'', $\bar{v}_P$ is not henselian. Therefore as $\bar{v}'_{P''}$ is henselian, B''/P'' is not a purely inseparable extension of B/P. Consequently,

$$1 < [B''/P'' \colon B/P]_s < +\infty .$$

By Endler's corollary of the Artin-Schreier theorem [3, Lemma, p. 187], B''/P'' is algebraically closed and has characteristic zero, B/P is real-closed, and $B''/P'' = (B/P)(i)$ where i is a root of X^2+1 in B''/P''. In particular, $[B''/P'' \colon B/P]=2$. If, under the canonical identification, B/P were all of B'/P', then $\bar{v}'_{P'}$ would be identical with $\bar{v}_P$. But $\bar{v}'_{P'}$ is henselian [10, Theorem 14; 18, Proposition 9, p. 210] whereas $\bar{v}_P$ is not. Consequently, $[B'/P' \colon B/P]>1$, so B'/P' is identified with all of B''/P''.

Let u' be the valuation $v'_{P'}$ of K', and let $u=v_P$, the restriction of u' to K. Since v' is henselian, u' is also [10, Theorem 10; 18, Proposition 9, p. 210], and therefore as the residue field B'/P' of u' is algebraically closed, the subring B' of K' contains a root i of X^2+1. As the residue field B/P of u is real-closed, $i \notin K$. Consequently, we have established the first half of 1° and (d) of 2°.

The restriction v'_h of v' to $K(i)$ has a finite-dimensional henselian extension, namely, v'. Applying what we have just proved to v'_h, we conclude that v'_h is henselian, since its domain contains i. Thus 1° is established.

Since the residue field of u is real-closed, K admits a compatible total ordering with which u is compatible by a theorem of Baer [1, Satz 4], expressed in the language of valuation theory by Krull [8, Satz 22], and later reproved by Lang [9, Theorem 6] and Nagata [11, Theorem 2]. Let L be a subfield of K such that the valuation ring V of u is $V_K(L)$. The ordered field K admits a real-closed algebraic extension R, and as $V_K(L)=K \cap V_R(L)$, u is the restriction of a valuation u_R of R whose valuation ring is $V_R(L)$. By Theorem 8, u_R is henselian, so the restriction of u_R to some subfield H of R that contains K is a henselization of u. Now u' is a henselian, finite-dimensional extension of u. Applying 1° to u, we conclude that u is henselian, since otherwise H would be K-isomorphic to $K(i)$, which is impossible as H is an ordered field. By definition of u and u', $u \leqq v$ and $u' \leqq v'$. However, u is henselian whereas v is not, so $u<v$, therefore also $u'<v'$, and the proof is complete.

Since $u'<v'$, the residue field of u' is a residual field of v'. Consequently, we conclude from Theorem 9:

Corollary 1. *Let v' be a henselian valuation of a field K'. One of the following two assertions holds:*

1°. The restriction of v' to every subfield K of K' such that K' is an algebraic extension of K and $[K'\colon K]_s < +\infty$ is henselian.

2°. There is a residual field of v' that is an algebraically closed field of characteristic zero.

In particular, 1° holds if K' has prime characteristic.

If, in Theorem 9, v has rank one, then as $u < v$, u is the improper valuation of K, so the residue field of u is K itself. Consequently, we obtain:

Corollary 2 (cf. [3, Theorem 2]). *Let v' be a rank one henselian valuation of a field K', and let v be the restriction of v' to a proper subfield K of K' such that K' is an algebraic extension of K and $[K'\colon K]_s < +\infty$. The following statements are equivalent:*

1°. v is not henselian.

2°. K' is an algebraically closed field of characteristic zero, K is a real-closed field, $K' = K(i)$, where i is a root of X^2+1, and there exists $a \in K$ such that $v'(a+i) \neq v'(a-i)$.

Corollary 3 (cf. [2, Exercise 17, § 8, Ch. 6, p. 194]). *If v' is a rank one henselian valuation of a field K', then either K' is an algebraically closed field of characteristic zero, or the restriction of v' to every subfield K of K' such that K' is an algebraic extension of K and $[K'\colon K]_s < +\infty$ is henselian.*

For complete rank one valuations of fields of characteristic zero, the two possibilities envisioned in Corollary 3 are mutually exclusive:

Theorem 10. *Let v be a complete rank one valuation of a field L of characteristic zero. The following assertions are equivalent:*

1°. The restriction of v to every finite-codimensional subfield of L is henselian.

2°. Every finite-codimensional subfield of L is closed.

3°. The closed subfields K of L such that L is an algebraic extension of K are precisely the finite-codimensional subfields.

4°. L is not algebraically closed.

Proof. As v is a complete rank one valuation, v is henselian. Thus 1° implies 2° by Theorem 4. We shall denote by v_K the restriction of v to a subfield K of L. To show that 2° implies 3° and 1°, let K be a closed subfield of L such that L is algebraic over K. Then K is complete, so v_K is henselian. By Theorem 4, K is finite-codimensional. Thus 3° holds. Moreover, as K is finite-codimensional and complete, L admits only one topology making it into a topological K-vector space. Hence any two extensions to L of v_K are dependent and thus identical, as v_K has rank one, so v_K is henselian as v is. Thus 1° holds. Clearly 3° implies 2°.

By Corollary 3 of Theorem 9, 4° implies 1°. To show that 1° implies 4°, we shall assume that 1° holds and that L is algebraically closed. Then L contains a real-closed field R such that $L=R(i)$, where i is a root of X^2+1. If σ is an automorphism of L and σ_R its restriction to R, then v_R and $v_{\sigma(R)}$ are henselian valuations of R and $\sigma(R)$ respectively by 1°, so v_R and $v_{\sigma(R)}\circ\sigma_R$ are henselian valuations of R. If they were independent, then R would be algebraically closed [2, Ch. 6, Exercise 15 c), p. 193], a contradiction. Consequently they are dependent and thus identical, as both are rank one valuations. As 1° implies 2°, R is a closed and hence complete, so L has only one topology making it into a topological R-space. Thus as v and $v\circ\sigma$ are both extensions of $v_R=v_{\sigma(R)}\circ\sigma_R$, they are dependent and hence identical.

Let $K=Q(B)$, where B is a transcendence base of L over the prime subfield Q, and let w be the restriction of v to K. As L is a normal extension of K, every extension of w to L is of the form $v\circ\sigma$ for some K-automorphism σ of L, so as $v\circ\sigma=v$ by what we have just proved, w is henselian. By Theorem 4, K is a finite-codimensional subfield of L, so by the Artin-Schreider theorem, K is real-closed and $L=K(i)$, a contradiction of the fact that K is a pure transcendental extension of Q.

A Galois extension L of a field K may contain an infinite strictly decreasing sequence of subfields of finite codimension between K and L. For example, if $(p_i)_{i\geq 1}$ is the sequence of primes and if

$$A_k = \{p_i^{\frac{1}{i}} : i \geqq k\},$$

then $Q(A_1)$ is a Galois extension of Q and $(Q(A_k))_{k\geqq 1}$ is a strictly decreasing sequence of subfields of $Q(A_1)$ of finite codimension. This cannot happen, however, if L is a Baire space for a henselian valuation:

THEOREM 11. *If v' is a henselian valuation of L, if L is a Baire space for the topology defined by v', and if L is a Galois extension of a subfield K, then the codimensions of all finite-codimensional subfields of L that contain K are bounded.*

PROOF. Let v be the restriction of v' to K, and for any subfield F of L, let v'_F be the restriction of v' to F. As v' is henselian and as L is an algebraic extension of K, L contains a subfield H such that v'_H is a henselization of v. By Theorem 4, as L is a separable extension of H, $[L\colon H] < +\infty$. Let $r = [L\colon H]$. If H' is another subfield of L such that $v'_{H'}$ is a henselization of v, then there is a K-isomorphism ϱ from H to H'. Therefore as L is a normal extension of K, there is a K-automorphism ϱ' of L whose restriction to H is ϱ. Thus

$$[L:H'] = [\varrho'(L):\varrho'(H)] = [L:H] = r .$$

Now let N be a finite-codimensional subfield of L that contains K. If v'_N is henselian, then N contains a subfield H' such that $v'_{H'}$ is a henselization of v, whence

$$[L:N] \leqq [L:H'] = r .$$

If v'_N is not henselian, then by Theorem 9, L contains a root i of X^2+1 and $v'_{N(i)}$ is henselian. Thus

$$[L:N] = [L:N(i)][N(i):N] \leqq 2r .$$

Therefore the codimension of a finite-codimensional subfield of L that contains K does not exceed $2r$.

We shall say that a valuation v of a field K is *half henselian* if v has precisely two extensions to the algebraic closure of K.

THEOREM 12. *Let v be a valuation of a field K, let Ω be an algebraic closure of K, and let v_H be a henselization of v defined on a subfield H of Ω. The following statements are equivalent*:

1°. *v is half henselian.*

2°. *v is not henselian but has only finitely many extensions to Ω.*

3°. *v is not henselian, but there is a henselian valuation extending v to a subfield L of Ω such that $[L:K] < +\infty$.*

4°. $1 < [H:K] < +\infty$.

5°. *K does not contain a root i of X^2+1, and $H = K(i)$.*

6°. *There is a (possibly improper) henselian valuation u of K such that $u < v$, the residue field k_u of u is a real-closed field, and the valuation $\bar{v}$ induced on k_u by v is not henselian.*

7°. *There is a real-closed subfield of Ω that contains K; if R is any such field, v has a unique extension v_R to R, and v_R is not henselian.*

PROOF. Clearly 1° implies 2°. To show that 2° implies 3°, let $v'_1, \ldots, v'_n$ be all the distinct extensions of v to Ω. For each $i \in [2, n]$ there exists $x_i \in \Omega$ such that $v'_1(x_i) \neq v'_i(x_i)$; let $L = K(x_2, \ldots, x_n)$. Then $[L:K] < +\infty$. Let w be the restriction of v'_1 to L. None of $v'_2, \ldots, v'_n$ is an extension of w. Therefore if w had another extension v'_0 to Ω besides v'_1, v would have the $n+1$ distinct extensions $v'_0, v'_1, \ldots, v'_n$, a contradiction. Consequently, w is henselian.

Clearly 3° and 4° are equivalent. By Theorem 9, 3° implies 5°. If 5° holds, then v has precisely two extensions, v'_1 and v'_2, to $K(i)$, and one of them, say v'_1, is henselian. But as $K(i)$ is a normal extension of K, $v'_2 = v'_1 \circ \sigma$, where σ is the conjugation K-automorphism of $K(i)$, so v'_2 is also henselian. As every extension of v to Ω is also an extension of either v'_1 or v'_2, 1° holds.

By Theorem 9, 3° implies 6°, for if $\bar{v}$ were henselian, then v would be also by [10, Lemma 5; 18, Proposition 10, p. 211]. Also, 6° implies 5°: As k_u is real-closed, K does not contain a root i of X^2+1. Let u' be the unique extension of u to $K(i)$, and let v' be an extension of v to $K(i)$. The valuation $\bar{v}'$ induced by v' on the residue field $k_u(i)$ of u' is trivially henselian, as $k_u(i)$ is algebraically closed. Therefore as u' is also henselian, v' is henselian by [10, Lemma 5; 18, Proposition 10, p. 211]. Moreover, v is not henselian as $\bar{v}$ is not [10, Theorem 14; 18, Proposition 9, p. 210]. Therefore $H = K(i)$, and 5° holds.

Thus 1°–6° are all equivalent. To show that they imply 7°, we first note that by 3° and Theorem 9, there is a compatible total ordering on K, so there is a real-closed extension R of K contained in Ω. Assume that v has distinct extensions, u'_1 and u'_2, to R. Then each of them can have only one extension to $R(i) = \Omega$ by 1°, so each of them is henselian. Consequently, R contains a subfield H' such that the restriction of u'_1 to H' is a henselization of v; therefore by 5°, H' is K-isomorphic to $K(i)$, a contradiction. Thus v has a unique extension v_R to R. If v_R had only one extension to Ω, then v would have only one extension to Ω, a contradiction; hence v_R is not henselian. Clearly 7° implies 1°, and the proof is complete.

Condition 6° of Theorem 12 suggests how to establish the existence of half henselian valuations.

THEOREM 13. *If u is a henselian valuation of a field K whose residue field k is real-closed, there is a half henselian valuation v of K such that $u < v$.*

PROOF. Let A and P be respectively the valuation ring and maximal ideal of u, and let φ be the canonical epimorphism from A to k. Let W be the valuation ring of an extension w to k of the 7-adic valuation of the prime subfield Q of k. As the residue field of w has characteristic 7, w is not henselian by Theorem 8. Let $V = \varphi^{-1}(W)$. Clearly V is a valuation subring of K (for if $x \in K \setminus A$, $x^{-1} \in P \subseteq V$). Let v be a valuation of K whose valuation ring is V. Then the valuation ring of $\bar{v}_P$ is W, so $\bar{v}_P$ is not henselian. As $V \subset A$, $u < v$. Thus by 6° of Theorem 12, v is half henselian.

THEOREM 14. *A field K admits a half henselian valuation if and only if it admits a henselian valuation whose residue field is real-closed.*

The assertion results from Theorems 12 and 13.

For example, let k be a real-closed field, G a totally ordered abelian group, and let $K=S(k,G)$, the field of all functions from G to k whose support is a well-ordered subset of G, with convolution as multiplication. The order valuation u of K (if $f\neq 0$, $u(f)$ is the smallest $\alpha\in G$ such that $f(\alpha)\neq 0$) is a maximal and hence henselian valuation of K whose value group is G and whose residue field is canonically isomorphic to k. Thus by Theorem 14, K admits a half henselian valuation v, and by Theorem 9, v has a henselian extension v' to $K(i)$. But if G is not divisible, K cannot be real-closed nor can $K(i)$ be algebraically closed. In particular, there exist fields admitting half henselian valuations that are not real-closed; moreover, the incorrectness of [2, Exercise 17, § 8, Ch. 6, p. 194] is established.

THEOREM 15. *If R is a real-closed field, every topology on R determined by a proper valuation is also determined by a half henselian valuation. If K is a field that is not real-closed, the topology determined by a half henselian valuation of K is the interval topology determined by a compatible total ordering.*

PROOF. Every nonhenselian valuation of R is half henselian, as $R(i)$ is algebraically closed. Let u be a proper henselian valuation of R, with valuation ring A and residue field k. By Theorem 8, k admits a compatible total ordering for which the canonical epimorphism from A to k is increasing. Therefore k is real-closed since R is. By Theorem 13, R admits a half henselian valuation v such that $u<v$, so as u is proper, u and v determine the same topology.

Let v be a half henselian valuation of a field K that is not real-closed. By Theorem 9, there exist a compatible total ordering of K and a henselian valuation u of K that is compatible with the ordering, has a real-closed residue field k, and satisfies $u<v$. Since k is real-closed but K is not, u is a proper valuation, and therefore u and v determine the same topology.

THEOREM 16. *If v is a half henselian valuation of a field K, then K is real-closed under any of the following conditions:*

1°. *v has rank one.*

2°. *The topology determined by v is not the interval topology given by a compatible total ordering on K.*

3°. *The value group of v is divisible.*

PROOF. By Corollary 2 of Theorem 9 and Theorem 15, we need only consider 3°. By Theorem 9, there is a henselian valuation u of K whose residue field is real-closed and whose value group is divisible, since it is a quotient group of the divisible value group of v. By a theorem of Krull [8, Satz 23], rediscovered by Nagata [11, Corollary 4], K is real closed. (By adjoining i to K, the conclusion of that theorem follows from another theorem of Krull [8, p. 190]: If v is a henselian valuatiom of a field K whose residue field is an algebraically closed field of characteristic zero and whose value group is divisible, then K is algebraically closed. This, in turn, is an immediate consequence of Ostrowski's "defekt" theorem [14, § 9, ¶55] in the characteristic zero case: If v is a henselian valuation of a field K whose residue field has characteristic zero, and if v' is the extension of v to a finite-dimensional extension K' of K, then $e(v'/v)f(v'/v) = [K':K]$.)

We conclude with a supplement to our earlier discussion of valuations complete by stages:

THEOREM 17. *Let v' be a valuation of a field K', and let v be the restriction of v' to a proper subfield K of K' such that K' is an algebraic extension of K. Of the following assertions,* 1° *implies* 2°, 2° *implies* 3°, *and all three are equivalent if each residual field of v' is separable over the corresponding residual field of v (a condition holding, for example, if the residue field of v has characteristic zero or if K is a perfect field of prime characteristic)*:

1°. *v is complete by stages and $[K':K] < +\infty$.*

2°. *v' is complete by stages, $[K':K] < +\infty$, and there do not exist $a \in K$ and a root i of X^2+1 in $K' \setminus K$ such that $v'(a+i) \neq v'(a-i)$.*

3°. *v' is complete by stages and v is henselian.*

PROOF. 1° implies 2° and 2° implies 3° by 3° and 4° of Theorem 3 and Theorem 9. The supplementary condition holds if the residue field of v has characteristic zero, for then every residual field of v has characteristic zero. It holds also if K is a perfect field of prime characteristic, for if B is any valuation subring of K strictly containing the valuation ring A of v, then a pth root in K of any element of B clearly belongs to B; the residual field determined by B is therefore perfect, so the corresponding residual field of v' is a separable extension of it.

We assume, finally, that 3° and the supplementary condition hold. Let A' be the valuation ring of v'. Let P be a nonmaximal prime ideal of A, P' the nonmaximal prime ideal of A' such that $P = P' \cap A$. The topology of the residual field $A'_{P'}/P'$ is given by $\bar{v}'_{P'}$, which is complete by stages by 1° of

Theorem 3. Consequently, $A'_{P'}/P'$ is a Baire space by the Corollary of Theorem 6. By [10, Theorem 14; 18, Proposition 9, p. 210], $\bar{v}_P$ is henselian as v is. Moreover, $A'_{P'}/P'$ is a separable extension of A_P/P by hypothesis. Therefore by Theorem 4,

$$[A'_{P'}/P' : A_P/P] < +\infty ,$$

and A_P/P is closed in $A'_{P'}/P'$. As $A'_{P'}/P'$ is complete, so is A_P/P. Thus v is complete by stages. Also $[K' : K] < +\infty$ as K' and K are the residual fields of v' and v determined by the zero ideal.

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