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FINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS OF C*-ALGEBRAS ASSOCIATED WITH TOPOLOGICAL DYNAMICAL SYSTEMS

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Abstract.

Let X be a compact space, σ a homeomorphism of X and $\mathscr{A}$ the C*-crossed product $C(X) \times_\sigma Z$ of $C(X)$ with respect to σ. It is proved that the space $\hat{\mathscr{A}}_n$ of unitary equivalence classes of n-dimensional irreducible representations of $\mathscr{A}$ is homeomorphic with the topological product $(X_n/\sim) \times T$, where X_n is the set of points x in X with $\sigma^n(x)=x$ and $\sigma^k(x) \neq x$ for $1 \leqq k \leqq n-1$, where $\sim$ is the orbit equivalence relation, and where $T=\{z \in \mathbf{C}\colon |z|=1\}$. As an application, we obtain a complete classification of the C*-algebras associated with Bernouilli shifts.

Introduction.

Let X be a compact (Hausdorff) space and σ a homeomorphism of X. Let $C(X)$ be the C*-algebra of all continuous functions on X. The *-automorphism of $C(X)$ induced by σ is denoted by σ again, i.e., $\sigma(f)(x)=f(\sigma(x))$ for f in $C(X)$ and x in X. Throughout this paper, $\mathscr{A}$ always denotes the C*-crossed product of $C(X)$ by the group $\mathbf{Z}$ of all integers with respect to the action $\{\sigma^n\colon n \in \mathbf{Z}\}$ on $C(X)$, that is, $\mathscr{A}=C(X) \times_\sigma \mathbf{Z}$, and $\hat{\mathscr{A}}_n$ the space of equivalence classes of n-dimensional irreducible representations of $\mathscr{A}$ $(1 \leqq n < \infty)$.

In this paper, we shall determine the topological structure of $\hat{\mathscr{A}}_n$ in terms of the topological dynamical system (X,σ). Let

$$X^n = \{x \in X : \sigma^n(x)=x\} \quad \text{and} \quad X_n = X^n \setminus \left(\bigcup_{m=1}^{n-1} X^m\right).$$

We say that two points x and y in X_n are equivalent if the two orbits of x and y coincide, i.e.,

$$\{\sigma^k(x) : 0 \leqq k \leqq n-1\} = \{\sigma^k(y) : 0 \leqq k \leqq n-1\}.$$

Let T be the unit circle in the complex plane. Then we have the following result.

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THEOREM A. *The space $\hat{\mathscr{A}}_n$ is homeomorphic with the product space $(X_n/\sim) \times T$.*

The representations of $\mathscr{A}$ have been studied by many authors. Especially the papers [3], [4], [7], and [9] are closely related to Theorem A. In [9: Theorem 5.3] Williams explicitly determined the topology on the primitive ideal space of a class of C*-crossed products ($E-H$ regular) associated with transformation groups on a locally compact space [9: Theorem 5.3]. When X is second countable, our theorem can be derived from his theorem, but we impose no condition on the dynamics (X, σ) to derive our result. The proof of Theorem A is elementary and is given in Section 1.

In Section 2, the theorem is applied to classify the C*-algebras associated with Bernoulli shifts, which were considered in [5] as a class of C*-algebras associated with topologically transitive dynamics (Theorem B). Moreover we discuss more general cases (Theorem C) and show that, in the case of Markov chains, C*-isomorphism determines the topological entropy of σ (Theorem D).

1. Proof of Theorem A.

Let $\mathfrak{H}_n$ be a fixed n-dimensional Hilbert space. We denote by $\mathrm{Irr}_n(\mathscr{A})$ the set of irreducible representations ϱ of $\mathscr{A}$ on $\mathfrak{H}_n$. We consider $\mathrm{Irr}_n(\mathscr{A})$ as a topological space with the topology of weak pointwise convergence over $\mathscr{A}$, which is equivalent to the norm convergence topology in this case. Since $\hat{\mathscr{A}}_n$ is homeomorphic with the quotient space $\mathrm{Irr}_n(\mathscr{A})/\sim$ by the unitary equivalence relation ([2: Theorem 3.5.8]), we consider the topological spaces $\mathrm{Irr}_n(\mathscr{A})$ and $\mathrm{Irr}_n(\mathscr{A})/\sim$ instead of $\hat{\mathscr{A}}_n$.

We now recall the product of covariant representations of the C*-crossed product $\mathscr{A} = C(X) \times_\sigma Z$. Let $K(\mathbf{Z}, C(X))$ be the set of all functions F of $\mathbf{Z}$ into $C(X)$ such that $F(n) = 0$ for all n in $\mathbf{Z}$ except finitely many n. Then $K(\mathbf{Z}, C(X))$ is naturally and densely embedded in $\mathscr{A}$. For a covariant representation (π, u) of $(C(X), \sigma)$ on a Hilbert space $\mathfrak{H}$, the representation $\pi \times u$ of $\mathscr{A}$ on $\mathfrak{H}$ is defined for $K(\mathbf{Z}, C(X))$ as follows;

$$(\pi \times u)(F) = \sum_{n \in \mathbf{Z}} \pi(F(n))u^n \qquad (F \in K(\mathbf{Z}, C(X))),$$

and the set $\{(\pi \times u)(F) \colon F \in K(\mathbf{Z}, C(X))\}$ is a dense subalgebra of $(\pi \times u)(\mathscr{A})$. Since every representation ϱ of $\mathscr{A}$ is of the form $\varrho = \pi \times u$ for some covariant representation (π, u) of $(C(X), \sigma)$ ([6: Proposition 7.6.4]), we consider only the set of covariant representations of $(C(X), \sigma)$ on $\mathfrak{H}_n$. Let $\{e_0, e_1, \ldots, e_{n-1}\}$ be a fixed basis for $\mathfrak{H}_n$. For x in X_n and z in T, we denote by π_x and u_z the representation of $C(X)$ and the unitary operator on $\mathfrak{H}_n$ defined by

$$\pi_x(f)e_k = f(\sigma^k(x))e_k \qquad (f \in C(X),\ 0 \leqq k \leqq n-1),$$

$$u_z e_k = e_{k+1} \qquad (0 \leqq k \leqq n-2) \quad \text{and} \quad u_z e_{n-1} = z e_0 .$$

Then the representation (π_x, u_z) becomes a covariant representation of $(C(X), \sigma)$ on $\mathfrak{H}_n$, that is, $u_z^* \pi_x(f) u_z = \pi_x(\sigma(f))$ for f in $C(X)$. In this paper, e_{in+k} (respectively z_{in+k}) means e_k (respectively z_k) for every integer i and k $(0 \leqq k \leqq n-1)$. For (x, z) in the product space $X_n \times T$, we put $\Phi(x,z) = \pi_x \times u_z$. Then Φ becomes a map of $X_n \times T$ into $\mathrm{Irr}_n(\mathscr{A})$.

LEMMA 1. *The map Φ is continuous.*

PROOF. Let (x, z) and (y, w) be in $X_n \times T$. For f in $C(X)$, we have

$$\|\pi_x(f)(u_z)^{in+k}e_0 - \pi_y(f)(u_w)^{in+k}e_0\|$$
$$= \|f(\sigma^k(x))z^i e_k - f(\sigma^k(y))w^i e_k\| = |f(\sigma^k(x))z^i - f(\sigma^k(y))w^i| \to 0$$
$$(\text{as } x \to y \text{ and } z \to w).$$

For F in $K(\mathbf{Z}, C(X))$, we have

$$\|(\pi_x \times u_z)(F)e_0 - (\pi_y \times u_w)(F)e_0\|$$
$$= \left\| \sum_{n \in \mathbf{Z}} (\pi_x(F(n))(u_z)^n e_0 - \pi_y(F(n))(u_w)^n e_0) \right\|$$
$$= \sum_{n \in \mathbf{Z}} \|\pi_x(F(n))(u_z)^n e_0 - \pi_y(F(n))(u_w)^n e_0\| \to 0$$
$$(\text{as } x \to y \text{ and } z \to w).$$

Since e_0 is cyclic for $(\pi_x \times u_z)(\mathscr{A})$ and $(\pi_x \times u_z)(K(\mathbf{Z}, C(X)))$ is dense in $(\pi_x \times u_z)(\mathscr{A})$, $(\pi_x \times u_z)(T)\xi$ converges to $(\pi_y \times u_w)(T)\xi$ for every T in $\mathscr{A}$ and every ξ in $\mathfrak{H}_n$ as (x, z) converges to (y, z) in $X_n \times T$.

For x in X_n, we put $O(x) = \{\sigma^k(x)\colon 0 \leqq k \leqq n-1\}$. Now we consider an equivalence relation in $X_n \times T$. Namely we say that (x, z) and (y, w) are equivalent if $O(x) = O(y)$ and $z = w$, and denote by φ the canonical map of $X_n \times T$ onto the quotient space $(X_n \times T)/\sim \, = (X_n/\sim) \times T$. The quotient map of $\mathrm{Irr}_n(\mathscr{A})$ onto $\mathrm{Irr}_n(\mathscr{A})/\sim$ is denoted by ψ. Then, since the equivalence relations in $X_n \times T$ and $\mathrm{Irr}_n(\mathscr{A})$ are open (cf. [2: 3.5.5]), φ and ψ are open. In the following, we prove that $\pi_x \times u_z$ and $\pi_y \times u_w$ are unitarily equivalent if and only if $O(x) = O(y)$ and $z = w$.

LEMMA 2. *Let (x, z) and (y, w) be in $X_n \times T$. Then*

$$\psi(\Phi(x,z)) = \psi(\Phi(y,w))$$

if and only if $\varphi(x,z)=\varphi(y,w)$.

PROOF. We first suppose that $O(x)=O(y)$ and $z=w$. Then $y=\sigma^i(x)$ for some i $(0\leqq i\leqq n-1)$. In the case $i\neq 0$, let v_1 and v_2 be the unitary operators on $\mathfrak{H}_n$ defined by

$$v_1 e_k = e_{k+i} \ (0\leqq k\leqq n-1) \quad \text{and}$$

$$v_2 e_k = z e_k \ (0\leqq k\leqq n-i-1), \quad v_2 e_k = e_k \ (n-i\leqq k\leqq n-1).$$

By the definition of the operators v_1 and v_2, one checks that

$$(v_1 v_2)^*((\pi_x\times u_z)(T))v_1 v_2 = (\pi_y\times u_w)(T) \quad (T\in\mathscr{A}).$$

Conversely we suppose that $O(x)\neq O(y)$, which is equivalent to $O(x)\cap O(y)=\emptyset$. Then there exists a continuous function f on X such that $f(x)=1$ but $f(\sigma^k(y))=0$ for all k $(0\leqq k\leqq n-1)$. Hence $\pi_x(f)\neq 0$ but $\pi_y(f)=0$. When $z\neq w$, we have that $(u_z)^n=z1\neq w1=(u_w)^n$. Therefore, in both cases, $\pi_x\times u_z$ is not unitarily equivalent to $\pi_y\times u_w$.

By the two lemmas above, we get an injective and continuous map Ψ of $(X_n/\sim)\times \boldsymbol{T}$ to $\mathrm{Irr}_n(\mathscr{A})/\sim$, by defining

$$\Psi(\varphi(x,z)) = \psi(\Phi(x,z)) \quad ((x,z)\in X_n\times \boldsymbol{T}).$$

Now, for a representation ϱ (respectively π) of $\mathscr{A}$ (respectively $C(X)$) on $\mathfrak{H}_n$ and a unitary operator u on $\mathfrak{H}_n$, we put

$$[\mathrm{Ad}\,u](\varrho)(T) = u^*\varrho(T)u \quad (T\in\mathscr{A})$$

$$[\mathrm{Ad}\,u](\pi)(f) = u^*\pi(f)u \quad (f\in C(X)).$$

In the following, we prove that

$$\{[\mathrm{Ad}\,u](\pi_x\times u_z) : (x,z)\in X_n\times \boldsymbol{T},\ u\in U(\mathfrak{H}_n)\} = \mathrm{Irr}_n(\mathscr{A}),$$

where $U(\mathfrak{H}_n)$ is the set of all unitary operators on $\mathfrak{H}_n$.

LEMMA 3. *The map* Ψ *is surjective.*

PROOF. Let $\varrho=\pi\times u$ be in $\mathrm{Irr}_n(\mathscr{A})$, where (π,u) is a covariant representation of $(C(X),\sigma)$. Then $\pi(C(X))$ is of course an abelian C*-algebra on $\mathfrak{H}_n$. Hence the spectrum S of $\pi(C(X))$ consists of m-points $\{p_0,p_1,\ldots,p_{m-1}\}$ $(1\leqq m\leqq n)$. For each characteristic function $\chi_k\in C(S)$ of $\{p_k\}$, there exists a function f_k in $C(X)$ such that $\pi\hat{}(f_k)=\chi_k$, where $\pi\hat{}(f_k)$ is the Gel'fand transform of $\pi(f_k)$. Hence

$\{\pi(f_k): 0 \leqq k \leqq m-1\}$ are orthogonal projections of $\mathfrak{H}_n$ and $\pi(C(X))$ is generated by them. Since the map $f \to \pi\hat{}(f)(p_k)$ is a character of $C(X)$, there is for each k a point x_k in X such that $\pi\hat{}(f)(p_k) = f(x_k)$ for all f in $C(X)$. It is easy to see that $\sigma(x_k)$ belongs to the set $\{x_i: 0 \leqq i \leqq m-1\}$ and that $u^*\pi(f_k)u = \pi(f_j)$ if and only if $\sigma(x_k) = x_j$. Moreover it follows that

$$O(x_k) = \{x_i : 0 \leqq i \leqq m-1\} \quad \text{for all } k,$$

because

$$\mathfrak{M} = \sum_{x_i \in O(x_k)} \oplus \ \pi(f_i)\mathfrak{H}_n$$

is a non-zero invariant subspace for $\{\pi(C(X)), u, u^*\}$, and ϱ is irreducible. Hence, rearranging the set $\{x_k: 0 \leqq k \leqq m-1\}$, we can assume that $\sigma^k(x_0) = x_k$ for each k $(0 \leqq k \leqq m-1)$ and $\sigma^m(x_0) = x_0$.

Next we shall show that $m = n$. For this, it is sufficient to prove that $\dim \pi(f_0)\mathfrak{H}_n = 1$. Suppose that $\dim \pi(f_0)\mathfrak{H}_n > 1$. Since $[\mathrm{Ad}\, u^m](\pi(f)) = \pi(f)$ for all f in $C(X)$, $\pi(f_0)$ commutes with u^m. Now consider the spectral decomposition of u^m on $\mathfrak{H}_n$, that is, $u^m = \sum_{j=1}^N e^{i\theta_j}E_j$. Then $E_j\pi(f_0) = \pi(f_0)E_j$ for $j = 1, \ldots, N$ and $\pi(f_0)E_j \neq 0$ for some j. By the hypothesis of $\pi(f_0)$, there exists a projection q on $\mathfrak{H}_n$ such that

$$(u^m)^*qu^m = q, \quad q \leqq \pi(f_0)E_j \quad \text{and} \quad q < \pi(f_0).$$

We put $p = q + u^*qu + \ldots + (u^{m-1})^*qu^{m-1}$. Then we have $p < 1$ and $up = pu$, so that the subspace $\mathfrak{M} = p\mathfrak{H}_n$ is invariant for $\{\pi(C(X)), u, u^*\}$. This is a contradiction. Hence

$$\dim \pi(f_0)\mathfrak{H}_n = \dim \pi(f_k)\mathfrak{H}_n = 1 \quad \text{for all } k \ (0 \leqq k \leqq m-1).$$

Since $\sum_{k=0}^{m-1} \pi(f_k) = 1$, we have $m = n$. Let d_0 be a unit vector in $\pi(f_0)\mathfrak{H}_n$ and put $d_k = u^k d_0$ for k $(0 \leqq k \leqq n-1)$. Then d_k belongs to the subspace $\pi(f_k)\mathfrak{H}_n$ for each k, and $ud_{n-1} = zd_0$ for some z in T. Let v be the unitary operator on $\mathfrak{H}_n$ defined by $ve_k = d_k$ for k $(0 \leqq k \leqq n-1)$. Then we have $[\mathrm{Ad}\, v](\pi \times u) = \pi_{x_0} \times u_z$.

LEMMA 1.4. *The map Ψ is open.*

PROOF. Since φ is a continuous map, it is sufficient to show that $\psi \circ \Phi$ is an open map of $X_n \times T$ onto $\mathrm{Irr}_n(\mathscr{A})/\sim$. Let U be an open set in $X_n \times T$. To prove that $\psi(\Phi(U))$ is an open set, we shall show that $W = \psi^{-1}(\psi(\Phi(U)))$ is an open set in $\mathrm{Irr}_n(\mathscr{A})$. Since

$$W = \{[\mathrm{Ad}\, u](\pi_x \times u_z) : (x, z) \in U, \ u \in U(\mathfrak{H}_n)\},$$

W is $[\mathrm{Ad}\, u]$-invariant, so that $\varrho = [\mathrm{Ad}\, u](\pi_x \times u_z)$ is an interior point of W if and only if $\pi_x \times u_z$ is an interior point of W.

Let (x, z) be in U. Then there exist $\varepsilon>0$ and a neighbourhood $U(x)$ of x in X such that

$$U^0 = \{(y, w) \in X_n \times T : y \in U(x), |z-w|<\varepsilon\} \subset U .$$

For this $U(x)$, there exists a continuous function f on X such that $f(x)=1$ and $f(y)=0$ for $y \notin U(x)$. Let W^0 be the set of representations ϱ in $\mathrm{Irr}_n(\mathscr{A})$ satisfying the following two conditions, in which F and G are functions in $K(\mathbf{Z}, C(X))$.

$$\text{(1)} \qquad \|\varrho(F)e_0-(\pi_x \times u_z)(F)e_0\| < 1 ,$$

where $F(0)=f$ and $F(m)=0$ for $m \neq 0$,

$$\text{(2)} \qquad \|\varrho(G)e_0-(\pi_x \times u_z)(G)e_0\| < \varepsilon ,$$

where $G(n)=1$ and $G(m)=0$ for $m \neq n$.

Let $\varrho=[\mathrm{Ad}\, u](\pi_y \times u_w)$ be in W^0. Condition (1) implies that $\|\varrho(F)e_0-e_0\|<1$, so that $\varrho(F)=u^*\pi_y(f)u \neq 0$. Hence $\pi_y(f) \neq 0$, that is,

$$\pi_y(f)e_k = f(\sigma^k(y))e_k \neq 0 \quad \text{for some } k \ (0 \leqq k \leqq n-1) .$$

This means that $\sigma^k(y)$ belongs to $U(x)$. On the other hand, Condition (2) implies that

$$\|\pi_y(1)(u_w)^n e_0-\pi_x(1)(u_z)^n e_0\| = \|we_0-ze_0\| = |w-z| .$$

Since $\pi_y \times u_w$ and $\pi_{\sigma^k(y)} \times u_w$ are unitarily equivalent, $\psi(\varrho)$ belongs to $\psi(\Phi(U^0))$. Hence W^0 is a neighbourhood of $\Phi(x, z)$ such that $W^0 \subset \psi^{-1}(\psi(\Phi(U^0))) \subset W$.

The preceding lemmas complete the proof of Theorem A.

2. Applications.

We consider three kinds of dynamical systems and C*-crossed products associated with them.

(2-1). Let $X(k)=\prod_{i \in \mathbf{Z}}\{0, \ldots, k-1\}$ and σ_k be the Bernoulli shift on $X(k)$, that is, $\sigma_k((x_i)_{i \in \mathbf{Z}})=(x_{i-1})_{i \in \mathbf{Z}}$. Then

$$X(k)_1 = \{x \in X(k) : \sigma_k(x)=x\} = \{x=(x_i)_{i \in \mathbf{Z}} : x_i=x_0 \text{ for all } i \in \mathbf{Z}\} .$$

Let $\mathscr{A}(k)=C(X(k)) \times_{\sigma_k} \mathbf{Z}$. By Theorem A, $\mathscr{A}(k)_1$ is homeomorphic with the topological sum of k-copies of the unit circle T, i.e.,

$$\mathscr{A}(k)_1 = \underbrace{T \oplus \ldots \oplus T}_{k}.$$

Therefore the C*-algebras $\mathscr{A}(k)$'s are completely classified.

THEOREM B. *$\mathscr{A}(k)$ is C*-isomorphic with $\mathscr{A}(j)$ if and only if $k=j$.*

(2–2). Let K be a compact space. Let $X_K=\prod_{i\in Z}K$ and $\sigma_K((x_i)_{i\in Z}) = (x_{i-1})_{i\in Z}$ on X_K. Let $\mathscr{A}_K=C(X_K)\underset{\sigma_K}{\times}Z$. Since $(X_K)_1$ is homeomorphic with $K\times T$, we have the following theorem by Theorem A.

THEOREM C. *Let K and L be compact spaces. If $\mathscr{A}_K$ and $\mathscr{A}_L$ are C*-isomorphic, then $K\times T$ and $L\times T$ are homeomorphic.*

The above theorem is of course a generalization of Theorem B. Moreover we find that $\mathscr{A}_{S^n}$ (respectively $\mathscr{A}_{T^n}$) is C*-isomorphic with $\mathscr{A}_{S^m}$ (respectively $\mathscr{A}_{T^m}$) if and only if $n=m$. However, from Theorem C one cannot conclude that K and L are homeomorphic. Indeed, there exist topological spaces K and L which are not homeomorphic but have homeomorphic product spaces, $K\times T$ and $L\times T$ (cf. [1: Theorem 4.1 and 6.6]).

(2–3). Let $X(k)$ and σ_k be as in (2–1). For a $k\times k$ matrix $M=(a_{i,j})_{i,j=0}^{k-1}$ with $a_{i,j}\in\{0,1\}$, let

$$X_M = \{(x_i)_{i\in Z}\in X(k) : a_{x_i,x_{i+1}}=1\}$$

and σ_M the restriction of σ_k to X_M. Let $\mathscr{A}_M=C(X_M)\underset{\sigma_M}{\times}Z$. Then, for each $n\geqq 1$, $(X_M)_n$ is a finite set and by Theorem A the cardinal numbers of $(X_M)^n$ are determined by the C*-algebra $\mathscr{A}_M$. Under the condition that M is irreducible, the topological entropy $h(\sigma_M)$ is determined by the cardinal numbers $N_n(\sigma_M)$ of $(X_M)^n$, i.e.,

$$h(\sigma_M) = \lim_{n\to\infty}(1/n)\log N_n(\sigma_M)$$

(cf. [9: Theorem 8.17]). Hence we have the following theorem.

THEOREM D. *Let M and N be irreducible matrices. If $\mathscr{A}_M$ and $\mathscr{A}_N$ are C*-isomorphic, then $h(\sigma_M)=h(\sigma_N)$.*

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