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ALMOST COMMUTING HERMITIAN MATRICES

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The purpose of this paper is to shed some light on an old problem in linear algebra and operator theory: If two norm one Hermitian matrices A and B have small commutator, are they close to a commuting pair of Hermitian matrices? The answer to this question is still unknown, but all the evidence points toward a negative solution. We give an equivalent reformulation of this problem as a question about a single self-adjoint operator, which may be easier to deal with. On the other hand, there are some positive results. Most notably, there is a quantitative absorption phenomenon which says roughly: If A and B have small commutator, there is a commuting pair of Hermitian matrices C and D so that $A \oplus C$ and $B \oplus D$ are close to a commuting pair of Hermitian matrices. In the addendum, this result is used to derive a *quantitative* version of the Brown–Douglas–Fillmore Theorem for the unit disc.

The question above has a number of analogues. The other interesting and important one replaces Hermitian matrices by arbitrary ones at every opportunity. This problem is also unsolved. The analogue for unitaries was shown to have a negative answer [10]. Voiculescu also showed that the corresponding question for triples of Hermitian matrices also has a negative solution [9]. We give an example which subsumes this last case, and also answers negatively the question of one Hermitian and one arbitrary matrix. More precisely, there is a sequence $\{A_n, B_n\}$ of matrices such that A_n is Hermitian, B_n is normal and $\lim_{n\to\infty} \|[A_n,B_n]\| = 0$, yet there are no commuting pairs $\{A'_n,B'_n\}$ with A'_n Hermitian such that

$$\lim_{n\to 0} \|A_n - A'_n\| + \|B_n - B'_n\| = 0.$$

There is no restriction here on the characteristics of B'_n. If one considers the triple $\{A_n, \operatorname{Re} B_n, \operatorname{Im} B_n\}$, one obtains an example with small commutators but for which there are no Hermitian triples $\{A'_n, C_n, D_n\}$ with $[A'_n, C_n] = 0 = [A'_n, D_n]$ and

$$\lim_{n\to\infty} \|A_n - A'_n\| + \|\operatorname{Re} B_n - C_n\| + \|\operatorname{Im} B_n - D_n\| = 0.$$

There are also infinite dimensional analogues of this problem, and they

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are known to have negative solutions for somewhat different reasons. M. D. Choi, among others, observed that the weighted shifts S_n defined on an orthonormal basis $\{e_k, k \geqq 1\}$ by

$$S_n e_k = \min\{k/n, 1\} e_{k+1}$$

have small self commutators, namely, $\lim_{n\to\infty} [S_n, S_n^*] = 0$, yet they are uniformly bounded away from the set of normal operators. This is because they are essentially unitary with non-zero Fredholm index. One can easily reformulate this in terms of the Hermitian pairs $A_n = \operatorname{Re} S_n$ and $B_n = \operatorname{Im} S_n$. Later, Berg and Olsen [3] showed that there were no operators (not necessarily Hermitian) A'_n and B'_n which commute and have

$$\lim_{n\to\infty} \|A_n - A'_n\| + \|B_n - B'_n\| = 0.$$

Their argument also relies on index, and so does not apply to the finite dimensional case. Nonetheless, one might view our example as having a quantitative, finite dimensional analogue of an index obstruction.

This problem has a bearing on the work of Brown, Douglas, and Fillmore [5]. An important consequence of their work is that the set of operators $\mathcal{N} + \mathcal{K}$ on Hilbert space which are the sum of a normal operator and a compact operator is norm closed. Operators of this form are both quasi-diagonal and essentially normal. It follows from [5] that they are the only quasidiagonal, essential normal operators and hence $\mathcal{N} + \mathcal{K}$ is the intersection of two closed sets. No direct operator theoretic proof of this exists. An approach one might take is the following: If T is quasidiagonal and essentially normal, it has a compact perturbation $T' = \oplus\Sigma_{n=1}^{\infty} T_n$ which is the direct sum of finite rank matrices T_n. Furthermore,

$$\lim_{n\to\infty} \|[T_n, T_n^*]\| = 0.$$

So $A_n = \operatorname{Re} T_n$ and $B_n = \operatorname{Im} T_n$ are almost commuting Hermitian matrices. If the first question has a positive answer, one could find normal matrices $T'_n = A'_n + iB'_n$ so that

$$\lim_{n\to\infty} \|T_n - T'_n\| = 0.$$

This would give a normal operator $\Sigma_{n=1}^{\infty} T'_n$ which is a compact perturbation of T. In [6], Douglas gives a heuristic argument why one may not be able to give a proof along these lines. An explanation of what is "really going on" would be most desirable.

Another consequence of [5] is the absorption phenomenon. Let A and B be Hermitian operators with compact commutator, one can find a commuting pair of Hermitian operators C and D so that $A \oplus C$ and $B \oplus D$ have Hermitian compact perturbations which exactly commute. Indeed, one just takes any commuting self-adjoint operators C and D whose joint spectrum equals the polynomially convex hull of the joint essential spectrum of A and B. (That is, it fills in the holes of $\sigma_e(A+iB)$.) There is now no index obstruction to prevent $(A \oplus C)+i(B \oplus D)$ being perturbed by a compact to a normal operator.

We give a quantitative, norm version of this phenomenon valid in both finite dimensional and infinite dimensional Hilbert space. A precise formulation is:

THEOREM 0.1. *Let A and B be Hermitian matrices. Then there are commuting pairs of Hermitian matrices $\{C,D\}$ and $\{A_1,B_1\}$ on Hilbert spaces of the same (double) dimension, respectively, so that $\|C\| \leqq \|A\|$, $\|D\| \leqq \|B\|$, and*

$$\max\{\|A \oplus C - A_1\|, \|B \oplus D - B_1\|\} \leqq 25\|AB-BA\|^{1/2}.$$

I would like to thank Chandler Davis for pointing out the relevance of [4], which has made possible certain sharper estimates in this paper.

1. Preliminaries

In this paper, $\mathscr{H}$ will denote a complex Hilbert space of finite or countably infinite dimension. $\mathscr{B}(\mathscr{H})$ will denote the algebra of all bounded operators on $\mathscr{H}$, and when $\mathscr{H}$ is infinite dimensional, $\mathscr{K} = \mathscr{K}(\mathscr{H})$ will denote the ideal of compact operators on $\mathscr{H}$. If $\mathscr{H}$ has finite dimension k, we may use $\mathscr{M}_k$ instead of $\mathscr{B}(\mathscr{H})$. Given two operators A and B, the commutator is $[A,B] = AB - BA$. Projections are always self-adjoint idempotents.

For A in $\mathscr{B}(\mathscr{H})$, the spectrum of A is denoted by $\sigma(A)$. When $\mathscr{H}$ is infinite demensional, π will denote the quotient map of $\mathscr{B}(\mathscr{H})$ onto the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}$. The essential spectrum of A is $\sigma_e(A) = \sigma(\pi A)$. An operator A is essentially normal (unitary) if πA is a normal (unitary) element in the quotient. An operator A is quasi-diagonal if there is an increasing sequence P_n of finite rank projections with supremum I such that $\lim_{n\to\infty}\|[A,P_n]\| = 0$. The set $\mathscr{Q}\mathscr{D}$ of quasidiagonal operators is closed, invariant under compact perturbations, and by the Weyl-von Neumann-Berg theorem [1], it contains all normal operators.

The question of concern in this paper is quantitatively described as:

(Q) For each $\varepsilon > 0$, is there a $\delta > 0$ so that if A and B are finite rank, Hermitian matrices of norm one with $\|[A,B]\| < \delta$, there are commuting Hermitian matrices A_1 and B_1 with $\|A - A_1\| < \varepsilon$ and $\|B - B_1\| < \varepsilon$?

2. An example

In this section, the example mentioned in the introduction will be constructed. We need two easy lemmas. The first (called a folk theorem in [10]) is the Corollary to Theorem 3.4 in [4]. For a self adjoint operator T, let $E_T[C]$ denote the spectral projection for T corresponding to the set C.

LEMMA 2.1. *Let ε and η be positive constants, and let C_1 and C_2 be closed intervals with* $\text{dist}(C_1, C_2) \geqq \eta$. *For any pair of self-adjoint operators S and T satisfying $\|S - T\| < \eta\varepsilon$, one has $\|E_S(C_1)E_T(C_2)\| < \varepsilon$.*

LEMMA 2.2. *Let $\varepsilon > 0$ be given. If E, F', and G are projections with $E \leqq G$, $\|EF'^{\perp}\| < \varepsilon$, and $\|F'G^{\perp}\| < \varepsilon$, then there is a projection F such that $E \leqq F \leqq G$ and $\|F' - F\| \leqq 5\varepsilon$.*

PROOF. Clearly, we may suppose that $\varepsilon \leqq 1/5$. Decompose the Hilbert space as $\mathscr{H} = EH \oplus (G - E)\mathscr{H} \oplus G^{\perp}\mathscr{H}$. With respect to this decomposition, F' has a matrix $(F_{ij})i,j = 1,2,3$. We have $F' = F'^2 = F'^*$ and that

$$\|(I - F_{11}, -F_{12}, -F_{13})\| < \varepsilon \text{ and } \|(F_{31}, F_{32}, F_{33})\| < \varepsilon.$$

In particular, $F_{22} - F_{22}^2 = F_{12}^* F_{12} + F_{23} F_{23}^*$ is positive and less than $2\varepsilon^2$. Since $0 \leqq F_{22} \leqq I$, we obtain that the spectrum of F_{22} is contained in $[0, 4\varepsilon^2] \cup [1 - 4\varepsilon^2, 1]$, so there is a projection P on $(G - E)\mathscr{H}$ satisfying $\|F_{22} - P\| < 4\varepsilon^2 < \varepsilon$. A simple estimate now gives $\|F' - (E + P)\| < 5\varepsilon$. So $F = E + P$ will suffice.

THEOREM 2.3. *There exist finite rank matrices A_n, B_n, $n \geqq 1$, of norm 1 such that A_n is self-adjoint, B_n is normal, and $\lim_{n\to\infty} \|[A_n, B_n]\| = 0$, yet there are no commuting pairs A_n', B_n' such that A_n' is self-adjoint and* $\lim_{n\to\infty} \|A_n - A_n'\| + \|B_n - B_n'\| = 0$.

PROOF. Define A_n and $\tilde{B}_n$ in $\mathscr{M}_{n^2+1}$ as follows. Let e_k, $0 \leqq k \leqq n^2$ be an orthonormal basis and let

$$A_n e_k = \frac{k}{n^2} e_k, \quad \tilde{B}_n e_k = \begin{cases} \dfrac{k+1}{n} e_{k+1} & 0 \leqq k < n \\ 1 e_{k+1} & n \leqq k \leqq n^2 - n \\ \dfrac{n^2 - k}{n} e_{k+1} & n^2 - n < k \leqq n^2. \end{cases}$$

Now $A_n = A_n^*$, B_n is a weighted shift and $\| [A_n, \tilde{B}_n] \| = n^{-2}$ tends to zero. Also, since $\lim_{n\to\infty} \| [\tilde{B}_n, \tilde{B}_n^*] \| = 0$, there are normal matrices B_n such that $\lim_{n\to\infty} \| \tilde{B}_n - B_n \| = 0$ [2]. So it suffices to use $\tilde{B}_n$ in lieu of B_n. In order to obtain a contradiction, we assume the existence of commuting pairs A'_n, B'_n with A'_n self adjoint and

$$\lim_{n\to\infty} \| A_n - A'_n \| + \| \tilde{B}_n - B'_n \| = 0.$$

Let F'_n be the spectral projection of A'_n for the interval $[0, \frac{1}{2}]$; let E_n and G_n be the spectral projections for A_n corresponding to the intervals $[0, \frac{1}{3}]$ and $[0, \frac{2}{3}]$, respectively. By Lemma 2.1, we see that

$$\lim_{n\to\infty} \| E_n F_n'^{\perp} \| = 0 = \lim_{n\to\infty} \| F'_n G_n^{\perp} \|.$$

Thus by Lemma 2.2, there are projections F_n with $E_n \leqq F_n \leqq G_n$ and

$$\lim_{n\to\infty} \| F'_n - F_n \| = 0.$$

Since $[F'_n, B'_n] = 0$, it follows that $\lim_{n\to\infty} \| [F_n, \tilde{B}_n] \| = 0$.

Let S_n be the shift in $\mathcal{M}_{n^2+1}$ on the basis $\{e_k, 0 \leqq k \leqq n^2\}$. Note that for $n \geqq 4$, $S_n(G_n - E_n) = \tilde{B}_n(G_n - E_n)$ and $(G_n - E_n)S_n = (G_n - E_n)\tilde{B}_n$. It follows that $\lim_{n\to\infty} \| [F_n, S_n] \| = 0$. However, for any projection P with $[\ker S_n^*] \leqq P \leqq [\ker S_n]^{\perp}$, one has $\| [P, S_n] \| = 1$ which is a contradiction.

REMARKS. This example puts the result of [7] into perspective. They give a δ depending on ε and n so that if $A = A^*$ and B are $n \times n$ matrices, and $\| [A, B] \| < \delta$, then there is a commuting pair $A_1 = A_1^*$ and B_1 such that $\| A - A_1 \| < \varepsilon$ and $\| B - B_1 \| < \varepsilon$. Their δ function goes to zero like n^{-1} for fixed ε. See also [8].

This example cannot be used to answer the question about self adjoint pairs. More precisely, the results of section 5 show that the pairs $(A_n, \operatorname{Re} B_n)$ and $(A_n, \operatorname{Im} B_n)$ are both asymptotically close to commuting pairs of self adjoint matrices.

3. A reformulation

We start with a general procedure for attempting to perturb almost commuting pairs to commuting pairs. Let $\varepsilon > 0$ be given. Suppose we have chosen a small positive number δ and an integer s. Let A and B be Hermitian operators (or matrices) such that $\| [A, B] \| < \delta$. Look for commuting self-adjoint matrices A_1 and B_1 so that $\| A - A_1 \| < \varepsilon$ and $\| B - B_1 \| < \varepsilon$.

Choose an integer $m > 2\varepsilon^{-1}\|A\|$, and let J_j be the disjoint intervals $[-\|A\| + (j-1)\varepsilon, \|A\| + j\varepsilon)$ for $1 \leqq j \leqq m$. Divide each J_j into s disjoint intervals I_i of length ε/s, $(j-1)s+1 \leqq i \leqq js$. Let $\mathscr{H}_i = E_A(I_i)$, $i = 1,\ldots,n = sm$, be the spectral subspaces for A. On $\mathscr{H} = \oplus\sum_{i=1}^n \mathscr{H}_i$, A has the diagonal form $A = \oplus\sum_{i=1}^n A_i$. Each A_i is self-adjoint with spectrum contained in $\bar{I}_i$. With respect to this decomposition, B has a matrix (B_{ij}). One can impose conditions on δ, s and ε to ensure that the B_{ij} terms for $|i-j| \geqq 2$ are small. To this end, we need the following lemma.

LEMMA 3.1. *Let A and B be self-adjoint operators. Let*

$$a_0 \leqq -\|A\| < a_1 < \ldots < a_n \leqq \|A\|$$

partition the spectrum of A into intervals $I_i = [a_i, a_{i+1})$, and write $A = \oplus\sum_{i=n}^n A_i$ such that $\sigma(A_i)$ is contained in I_i. Suppose that $a_{i+1} - a_i \geqq \varepsilon$ for $i = 1,\ldots,n$, $\|[A,B]\| < \delta$ and $E_A(I_i)BE_A(I_j) = 0$ for $|i-j| \leqq 1$. Then dist $(B,\{A\}') < 2\delta/\varepsilon$.

PROOF. The proof is an easy adaption of the proof of Theorem 4.1 in [4]. We sketch the ideas briefly here.

There is a function f in $L^1(\mathrm{R})$, continuous except at 0, such that $\hat{f}(x) = x^{-1}$ for $|x| \geqq \varepsilon$ and $\|f\|_1 < 2\varepsilon^{-1}$. (Here $\hat{f}$ denotes the Fourier transform.) Define

$$Q = \int_{-\infty}^{\infty} e^{-isA}[A,B]e^{isA}f(s)ds.$$

It is immediate that

$$\|Q\| \leqq \|[A,B]\| \, \|f\|_1 < 2\delta\varepsilon^{-1}.$$

So it suffices to prove that $[A,Q] = [A,B]$.

Proceed as in [4] by taking eigenvectors u and v for A, with $Au = \alpha u$ and $Av = \beta v$. Then, as computed there,

$$u^*(AQ - QA)v = u^*[A,B]v\,(\alpha - \beta)\hat{f}(\alpha - \beta).$$

If $|\alpha - \beta| < \varepsilon$, the hypothesis on B guarantees that $u^*[A,B]v = 0$. When $|\alpha - \beta| \geqq \varepsilon$, $\hat{f}(\alpha - \beta) = (\alpha - \beta)^{-1}$ so we obtain

$$u^*[A,Q]v = u^*[A,B]v$$

for all eigenvectors u and v of A. Linearity then shows that $[A, B - Q] = 0$

in the finite dimensional case. In general, a straightforward approximation argument completes the proof.

Let B' be the operator with matrix entries $B'_{ij} = B_{ij}$ if $|i-j| \leqq 1$, and $B'_{ij} = 0$ otherwise. We immediately obtain that

$$\|[A, B']\| \leqq 3\|[A, B]\| < 3\delta.$$

Hence $\|[A, B-B']\| < 4\delta$. By applying Lemma 3.1 to $B-B'$, we obtain an operator C in $\{A\}'$ such that $\|B-(B'+C)\| < 8\delta s\varepsilon^{-1}$. Let $\tilde{B} = B'+C$, and note that like B', this is also tridiagonal with respect to the spectral decomposition of A. So

$$\|[A, \tilde{B}]\| < 3\delta \text{ and } \|B - \tilde{B}\| < 8\delta s\varepsilon^{-1}.$$

Let R_j be the projection onto $E_A(J_j)$. Look for projections L_j with

$$\sum_{j<l} R_l \leqq L_j \leqq \sum_{j<l} R_l$$

and $\|[\tilde{B}, L_j]\| < \varepsilon/5$. If this can be accomplished, let

$$A_1 = \sum_{j=l}^{m} \mu_j(L_{j-1} - L_j)$$

where μ_j is the left endpoint of the interval J_j. And let

$$B_1 = \sum_{j=1}^{m} (L_{j-1} - L_j)\tilde{B}(L_{j-1} - L_j).$$

It is clear that A_1 and B_1 commute. Also, since $\{L_j\}$ commutes with $\{R_j\}$, it is straight forward spectral estimate to obtain

$$\|A - A_1\| \leqq \varepsilon.$$

Now,

$$B_1 - \tilde{B} = \sum_{j=2}^{m} (L_{j-1} - L_j)\tilde{B}L_{j-1}^{\perp} + L_{j-1}^{\perp}\tilde{B}(L_{j-1} - L_j).$$

However, since B is tridiagonal,

$$(L_{j-1} - L_j)\tilde{B}L_{j-1}^{\perp} = (R_{j-1} + R_j)(L_{j-1} - L_j)\tilde{B}L_{j-1}^{\perp}(R_{j-2} + R_{j-1}).$$

These operators have norm at most $\|[\dot{B}, L_{j-1}]\| > \varepsilon/5$. The set of even terms (respectively, odd terms) have pairwise orthogonal domains and ranges. So adding up the pieces and dealing with the second term by symmetry, we obtain $\|B_1 - B\| < 4\varepsilon/5$. Thus

$$\|B - B_1\| < 4\varepsilon/5 + 8\delta s \varepsilon^{-1}.$$

Consequently, the choice $\delta \leqq \varepsilon^2/40s$ will suffice.

To further facilitate our search for the projections L_j, we make a further restriction. Note that $\mathscr{R}_j\mathscr{H}$ consists of s blocks $\mathscr{H}_i$, $sj - s < i \leqq sj$. Since $\tilde{B}$ is block tridiagonal, the compression $\tilde{B}_j$ to $\mathscr{R}_j\mathscr{H}$ is also block tridiagonal. Furthermore,

$$R_{j+1}\tilde{B}R_j = P_{sj+1}\tilde{B}P_{sj} = \tilde{B}_{sj,sj+1}.$$

So, write $L_j = \sum_{j<l} R_l + \tilde{L}_j$, where $\tilde{L}_j$ is a projection acting in $\mathscr{R}_j\mathscr{H}$. We stipulate that $P_{sj-s+1} \leqq \tilde{L}_j$, $\tilde{L}_j \perp P_{sj}$, and $\|[\tilde{B}_j, \tilde{L}_j]\| < \varepsilon/5$. This requirement ensures that we need not worry about $\tilde{B}_{sj,sj+1}$ or $\tilde{B}_{sj-s,sj-s+1}$ in the commutator $[\tilde{B}, L_j]$. What remains is to find such projections L_j. We attempt to do this by making s very large.

Our requirements are formulated in the following question:

(Q′) For each $\varepsilon > 0$, is there a positive integer $s = s(\varepsilon)$ so that if B is a finite rank Hermitian, norm one, block tridiagonal matrix acting on $\mathscr{H} = \oplus\sum_{i=0}^{s}\mathscr{H}_i$, there is a projection L with $\mathscr{H}_0$ in the range of L and $\mathscr{H}_s$ in the kernel of L such that $\|[B,L]\| < \varepsilon$?

The significance of (Q′) lies in the fact that, though it appears to be a much simpler question than (Q), it is in fact equivalent.

THEOREM 3.2. *Questions* (Q) *and* (Q′) *are equivalent.*

PROOF. The construction above shows that a positive answer to (Q′) also settles (Q) affirmative and $\varepsilon > 0$. Let δ be supplied by (Q) for $\varepsilon' = \varepsilon/21$.

Let $s = 2[\delta^{-1}] + 2$. Given an appropriate Hermitian matrix B acting on $\mathscr{H} = \oplus\sum_{i=0}^{s}\mathscr{H}_i$, let A be the block diagonal matrix $\sum_{i=0}^{s} \frac{i}{s}P_i$, where P_i is the projection onto $\mathscr{H}_i$. Then $\|[A,B]\| \leqq 2s^{-1} < \delta$. So there are commuting Hermitian matrices A_1 and B_1 with

$$\|A - A_1\| < \varepsilon/21 \text{ and } \|B - B_1\| < \varepsilon/21.$$

Let $E = E_{A_1}[0,1/2]$ be the spectral projection for A_1. Since $P_0 = E_A\{1\}$, Lemma 2.1 implies that $\|P_0E^\perp\|$ and $\|P_sE\|$ are both less than $2\varepsilon/21$. By

Lemma 2.2, there is a projection L such that $P_0 \leqq L$ and $P_s L = 0$ so that $\|L - E\| < 10\varepsilon/21$. Thus

$$\begin{aligned}\|[B,L]\| &\leqq \|[B - B_1, L]\| + \|[B_1, E]\| + \|[B, L - E]\| \\ &\leqq \|B - B_1\| + 0 + 2\|L - E\| < \varepsilon.\end{aligned}$$

In subsequent sections, some partial answers to (Q′) will be obtained. First, there are a few more pertinent remarks. Suppose B is tridiagonal on $\mathscr{H} = \oplus\sum_{i=0}^{s}\mathscr{H}_i$ and $\mathscr{H}_0$ is k-dimensional. Let $\mathscr{M}_j$ be the span of $B^i\mathscr{H}_0$ for $0 \leqq i \leqq j$, and $\mathscr{H}'_i = \mathscr{M}_i \ominus \mathscr{M}_{i-1}$. Then B is tridiagonal with respect to the decomposition $\mathscr{M} = \oplus\sum_{i=0}^{\infty}\mathscr{H}'_i$. The space $\mathscr{M}$ is reducing for B, and clearly $\mathscr{H}_s$ is orthogonal to $\oplus\sum_{i=0}^{s-1}\mathscr{H}'_i$. So for the purpose of answering (Q′), one may as well suppose that each $\mathscr{H}_i$ has dimension $\dim \mathscr{H}_i \leqq \dim \mathscr{H}_0$.

Now, let M be any fixed self-adjoint operator with spectrum $\sigma(M) = [0,1]$. (One might take M to be multiplication by x on $L^2(0,1)$.) It follows from the Weyl-von Neumann theorem that given any positive matrix B of norm one and an $\varepsilon = 0$, there is a subspace $\mathscr{M}$ so that $\|[P_{\mathscr{M}}, M]\| < \varepsilon$ and $\|B - M|_{\mathscr{M}}\| < \varepsilon$. Consequently, by the previous remarks, one readily sees that (Q′) is equivalent to (Q″).

(Q″) For every $\varepsilon > 0$, is there a positive integer s so that if $\mathscr{K}$ is any finite dimensional subspace of $L^2(0,1)$, there is a projection P with

$$\mathscr{K} \subsetneqq \text{range}(P) \subsetneqq \text{span}\{M^i\mathscr{K}, 0 \leqq i \leqq s\}$$

such that $\|[M,P]\| < \varepsilon$.

This might be thought of as asking if self-adjoint operators are uniformly quasidiagonal.

4. The absorption theorem.

The purpose of this section is to prove the theorem stated in the introduction. First, some lemmas:

LEMMA 4.1. *Let*

$$A = \begin{pmatrix} A_- & 0 \\ 0 & A_0 \end{pmatrix}$$

be self-adjoint with $A_- \leqq 0$ and $0 \leqq A_0 \leqq I$. Suppose that $\|[B,A]\| \leqq \varepsilon$. Then

$$C = \begin{bmatrix} 0 & 0 \\ 0 & A_0^2 \end{bmatrix}$$

satisfies $\|[B,C]\| \leqq 6\varepsilon$.

PROOF. Write $B = (B_{ij}), i,j = 1.2$. Then

$$\varepsilon = \|[B,A]\| = \left\|\begin{bmatrix} [B_{11},A_-] & B_{12}A_0 - A_-B_{12} \\ B_{21}A_- - A_0B_{21} & [B_{22},A_0] \end{bmatrix}\right\|$$

and,

$$[B,C] = \begin{bmatrix} 0 & B_{12}A_0^2 \\ -A_0^2B_{21} & [B_{22},A_0^2] \end{bmatrix}.$$

Now,

$$\|[B_{22},A_0^2]\| \leqq 2\|A_0\|\,\|[B_{22},A_0]\| \leqq 2\varepsilon.$$

Let $P_k = E_A(2^{-k-1}, 2^{-k}]$ be the spectral projection of A for the interval $(2^{-k-1}, 2^{-k}]$ for $k \geqq 0$. Then

$$\|(B_{12}P_k)(A_0P_k) - A_-(B_{12}P_k)\| < \varepsilon,$$

and since $\sigma(A_0|P_k\mathscr{H})$ is at least 2^{-k-1} from $\sigma(A_-)$, we obtain from [4] that $\|B_{12}P_k\| \leqq 2^{k+1}\varepsilon$. Hence

$$\|B_{12}P_kA_0^2\| \leqq 2^{k+1}\varepsilon 2^{-2k} = 2\varepsilon/2^k.$$

It is immediate that $\|B_{12}A_0^2\| \leqq 4\varepsilon$. Similarly, $\|A_0^2B_{21}\| \leqq 4\varepsilon$, and thus $\|[B,C]\| \leqq 4\varepsilon + 2\varepsilon = 6\varepsilon$.

REMARK. It is not just to fit the proof that A_0^2 is used. The choice of

$$C' = \begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix}$$

would not work. To see this, consider the map from $[A,B]$ to $[C',B]$ for B of the form $\begin{pmatrix} 0 & 0 \\ B_1 & 0 \end{pmatrix}$. This then amounts to the map from $B_1A_0 - A_-B_1$ to B_1A_0. If we take A_0 to be diagonal with weights $\rho^n, n \geqq 0$ and $A_- = -A_0$, then this map becomes the Schur product of $B_1A_0 - A_-B_1$ by the matrix $Z_\rho = (z_{ij})$ where

$$z_{ij} = \frac{\rho^j}{\rho^i + \rho^j} = \frac{1}{\rho^{i-j} + 1}.$$

For ρ very small, Z_p is very close to the matrix $T = (t_{ij})$ where $t_{ii} = 1/2$, $t_{ij} = 1$ if $i < j$, and $t_{ij} = 0$ if $i > j$. Schur multiplication by T is unbounded since triangular truncation is unbounded. Thus, there is no constant $K < \infty$ so that

$$\|[C',B]\| \leqq K\|[A,B]\| \quad \text{for all } B.$$

It is apparently the fact that

$$g(x) = \begin{cases} 0 & x \leqq 0 \\ x^2 & x \geqq 0 \end{cases}$$

is differentiable at 0 that makes this work. Indeed, if f is a function such that

$$f(x) - f'(0)x - f(0) = O(x^{1+\varepsilon})$$

for some $\varepsilon > 0$, then there is a constant K so that

$$\|f(A_-)B - Bf(A_0)\| \leqq K\|A_-B - BA_0\|$$

for all B. The proof is a simple modification of the special case in the lemma.

LEMMA 4.2. *Suppose*

$$A = \begin{bmatrix} A_- & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & A_+ \end{bmatrix}$$

is self-adjoint, $A_- \leqq L$, $L \leqq A_0 \leqq L+\varepsilon$, *and* $A_+ \leqq L+\varepsilon$. *Suppose* $\|[B,A]\| \leqq \delta$. *Let*

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

where

$$D_0 = \sin^2(\tfrac{1}{2}\pi\varepsilon^{-1}(A_0 - L)).$$

Then $\|[B,D]\| \leqq 14\delta/\varepsilon$.

PROOF. We will scale A so that $L = 0$ and $\varepsilon = 1$ (so δ now represents δ/ε). By [4], $\|B_{13}\| \leqq \delta$ and $\|B_{31}\| \leqq \delta$. And

$$\begin{aligned} \|[B_{22}, D_0]\| &\leqq 2\|[B_{22}, \sin(\tfrac{1}{2}\pi A_0)]\| \\ &\leqq 2\sum_{k=0}^{\infty}(2k+1)\frac{\|\frac{1}{2}\pi A_0\|^{2k}}{(2k+1)!}\|[B_{22}, A_0]\| \end{aligned}$$

$$\leqq 2 \sum_{k=0}^{\infty} \frac{(\frac{1}{2}\pi)^{2k}}{(2k)!}\delta = 2\cosh(\tfrac{\pi}{2})\delta.$$

Let $f(x) = (\frac{1}{2}\pi x)^{-1} \sin \frac{1}{2}\pi x$, which is analytic and entire, and $\| f | \mathrm{R} \|_\infty = 1$. By the previous lemma,

$$\| B_{12} D_0 \| \leqq \| B_{12} A_0^2 \| \ \| f(A_0) \|^2 \leqq 4\delta.$$

Similarly,

$$\begin{aligned} \| B_{32}(I - D_0) \| &= \| B_{32} \cos^2(\tfrac{1}{2}\pi \mathrm{A}_0) \| \\ &= \| B_{32}(I - A_0)^2 \| \ \| f(I - A_0) \|^2 \leqq 4\delta. \end{aligned}$$

Now,

$$[B,D] = \begin{bmatrix} 0 & B_{12} D_0 & B_{13} \\ -D_0 B_{21} & [B_{22}, D_0] & -(I - D_0) B_{23} \\ -B_{31} & B_{32}(I - D_0) & 0 \end{bmatrix}$$

So $\| [B,D] \| \leqq 4\delta + 2\cosh(\frac{1}{2}\pi)\delta + 4\delta \leqq 14\delta$.

COROLLARY 4.3. *Let A, B, and D be as above. Define*

$$C = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & C_3 \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} 0 \oplus 0 & 0 \oplus 0 & 0 \oplus 0 \\ 0 \oplus 0 & P_0 & 0 \oplus 0 \\ 0 \oplus 0 & 0 \oplus 0 & I \oplus I \end{bmatrix}$$

where

$$P_0 = \begin{bmatrix} \sin^2(\frac{1}{2}\pi D_0) & \sin(\frac{1}{2}\pi D_0)\cos(\frac{1}{2}\pi D_0) \\ \sin(\frac{1}{2}\pi D_0)\cos(\frac{1}{2}\pi D_0) & \cos^2(\frac{1}{2}\pi D_0) \end{bmatrix}$$

Then P is an orthogonal projection such that $\| [B \oplus C, P] \| \leqq 61\delta/\varepsilon$.

PROOF. A direct computation and the previous proof shows that the (1,2) (and similarly the (2,1)) entry of $[B \oplus C, P]$ have norm dominated by

$$\| B_{12}(\tfrac{1}{2}\pi D_0) f(\tfrac{1}{2}\pi D_0) \| \leqq 2\pi\delta/\varepsilon.$$

And the (3,2) (and similarly the (2,3)) term are likewise bounded, as

$$\| B_{32} \tfrac{1}{2}\pi(1 - D_0) f(\tfrac{1}{2}\pi(1 - D_0)) \| \leqq 2\pi\delta/\varepsilon.$$

Again, $\| B_{13} \| \leqq \delta/\varepsilon$ and $\| B_{31} \| \leqq \delta/\varepsilon$. Finally, as above,

$$\|[B_{22}\oplus B_{22},P_0]\| \leqq \|[B_{22},\sin(\tfrac{1}{2}\pi D_0)]\| + \|[B_{22},\sin(\tfrac{1}{2}\pi D_0)\cos(\tfrac{1}{2}\pi D_0)]\|$$
$$\leqq (2\cosh\tfrac{1}{2}\pi)^2\delta/\varepsilon + (2\cosh\tfrac{1}{2}\pi)(2\sinh\tfrac{1}{2}\pi)\delta/\varepsilon$$

Thus

$$\|[B\oplus C,P]\| \leqq 4(\cosh^2\tfrac{1}{2}\pi + \cosh\tfrac{1}{2}\pi\sinh\tfrac{1}{2}\pi + \pi)\delta/\varepsilon \leqq 61\delta/\varepsilon.$$

We are now ready to prove the theorem mentioned in the introduction.

THEOREM 4.4. *For every pair A,B of self-adjoint operators on $\mathscr{H}$, there are commuting pairs C,D and A_1,B_1 of self-adjoint operators on $\mathscr{H}$ and $\mathscr{H}\oplus\mathscr{H}$, respectively, such that $\|C\| \leqq \|A\|$, $\|D\| \leqq \|B\|$, and*

$$\max(\|A\oplus C - A_1\|,\ \|B\oplus D - B_1\|)\} \leqq 25\,\|AB-BA\|^{1/2}.$$

PROOF. Let $\|AB-BA\| = \delta = (25)^{-2}\varepsilon^2$. Taking $s=1$, perform the reduction of section 3. That is, write $A = \oplus\sum_{i=1}^n A_i$ as a block diagonal matrix with $\sigma(A_i)$ contained in disjoint intervals I_i of length ε. Obtain the tridiagonal matrix $\tilde{B}$ with $\|B-\tilde{B}\| \leqq 8\delta\varepsilon^{-1}$ and $\|[A,\tilde{B}]\| \leqq 3\delta$. Define $D = \oplus\sum_{i=1}^n \tilde{B}_{ii}$ to be block diagonal with the same diagonal entries as $\tilde{B}$, and $C = \oplus\sum_{i=1}^n \lambda_i R_i$ where $R_i = E_A(I_i)$ and λ_i is the midpoint of I_i. Clearly, $\|C\| \leqq \|A\|$ and $\|D\| \leqq \|B\|$.

Then C and D commute, and $A\oplus C$ and $\tilde{B}\oplus D$ satisfy $\|[A\oplus C,\tilde{B}\oplus D]\| \leqq 3\delta$, $\|B\oplus D - \tilde{B}\oplus D\| \leqq 8\delta\varepsilon^{-1}$, and the matrix decomposition conform to section 3. So it suffices to define the projections L_j as required there. That is, $\|[\tilde{B}\oplus D,L_j]\| \leqq \varepsilon/5$ and

$$\sum_{l>j} R_l\oplus R_l \leqq L_j \leqq \sum_{l\geqq j} R_l\oplus R_l.$$

Now, Corollary 4.3 provides such a projection

$$L_j = P_j\oplus\sum_{l>j} R_l\oplus R_l,$$

where

$$D_j = \sin^2(\tfrac{1}{2}\pi\varepsilon(A_j - c_j)),$$

$c_j = \lambda_j - \varepsilon/2$ is the left end point of I_i, and

$$P_j = \begin{bmatrix} \sin^2(\tfrac{1}{2}\pi D_j) & \sin(\tfrac{1}{2}\pi D_j)\cos(\tfrac{1}{2}\pi D_j) \\ \sin(\tfrac{1}{2}\pi D_j)\cos(\tfrac{1}{2}\pi D_j) & \cos^2(\tfrac{1}{2}\pi D_j) \end{bmatrix}$$

which acts on $R_j\mathscr{H} \oplus R_j\mathscr{H}$. This projection satisfies

$$\|[\tilde{B} \oplus D, L_j]\| \leqq 61(3\delta)/\varepsilon = \tfrac{123}{625}\varepsilon \leqq \varepsilon/5.$$

Hence, by section 3, there are commuting self-adjoint operators A_1 and B_1 such that

$$\max\{\|A \oplus C - A_1\|, \|B \oplus D - B_1\|\} \leqq \varepsilon = 25\delta^{1/2} = 25\|AB - BA\|^{1/2}.$$

This yields an immediate corollary. Let $\mathscr{N}$ denote the set of normal operators.

COROLLARY 4.5. *If T is an operator on $\mathscr{H}$, then there is a normal operator N on $\mathscr{H}$ so that $\|N\| \leqq \|T\|$ and*

$$\operatorname{dist}(T \oplus N, \mathscr{N}) \leqq 50\sqrt{2}\|[T, T^*]\|^{1/2}.$$

PROOF. Since both functions are homogeneous, we may suppose that $\|T\| = 1$. Write $T = A + iB$, where A and B are self-adjoint. Note that $\|[T, T^*]\| = 2\|[A,B]\|$. By Theorem 4.4, we obtain commuting pairs $\{C,D\}$ and $\{A_1, B_1\}$ so that

$$\max\{\|A \oplus C - A_1\|, \|B \oplus D - B_1\|\} < \varepsilon = 25\|[A,B]\|^{1/2}.$$

Let $N' = C + iD$ and $M = A_1 + iB_1$. If $\|N'\| > 1$, replace it by $N = N'/\|N'\|$. From the proof of Theorem 4.4, one has

$$\|N' - \sum_{i=1}^{n} R_i T R_i\| < 2\varepsilon,$$

whence $\|N' - N\| \leqq \|N'\| - 1 < 2\varepsilon$. So

$$\operatorname{dist}(T \oplus N, \mathscr{N}) \leqq \|T \oplus N - M\| \leqq \|N - N'\| + \|T \oplus N' - M\|$$
$$< 2\varepsilon + 2\varepsilon = 100\|[A,B]\|^{1/2} = 50\sqrt{2}\|[T,T^*]\|^{1/2}.$$

REMARK. This estimate is good because it is homogeneous (of order 1), and as such, it is the best distance estimate possible up to a constant that can be determined from $\|[T^*, T]\|$ alone. However, the following examples show that this is not equivalent to the distance itself.

Let S be the unilateral shift. Then $\|[S,S^*]\| = 1$ and $\operatorname{dist}(S \oplus N, \mathscr{N}) \geqq \sqrt{5} - 2 > 1/5$. So

$$\operatorname{dist}((I + \varepsilon S) \oplus N, \mathscr{N}) \geqq \varepsilon/5 = \tfrac{1}{5}\|[I + \varepsilon S, I + \varepsilon S^*]\|^{1/2}.$$

However, if $T_\varepsilon = \operatorname{Re} S + i\varepsilon \operatorname{Im} S$, it is clear that

$$\operatorname{dist}(T_\varepsilon, \mathcal{N}) \leqq \varepsilon = \| [T_\varepsilon, T_\varepsilon^*] \| = \varepsilon^{1/2} \| [T_\varepsilon, T_\varepsilon^*] \|^{1/2}.$$

These two examples show that some information other than $\| [T, T^*] \|$ is needed. Modifications of these examples work in finite dimensions as well.

It is also of interest to obtain an anelogue og Theorem 4.1 for compact perturbations. The techniques here do apply, but we have not obtained a completely satiafying result. Independent of [5], we can obtain the following. Let C and D be commuting, self-adjoint operators with joint spectrum equal to $[-1,1] \times [-1,1]$.

THEOREM 4.6. *If A and B are norm one, self-adjoint operators with compact commutator, then $(A \oplus C) + i(B \oplus D)$ is in the norm closure of $\mathcal{N} + \mathcal{K}$ (normals plus compacts).*

REMARK. What is unsatisfying about this result is that it is not quantitative. There is no control on the norm of the compact perturbation required. This deficiency is independent of the fact that [5] is required to show that $\mathcal{N} + \mathcal{K}$ is closed. Compare with section 6.

PROOF. For each $\varepsilon > 0$, proceed as in section 3, but ignore the role of δ (and use $s = 1$). We obtain a diagonal form $A = \oplus\sum_{i=1}^n A_i$ for A, and a matrix $B = (B_{ij})$ for B. The compactness of $[A,B]$ implies that B_{ij} is compact for $|i-j| \geqq 2$. So the matrix B' with entries $B'_{ij} = B_{ij}$ if $|i-j| \leqq 1$ and $B'_{ij} = 0$ for $|i-j| \geqq 2$ is a compact perturbation of B. (Here the norm just got out of hand.) Let

$$A' = \sum_{i=1}^n \lambda_i R_i,$$

where λ_i is the midpoint of I_i. Then $\| A - A' \| < \varepsilon/2$ and $\| [A', B'] \| < 2\varepsilon$. By Theorem 4.4, there are commuting pairs of norm one self-adjoint operators $\{C', D'\}$ and $\{A_1, B_1\}$ such that

$$\max\{ \| A' \oplus C' - A_1 \|, \| B' \oplus D' - B_1 | \| < 25(2\varepsilon)^{1/2}.$$

Furthermore, the Weyl-von Neumann-Berg Theorem [1] shows that $(C' \oplus C) + i(D' \oplus D)$ is unitarily equivalent to a normal operator N with $\| N - (C + iD) \| < \varepsilon$. Thus, there are commuting self-adjoints $\tilde{A}_1$ and $\tilde{B}_1$ (unitarily equivalent to $A_1 \oplus C$ and $B_1 \oplus D$) such that

$$\max\{ \| A \oplus C - \tilde{A}_1 \|, \| B' \oplus D - \tilde{B}_1 \| \} < 2\varepsilon + 25(2\varepsilon)^{1/2}.$$

Hence $(A \oplus C) + i(B \oplus D)$ is a compact perturbation of something close to normal, and thus belongs to the closure of $\mathcal{N} + \mathcal{K}$.

5. Partial results

In this section we obtain a partial answer to question (Q′) which serves to eliminate from consideration many "natural" candidates for a counterexample. In particular, it will follow that the pairs $(A_n, \text{Re}B_n)$ and $(A_n, \text{Im}B_n)$ from section 2 are asymptotically close to commuting pairs of Hermitian matrices.

Say that a matrix T has a band width at most k with respect to an orthonormal basis $e_1, e_2, \ldots$ if $(Te_i, e_j) = 0$ for $|i-j| > k$.

LEMMA 5.1. *Given $\varepsilon > 0$ and a positive integer k, there exists an integer $N = N(\varepsilon,k)$ so that for every norm one, Hermitian matrix (or operator) T of band width k, there is a projection P such that $P\mathcal{H}$ contains $e_1, \ldots, e_k$, is contained in* span$\{e_1, \ldots, e_N\}$, *and* $\|[T,P]\| < \varepsilon$.

PROOF. Suppose, to the contrary, that there is a sequence T_m of $(n_m \times n_m)$ matrices and integers N_m tending to infinity so that for any projection P with the required range, one has $\|[T_m,P]\| \geqq \varepsilon$. Consider each T_m as acting on a fixed Hilbert space with basis $e_1, e_2, \ldots$ by setting it equal to zero on the orthogonal complement of span$\{e_1, \ldots, e_{n_m}\}$. The coefficients $t_{ij}^{(m)} = (T_m e_i, e_j)$ are bounded, so we can drop to a subsequence and assume that limits

$$\lim_{m \to \infty} t_{ij}^{(m)} = t_{ij}$$

exist for all $i,j \geqq 1$.

Let T be the operator with matrix entries t_{ij}. Then T is self-adjoint with norm at most one. Hence T is quasi-diagonal. So there is a finite rank projection P with the range of P containing $e_1, \ldots, e_k$ so that $\|[T,P]\| < \varepsilon/2$. It can readily be arranged that the range of P is contained on span $\{e_1, \ldots, e_N\}$ for some very large integer N.

Now, choose m_0 sufficiently large that $N_{m_0} \geqq N$ and

$$|t_{ij}^{(mo)} - t_{ij}| < \varepsilon/4k+2 \quad \text{for} \quad i,j \leqq N+k.$$

Let Q be the projection with range span$\{e_1, \ldots, e_{N+k}\}$. One obtains that $\|Q(T - T_{m_0})Q\| < \varepsilon/2$. So

$$\begin{aligned} \|[T_{m_0}, P]\| &= \|Q[T_{m_0}, P]Q\| \\ &= \|[T,P] + [Q(T_{m_0} - T)Q, P]\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This is a contradiction, which establishes our claim.

THEOREM 5.2. *Given $\varepsilon > 0$ and a positive integer k, there is a $\delta = \delta(\varepsilon,k) > 0$ with the following property: Fix an orthonormal basis $e_1, e_2, \ldots$. Let A and B be Hermitian matrices of norm one such that A is diagonal with monotone increasing entries and B is of band width k and such that $\|[A,B]\| < \delta$. Then there are commuting self-adjoint matrices A_1 and B_1 with $\|A - A_1\| < \varepsilon$ and $\|B - B_1\| < \varepsilon$.*

PROOF. Follow the procedure in section 3 with $s = N(\varepsilon/5,k) + k$ and $\delta = \varepsilon^2/40s$. The operators B'_j are block tridiagonal on

$$\mathscr{K}_j = R_j \mathscr{H} = \oplus \sum_{i=js-s+1}^{js} \mathscr{H}_i,$$

and are also band width k. If any $\mathscr{H}_{i_0}$ is zero space, the projection $\tilde{L}_j$ onto $\oplus\sum_{i=i_0}^{sj} \mathscr{H}_i$ commutes exactly with B'_j, and $L_j = \sum_{j<l} R_l + \tilde{L}_j$ commutes with B'. Otherwise, all $\mathscr{H}_i$ have dimension at least one, so Lemma 5.1 applies, and provides a projection $\tilde{L}_j$ such that span $\{e_1, \ldots, e_k\}$ is contained in the range of $\tilde{L}_j$, and span $\{e_s, e_{s-1}, \ldots, e_{s+1-k}\}$ is in the kernel, and $\|[B'_j,\tilde{L}_j]\| < \varepsilon/5$. Thus

$$L_j = \sum_{j<l} R_l + \tilde{L}_j^{\perp}$$

satisfies $\|[B',L_j]\| < \varepsilon/5$. The method of section 3 thus provides the desired perturbation.

REMARK 5.3. In view of the remarks in Section 3, it follows that (Q′) has a positive solution if the dimension of $\mathscr{H}_0$ is bounded by k. The whole problem is to find a bound independent of this (finite) dimension. However, if the dimension of $\mathscr{H}_0$ is infinite, it is not in general possible. Indeed, there is a norm one, self-adjoint, block tridiagonal operator A on $\mathscr{H} = \oplus\sum_{n=0}^{\infty} \mathscr{H}_n$ so that if P is any projection with

$$\mathscr{H}_0 \subsetneq P\mathscr{H} \subsetneq \oplus\sum_{n=0}^{N} \mathscr{H}_n \quad \text{for any } N,$$

then $\|[A,P]\| \geqq 10^{-2}$.

Such an operator can be constructed using the real and imaginary parts of the unilateral shift. The details are omitted.

6. Addendum

In this section, we obtain a quantitative version of the Brown-Douglas-Fillmore Theorem [5] valid for nice simply connected regions.

THEOREM 6.1. *Let T be an essentially normal operator with $\|T\| = 1$ and $\sigma_e(T)$ equal to the unit disc. Then there is a compact operator K such that $T - K$ is normal, and $\|K\| \leqq 75 \|TT^* - T^*T\|^{1/2}$.*

PROOF. By [5], T is quasidiagonal, and thus there is a compact operator K_0 so that $\|K_0\| \leqq \varepsilon = \|TT^* - T^*T\|^{1/2}$ and

$$T - K_0 \cong \sum_{n=1}^{\infty} \oplus T_n \oplus D$$

where D is a diagonal normal with spectrum the unit disc, T_n are finite rank, $\|T_n\| \leqq 1$, and

$$\sup \|T_n T_n^* - T_n^* T_n\| \leqq \varepsilon^2$$

and

$$\lim_{n \to \infty} \|T_n T_n^* - T_n^* T_n\| = 0.$$

By Corollary 4.5, there are normal operators N_n of norm one and finite rank operators K_n with $\|K_n\| \leqq 50\sqrt{2}\|T_n T_n^* - T_n^* T_n\|^{1/2}$ and $(T_n \oplus N_n) - K_n$ is normal of norm one. There is a diagonal compact operator K_∞ with $\|K_\infty\| < \varepsilon$ so that

$$D - K_\infty \approx \sum_{n=1}^{\infty} \oplus N_n \oplus D.$$

Let $K' = K_0 + (0 \oplus K_\infty)$, so that

$$T - K' \cong \sum_{n=1}^{\infty} \oplus (T_n \oplus N_n) \oplus D.$$

Let $K = K' + \left(\sum_{n=1}^{\infty} \oplus K_n \oplus 0 \right)$. Then

$$\|K\| \leqq (50\sqrt{2} + 2)\varepsilon \leqq 75\varepsilon,$$

and

$$T - K \cong \sum_{n=1}^{\infty} \oplus (T_n \oplus N_n - K_n) \oplus D \cong D.$$

This result can be extended using the continuity of the functional calculus to the following:

COROLLARY 6.2. *Let f be univalent in a neighbourhood of the closed disc $\bar{D}$, and let $\Omega = f(\bar{D})$. There is a constant C depending on Ω so that: if T is an essentially normal operator such that $\Omega = \sigma_e(T)$ is a spectral set for T, then there is a compact operator K so that $T - K$ is normal and*

$$\|K\| \leqq C \|TT^* - T^*T\|^{1/2}.$$

OPEN PROBLEM. Find a quantitative version of BDF valid for an annulus.

ADDED IN PROOF. 1. The author and I. D. Berg have now proven Theorem 6.1 without using BDF [5].

2. Berg pointed out that Lemma 3.1 immediately yields the conclusion $\|B\| < 4\delta\varepsilon^{-1}$.

3. The remark after Lemma 4.1 is closely related to an example of A. McIntosh, Proc. Amer. Math. Soc. 29 (1971), 337–340.

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