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SHAPE THEORY FOR C*-ALGEBRAS

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Abstract.

A shape theory is developed for separable C*-algebras, generalizing the topological theory. Several related results about homotopy and lifting homomorphisms into quotients are obtained. The relationship between shape equivalence and K-theory and Kasparov theory is explored.

Shape theory has played an important role in topology in recent years. Roughly speaking, the goal of shape theory is to separate out the global properties of a topological space X which can be measured by the homology or cohomology groups of X from the possibly pathological local structure of X. The idea is to write X as a projective limit $X = \varprojlim X_n$ of "nice" spaces (specifically absolute neighborhood retracts or ANR's), and then consider only those topological properties of X which can be determined from the homotopy type of the X_n and the connecting maps. A general reference for shape theory is [4].

Shape theory for C*-algebras was first introduced by Effros and Kaminker [9]. The idea here is to write a general C*-algebra A as an inductive limit $A = \varinjlim A_n$ of "nice" (semiprojective—see below) C*-algebras and classify the algebras up to homotopy equivalence of the associated inductive systems.

The theory of [9] has already proved useful in classifying C*-algebras which are inductive limits of algebras of the form $C(S^1) \otimes F$, F finite-dimensional, and it is clear that C*-shape theory will be important in the future.

It is desirable for potential applications of shape theory to extend the theory to apply to C*-algebras which are not obviously covered by [9]. Also, the theory of [9], although analogous to topological shape theory, is not a direct noncommutative generalization.

In this paper, we develop a shape theory for general (separable) C*-algebras which exactly restricts to topological shape theory in the commutative case, and prove analogs and generalizations of some of the principal results of the topological theory as well as those of [9]. It is hoped that this shape theory will

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play a role in noncommutative topology similar to that played by ordinary shape theory in the commutative case.

The organization of the paper is as follows. In section 1 we give a systematic treatment of universal C*-algebras on sets of generators and relations, which has heretofore been missing from the literature. In section 2, we define projective and semiprojective C*-algebras and maps between C*-algebras. Our definition of a semiprojective C*-algebra differs from that of [9] and is more in keeping with the definition of an ANR in topology. We prove that many "standard" C*-algebras are semiprojective. Section 3 contains some results about semiprojectives analogous to properties of ANR's, and a comparison of our definition with [9]. Section 4 contains the definition and basic results about shape systems and shape equivalence of C*-algebras. Then in section 5 we explore the relationship of shape theory with K-theory and Kasparov theory.

Throughout the paper, except in section 1, we will assume that all C*-algebras are *separable*. This is probably an unnecessary assumption, and most of the results carry through to the nonseparable case with only obvious modifications. The only results which do not obviously generalize are 3.1 and 3.6. We will have occasion to refer to several subcategories of the category $\mathscr{S}$ of separable C*-algebras: we will denote by $\mathscr{S}_1$ the category of separable unital C*-algebras and unital homomorphisms, $\mathscr{S}\mathscr{C}$ category of separable commutative C*-algebras (equivalent to the category of pointed compact metrizable spaces), and $\mathscr{S}\mathscr{C}_1$ the separable commutative unital C*-algebras (equivalent to the category of compact metrizable spaces.) We will use $\mathscr{C}$ to denote a general subcategory of $\mathscr{S}$. The term "ideal" will always mean "closed two-sided ideal." As in [9], if $\varphi, \psi: A \to B$ we will write $\varphi \simeq \psi$ if φ and ψ are homotopic. We will denote the unital extension of A by A^1, and the smallest unital C*-subalgebra of A^1 containing A by $\tilde{A}$. $\mathscr{K}$ will denote the compact operators.

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1. Universal C*-algebras.

We wish to define a universal C*-algebra on a set $\mathscr{G} = \{x_\alpha\}$ of generators and a set of relations $\mathscr{R}$. We could allow the relations in $\mathscr{R}$ to be any kind of relations which could be formulated for operators on a Hilbert space or for elements of a C*-algebra, but for specificity we will only consider relations of the form $(\|p(x_{\alpha_1}, \ldots, x_{\alpha_n}, x_{\alpha_1}^*, \ldots, x_{\alpha_n}^*)\| \leqq \eta)$, where p is a polynomial in $2n$ noncommuting variables with complex coefficients, $x_{\alpha_1}, \ldots, x_{\alpha_n} \in \mathscr{G}$, and $\eta \geqq 0$.

If $\eta=0$, the relation may be rewritten as an algebraic relation among $x_{\alpha_1},\ldots,x_{\alpha_n}, x^*_{\alpha_1},\ldots,x^*_{\alpha_n}$, and the scalars. In the relations, it is important to distinguish between functions of the generators which behave as scalars and scalars themselves (which are not assumed to be in the generated algebra.)

A representation of $(\mathscr{G},\mathscr{R})$ is a set of operators $\{y_\alpha\}$ on a Hilbert space $\mathscr{H}$ which satisfy

$$\|p(y_{\alpha_1},\ldots,y_{\alpha_n}, y^*_{\alpha_1},\ldots,y^*_{\alpha_n})\| \leqq \eta$$

whenever $(\|p(x_{\alpha_1},\ldots,x_{\alpha_n}, x^*_{\alpha_1},\ldots,x^*_{\alpha_n})\| \leqq \eta) \in \mathscr{R}$ (where complex coefficients are interpreted as scalar multiples of the identity.) A representation ϱ of $(\mathscr{G},\mathscr{R})$ extends uniquely to a $*$-homomorphism, also denoted ϱ, from the free $*$-algebra $\mathscr{F}(\mathscr{G})$ generated by $\mathscr{G}$ into $\mathscr{L}(\mathscr{H})$.

DEFINITION 1.1. A set $(\mathscr{G},\mathscr{R})$ is admissible if

(a) There exists a representation of $(\mathscr{G},\mathscr{R})$.

(b) Whenever $\{y^\beta_\alpha\}$ is a representation of $(\mathscr{G},\mathscr{R})$ on $\mathscr{H}^\beta$ for each $\beta\in\Omega$, then $\oplus_\beta y^\beta_\alpha \in \mathscr{L}(\oplus_\beta \mathscr{H}^\beta)$ for each α (and $\{\oplus_\beta y^\beta_\alpha\}$ is a representation of $(\mathscr{G},\mathscr{R})$).

Condition 1.1(a) insures that the relations in $\mathscr{R}$ are not inconsistent with each other or with the C*-axioms. (If representations on a 0-dimensional space are allowed, then 1.1(a) is vacuous, but the universal C*-algebra may be $\{0\}$.) 1.1(b) implies that the relations in $\mathscr{R}$ at least implicitly place a bound on $\|x_\alpha\|$ for each α. For relations of the form we are considering, the second part of 1.1(b) follows automatically from the first. The two conditions together imply that for any $z\in\mathscr{F}(\mathscr{G})$,

$$|||z||| = \sup\{\|\varrho(z)\| : \varrho \text{ a representation of } (\mathscr{G},\mathscr{R})\}$$

is a well-defined finite number, and that $|||\cdot|||$ is a C*-seminorm on $\mathscr{F}(\mathscr{G})$.

DEFINITION 1.2. The completion of $\mathscr{F}(\mathscr{G})/\{z: |||z|||=0\}$ under $|||\cdot|||$ is called the universal C*-algebra of $(\mathscr{G},\mathscr{R})$, denoted $C^*(\mathscr{G},\mathscr{R})$. $C^*(\mathscr{G},\mathscr{R})$ has the property that any representation of $(\mathscr{G},\mathscr{R})$ extends uniquely to a representation of $C^*(\mathscr{G},\mathscr{R})$, and any representation of $C^*(\mathscr{G},\mathscr{R})$ gives a representation of $(\mathscr{G},\mathscr{R})$.

EXAMPLES 1.3. (a) Let A be any C*-algebra, $\mathscr{G}=A$, $\mathscr{R}$ the set of all $*$-algebraic relations in A. Then $C^*(\mathscr{G},\mathscr{R})\cong A$.

(b) Let A be any C*-algebra, $\mathscr{G}=A_0$ a dense $*$-subring of A which is an algebra over a dense subfield of $\mathbf{C}$, $\mathscr{R}$ the set of all $*$-algebraic relations on A_0 plus the scalar multiple relations between elements of A_0 plus a relation $\|x\| \leqq \|x\|_A$ for each $x\in A_0$. Then $C^*(\mathscr{G},\mathscr{R})\cong A$. So in particular a separable C*-

algebra is the universal C*-algebra on a countable set of generators and relations.

(c) Let A be any Banach $*$-algebra, $\mathcal{G}=A$, $\mathcal{R}$ the $*$-algebraic relations. Then $C^*(\mathcal{G},\mathcal{R})$ is the enveloping C*-algebra of A. One may also work with a dense $*$-subring as in (b).

(d) Let G be a locally compact group. Applying (c) to $A=L^1(G)$, we get $C^*(\mathcal{G},\mathcal{R})=C^*(G)$. If G is discrete, $C^*(G)$ is also $C^*(\mathcal{G},\mathcal{R})$, where $\mathcal{G}=G$,

$$\mathcal{R} = \{x^*x=1, xx^*=1, \quad xy=z \text{ for all } x,y,z \in G \text{ with } xy=z\} .$$

(e) Many well-known C*-algebras occur naturally as universal C*-algebras:

(1) $\mathcal{G}=\{x\}$,

$$\mathcal{R} = \{x=x^*,\ \|x\|\leqq 1,\ \|1-x^2\|\leqq 1\} .$$

$C^*(\mathcal{G},\mathcal{R})\cong C_0((0,1])$.

(2) $\mathcal{G}=\{x,1\}$,

$$\mathcal{R} = \{x=x^*,\ \|x\|\leqq 1,\ \|1-x^2\|\leqq 1,\ 1=1^*=1^2,\ x1=1x=x\} .$$

$C^*(\mathcal{G},\mathcal{R})\cong C([0,1])$.

(1) and (2) are the "universal positive contraction C*-algebras."

(3) $\mathcal{G}=\{x\}$,

$$\mathcal{R} = \{x^*x=1,\ xx^*=1\} .$$

$C^*(\mathcal{G},\mathcal{R})\cong C(S^1)\cong C^*(\mathbf{Z})$, "universal unitary algebra."

(4) $\mathcal{G}=\{x\}$,

$$\mathcal{R} = \{x^*x=xx^*,\ \|x\|\leqq 1\} .$$

$C^*(\mathcal{G},\mathcal{R})\cong C_0(\mathcal{D}_0)$, where $\mathcal{D}_0$ is the punctured unit disk.

(5) $\mathcal{G}=\{x,1\}$,

$$\mathcal{R} = \{x^*x=xx^*,\ \|x\|\leqq 1,\ 1=1^*=1^2,\ x1=1x=x\} .$$

$C^*(\mathcal{G},\mathcal{R})\cong C(\mathcal{D})$.

(4) and (5) are the "universal normal contraction C*-algebras."

(6) $\mathcal{G}=\{x\}$, $\mathcal{R}=\{x^*x=1\}$. $C^*(\mathcal{G},\mathcal{R})\cong C^*(u)$, where u is the unilateral shift. This is the "universal isometry algebra" or "Toeplitz algebra," denoted $\mathcal{T}$.

(7) $\mathcal{G}=\{x\}$, $\mathcal{R}=\{\|x\|\leqq 1\}$, "universal contraction algebra."

(8) $\mathcal{G}=\{x,1\}$,

$$\mathcal{R} = \{\|x\|\leqq 1,\ 1=1^*=1^2,\ x1=1x=x\}\ ,$$

"universal unital contraction algebra."

(9) Let $A=(a_{ij})$ be an $n\times n$ matrix of 0's and 1's, $\mathcal{G}=\{s_1,\ldots,s_n\}$,

$$\mathcal{R} = \{s_i^*s_i=(s_i^*s_i)^2,\ s_i^*s_i=\sum_{j=1}^{n} a_{ij}s_js_j^*,\ s_k^*s_i=0 \quad \text{for all } i,k,i\neq k\}\ .$$

$C^*(\mathcal{G},\mathcal{R})\cong\mathcal{O}_A$ [7; 6.7].

(10) Let α be an irrational number, $0<\alpha<1$, $\mathcal{G}=\{u,v\}$,

$$\mathcal{R} = \{u^*u=uu^*=v^*v=vv^*=1,\ uv=e^{2\pi i\alpha}vu\}\ .$$

$C^*(\mathcal{G},\mathcal{R})$ is the irrational rotation algebra R_α.

(11) $\mathcal{G}=\{x_{ij}: 1\leqq i,j\leqq n\}$,

$$\mathcal{R} = \{x_{ij}=x_{ji}^*,\ x_{ij}x_{kl}=\delta_{jk}x_{il}\ :\ 1\leqq i,j,k,l\leqq n\}\ .$$

$C^*(\mathcal{G},\mathcal{R})\cong M_n(\mathbf{C})$.

(12) $\mathcal{G}=\{x_{ij}, 1: 1\leqq i,j\leqq n\}$,

$$\mathcal{R} = \{x_{ij}=x_{ji}^*=\sum_{k=1}^{n} x_{ik}x_{kj},\ 1=1^*=1^2,\ x_{ij}1=1x_{ij}=x_{ij}\}\ .$$

$C^*(\mathcal{G},\mathcal{R})$ is the "noncommutative Grassmanian" G_n^{nc} [5].

(13) $\mathcal{G}=\{x_{ij}: 1\leqq i,\ j\leqq n\}$,

$$\mathcal{R} = \left\{\sum_{k=1}^{n} x_{ki}^*x_{kj}=\delta_{ij}1,\ \sum_{k=1}^{n} x_{ik}x_{jk}^*=\delta_{ij}1\right\}.$$

$C^*(\mathcal{G},\mathcal{R})=U_n^{\mathrm{nc}}$ [5]. $U_1^{\mathrm{nc}}\cong C(S^1)$.

(f) Amalgamated free products. Given C*-algebras A_α ($\alpha\in\Omega$), D, with $\varphi_\alpha: D\to A_\alpha$ an injective map for each α, let $\mathcal{G}=\bigcup A_\alpha$, $\mathcal{R}$ the set of $*$-algebraic relations in all of the A_α, along with $\{\varphi_\alpha(x)=\varphi_\beta(x): x\in D,\ \alpha,\beta\in\Omega\}$. $C^*(\mathcal{G},\mathcal{R})$ is called the amalgamated free product of the A_α over D, denoted $*(A_\alpha,D,\varphi_\alpha)$ or $*_D A_\alpha$ when the φ_α are understood. These C*-algebras are considered in [5], [8], and [9]. One important (although not very difficult) point not mentioned in these references is the fact that $(\mathcal{G},\mathcal{R})$ has a representation. A proof can be found in [2; 3.1].

If $D=\{0\}$, then $*_D A_\alpha=*A_\alpha$ is the free product of the A_α. If $D=\mathbf{C}$, each A_α is unital, and φ_α maps D onto the scalars, then $*_D A_\alpha=*_{\mathbf{C}} A_\alpha$ is the unital free product. The universal C*-algebra generated by a set of unitaries, isometries, contractions, positive contractions, normal contractions, etc., can be written as a free product of algebras in (e).

(g) Tensor products. The unital (maximal) tensor product of a family of unital C*-algebras can be defined in a universal way as in the unital free product, with additional relations making the different algebras commute. However, if the algebras are nonunital, the commuting free product defined in this way generally contains the tensor product as a proper ideal. For example, the commuting free product of two copies of $C_0(\mathbf{R})$ is isomorphic to the algebra of continuous functions on the 2-torus vanishing at one point, which is not isomorphic to the tensor product $C_0(\mathbf{R}^2)$. Although arbitrary commuting free products can be defined, there is no reasonable general definition of a tensor product of an infinite number of nonunital C*-algebras [1]. A theory of amalgamated tensor products of C*-algebras over commutative subalgebras can be developed, although there appear to be technical problems in obtaining a satisfactory theory.

While universal C*-algebras often have a very complicated internal structure and are usually regarded as pathological, they are in some respects tractable objects for study. For example, it is often relatively easy to compute the K-theory of such algebras ([5], [7], [8]).

2. Projective and semiprojective C*-algebras.

In this section, we will denote by $\mathscr{C}$ a fixed subcategory of $\mathscr{S}$ which is closed under quotients.

DEFINITION 2.1. Let $A, B \in \mathscr{C}$ and $\varphi: A \to B$ a $\mathscr{C}$-morphism. φ is a projective morphism in $\mathscr{C}$ if for any $C \in \mathscr{C}$, ideal $J \subseteq C$, and morphism $\sigma: B \to C/J$, there is a morphism $\psi: A \to C$ with $\pi \circ \psi = \sigma \circ \varphi$, where $\pi: C \to C/J$ is the quotient map. A is projective in $\mathscr{C}$ if the identity map on A is projective.

This definition of projective C*-algebras agrees with that of [9]. If either A or B is projective, then any morphism from A to B is projective. More generally, a composition of a projective morphism with any other morphism (in either order) is projective.

EXAMPLES 2.2. (a) $C_0((0,1])$ is projective in $\mathscr{S}$ [9].

(b) $\mathbf{C}$ is projective in $\mathscr{S}_1$ but not in $\mathscr{S}$.

(c) $C([0,1]^2)$ is projective in $\mathscr{S}\mathscr{C}_1$ but not in $\mathscr{S}_1$ (the real and imaginary parts of the image of the unilateral shift u in the Calkin algebra give a homomorphism of $C([0,1]^2)$ into $C^*(u)/\mathscr{K}$ which cannot be lifted).

We now list some general facts about projective C*-algebras.

PROPOSITION 2.3. [9; 3.1] *A projective C*-algebra is contractible. In particular, a C*-algebra which is projective in $\mathscr{S}$ cannot be unital.*

PROPOSITION 2.4. *If A is projective, and $\varphi, \psi: A \to B$, then $\varphi \simeq \psi$.*

PROOF. Let $C = C([0,1], B)$, $J = C_0((0,1), B)$. $C/J \cong B \oplus B$. Lift $\varphi \oplus \psi$ to C.

PROPOSITION 2.5. *A is projective in $\mathscr{S}$ if and only if A^1 is projective in $\mathscr{S}_1$.*

PROOF. See [9; § 4] for one direction. For the other, note that any $\varphi: A \to B$ extends uniquely to a unital homomorphism from A^1 to $\tilde{B}$.

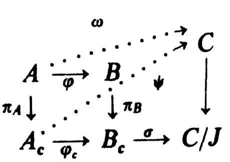
PROPOSITION 2.6. *If each A_n is projective in $\mathscr{S}$ (respectively $\mathscr{S}_1$), then $*A_n$ is projective in $\mathscr{S}$ (respectively $*_{\mathbf{C}}A_n$ is projective in $\mathscr{S}_1$).*

PROPOSITION 2.7. *A is projective in $\mathscr{S}\mathscr{C}$ (respectively $\mathscr{S}\mathscr{C}_1$) if and only if $A = C_0(X)$ for a locally compact (respectively compact) absolute retract X.*

DEFINITION 2.8. If A is a C*-algebra, let I be the commutator ideal of A, i.e. the (possibly improper) ideal generated by $\{xy - yx : x, y \in A\}$. $A_c = A/I$ is a commutative C*-algebra, called the abelianization of A. We can have $A_c = \{0\}$. A_c has the universal property that any homomorphism from A into a commutative C*-algebra factors through A_c. If $\varphi: A \to B$, then φ induces a unique morphism φ_c from A_c to B_c, called the abelianization of φ.

PROPOSITION 2.9. *If $\varphi: A \to B$ is projective in $\mathscr{S}$ (respectively $\mathscr{S}_1$), then $\varphi_c: A_c \to B_c$ is projective in $\mathscr{S}\mathscr{C}$ (respectively $\mathscr{S}\mathscr{C}_1$). So if A is projective in $\mathscr{S}$ (respectively $\mathscr{S}_1$), then A_c is projective in $\mathscr{S}\mathscr{C}$ (respectively $\mathscr{S}\mathscr{C}_1$).*

PROOF. Let C be commutative, and $\sigma: B_c \to C/J$. Consider the following diagram:



By projectivity, $\sigma \circ \pi_B \circ \varphi$ lifts to ω; since C is commutative, ω factors through A_c to give ψ.

More generally, any commutative diagram of maps abelianizes to a commutative diagram.

DEFINITION 2.10. Let $A, B \in \mathscr{C}$, $\varphi: A \to B$. φ is semiprojective in $\mathscr{C}$ if, for any $C \in \mathscr{C}$ and increasing sequence J_n of ideals of C with $J = \overline{\bigcup J_n}$, and for any morphism $\sigma: B \to C/J$, there is an n and a morphism $\psi: A \to C/J_n$ with $\pi \circ \psi = \sigma \circ \varphi$, where $\pi: C/J_n \to C/J$ is the quotient map. A is semiprojective in $\mathscr{C}$ if the identity map on A is semiprojective.

If A or B is semiprojective, then any morphism from A to B is semiprojective. More generally, a composition of a semiprojective morphism with any other morphism (in either order) is semiprojective. Any projective C*-algebra or morphism is semiprojective.

PROPOSITION 2.11. *$C_0(X)$ (respectively $C(X)$) is semiprojective in $\mathscr{S}\mathscr{C}$ (respectively $\mathscr{S}\mathscr{C}_1$) if and only if X is an* ANR.

The proof is an easy exercise, and is omitted. In the nonunital case one works in the category of pointed spaces. See [10] for detailed information about ANR's.

This shows that our definition of semiprojective does not agree with the definition in [9]. For any contractible C*-algebra is semiprojective in the sense of [9] by [9; 3.5], so if X is compact and contractible $C(X)$ is semiprojective in the sense of [9]. But there exist contractible spaces which are not ANR's, such as the cone over the Cantor set. We will show in 3.2 that our definition is actually more restrictive than the one in [9].

PROPOSITION 2.12. *If $\varphi: A \to B$ is semiprojective in $\mathscr{S}$ (respectively $\mathscr{S}_1$), then $\varphi_c: A_c \to B_c$ is semiprojective in $\mathscr{S}\mathscr{C}$ (respectively $\mathscr{S}\mathscr{C}_1$). So if A is semiprojective in $\mathscr{S}$ (respectively $\mathscr{S}_1$), then A_c is semiprojective in $\mathscr{S}\mathscr{C}$ (respectively $\mathscr{S}\mathscr{C}_1$).*

Thus, a commutative C*-algebra which is semiprojective in $\mathscr{S}$ or $\mathscr{S}_1$ must be $C_0(X)$ for an ANR X. The converse is false (2.33).

We now prove that many standard C*-algebras are semiprojective, using a long series of propositions.

LEMMA 2.13. *Let C be a C*-algebra, J_n an increasing sequence of ideals of C with $J = \overline{\bigcup J_n}$. Let $\pi_n: C \to C/J_n$, $\pi: C \to C/J$ be the quotient maps. Then, for any $x \in C$, $\|\pi(x)\| = \inf \|\pi_n(x)\|$.*

PROOF. Follows easily from the definition of the quotient norm.

PROPOSITION 2.14. *Let C, J_n, J be as in 2.13, and let $\bar{p}$ be a projection in C/J. Then there is a projection $p \in C/J_n$ for sufficiently large n with $\pi(p)=\bar{p}$.*

PROOF. Let $0 \leqq x \in C$ with $\pi(x)=\bar{p}$. Then $\pi(x-x^2)=0$, so $\|\pi_n(x-x^2)\|<1/4$ for sufficiently large n. Thus for large n $\pi_n(x)$ has disconnected spectrum at $1/2$ and p may be constructed from $\pi_n(x)$ by functional calculus.

COROLLARY 2.15. *If $\varphi\colon A \to B$ is semiprojective in $\mathcal{S}_1$ (respectively $\mathcal{S}\mathcal{C}_1$), then it is semiprojective in $\mathcal{S}$ (respectively $\mathcal{S}\mathcal{C}$). So if A is semiprojective in $\mathcal{S}_1$, it is semiprojective in $\mathcal{S}$. Conversely, if φ (respectively A) is unital and semiprojective in $\mathcal{S}$, then φ (respectively A) is semiprojective in $\mathcal{S}_1$.*

PROOF. Given C, J_n, J, $\sigma\colon B \to C/J$, let $\bar{p}=\sigma(1_B)$. Lift $\bar{p}$ to $p \in C/J_n$, and replace C by $p(C/J_n)p$, J_k by $p(J_k/J_n)p$ for $k \geqq n$, J by $p(J/J_n)p$. Conversely, if φ is semiprojective in $\mathcal{S}$, C is unital, and $\sigma\colon B \to C/J$ is unital, let $\psi\colon A \to C/J_k$ be a lift. Set $q=\psi(1_A)$. Since $\pi(1_{C/J_k}-q)=0$, we have $\|\pi_n(1_{C/J_k}-q)\|<1$ for large n. But $\pi_n(1_{C/J_k}-q)$ is a projection, so $\pi_n(q)=1_{C/J_n}$ for large n, that is $\pi_n \circ \psi$ is a unital lift.

It follows that C is semiprojective in $\mathcal{S}$. From now on, unless otherwise qualified, "semiprojective" will mean "semiprojective in $\mathcal{S}$."

COROLLARY 2.16. *A is semiprojective if and only if $\tilde{A}$ is semiprojective in $\mathcal{S}_1$.*
PROOF. If A is unital, this follows trivially from 2.15; if A is nonunital, the argument is similar to 2.5.

PROPOSITION 2.17. *G_n^{nc} is semiprojective.*

PROOF. We may work in $\mathcal{S}_1$. If $\sigma\colon G_n^{\mathrm{nc}} \to C/J$, then σ extends to a morphism, also denoted σ, from $M_n(G_n^{\mathrm{nc}})$ to $M_n(C/J) \cong M_n(C)/M_n(J)$. If x is the matrix $(x_{ij}) \in M_n(G_n^{\mathrm{nc}})$, then x is a projection, so $\sigma(x)$ is a projection $\bar{p}$ in $M_n(C)/M_n(J)$. Lift $\bar{p}$ to a projection

$$p = (p_{ij}) \in M_n(C)/M_n(J_k) \cong M_n(C/J_k)$$

for some k; then the map $x_{ij} \mapsto p_{ij}$ gives a lift of σ to C/J_k.

PROPOSITION 2.18. *Let C, J_n, J be as in 2.13, and $\bar{p}_1,\ldots,\bar{p}_r$ orthogonal projections in C/J. Then for sufficiently large n there are orthogonal projections $p_1,\ldots,p_r \in C/J_n$ with $\pi(p_i)=\bar{p}_i$ for all i. If C is unital and $\bar{p}_1+\ldots+\bar{p}_r=1$, then we may choose the p_i* so that $p_1+\ldots+p_r=1$.

PROOF. We apply 2.14 inductively. First lift $\bar{p}_1$ to $p_1 \in C/J_{n_1}$. Replace C by $(1-p_1)(C/J_{n_1})(1-p_1)$, J_n by $(1-p_1)(J_n/J_{n_1})(1-p_1)$, and J by $(1-p_1)(J/J_{n_1})(1-p_1)$ (these make sense even if C is nonunital). Now lift $\bar{p}_2$, and continue inductively. If C is unital and $\bar{p}_1+\ldots+\bar{p}_r=1$, then $\bar{p}_r=1-\bar{p}_1-\ldots-\bar{p}_{r-1}$. In this case, stop the induction after $r-1$ steps and set $p_r=1-p_1-\ldots-p_{r-1}$; otherwise continue the induction through all r steps.

COROLLARY 2.19. *If φ_i: $A_i \to B_i$ $(i=1,\ldots,r)$ are morphisms in $\mathcal{S}_1$, then*

$$\bigoplus_{i=1}^{r} \varphi_i : \bigoplus_{i=1}^{r} A_i \to \bigoplus_{i=1}^{r} B_i$$

is semiprojective if and only if each φ_i is semiprojective. So if $A_1,\ldots,A_r$ are unital, then $A_1 \oplus \ldots \oplus A_r$ is semiprojective if and only if each A_i is semiprojective.

PROOF. First lift $\bar{p}_i=\sigma(1_{B_i})$ to $p_i \in C/J_k$, and then lift φ_i to

$$\pi_{n_i}(p_i)[(C/J_k)/(J_{n_i}/J_k)]\pi_{n_i}(p_i) \qquad \text{for some } n_i \geqq k\ .$$

Let $n=\max n_i$. The converse is trivial.

REMARK 2.20. It is not known, and quite possibly false, that a direct sum of nonunital semiprojective C*-algebras is always semiprojective.

PROPOSITION 2.21. *Let C, J_n, J be as in* 2.13, *with C unital. If $\bar{u}$ is an isometry in C/J, then there is an isometry $u \in C/J_n$ for sufficiently large n with $\pi(u)=\bar{u}$. If $\bar{u}$ is unitary, then u can be chosen to be unitary.*

PROOF. Let $x \in C$ with $\pi(x)=\bar{u}$. Then

$$\lim \|\pi_n(x)^*\pi_n(x)-1\| = 0\ ,$$

so for large n, $\pi_n(x^*x)$ is invertible. Set $u_n=\pi_n(x)[\pi_n(x^*x)]^{-1/2}$. Then u_n is an isometry with $\pi(u_n)=\bar{u}$. If $\bar{u}$ is unitary, we also have

$$\lim \|\pi_n(x)\pi_n(x)^*-1\| = 0\ ,$$

so $\pi_n(x)$ is invertible for large n, and hence u_n is unitary for large n.

COROLLARY 2.22. *$\mathcal{T}$, $C(S^1)$, and U_n^{nc} are semiprojective.*

PROOF. Trivial for $\mathcal{T}$ and $C(S^1)$, and as in 2.17 for U_n^{nc}.

PROPOSITION 2.23. *Let C, J_n, J be as in* 2.13. *Let $\bar{p}$ and $\bar{q}$ be projections in C/J,* *and let $\bar{u}$ be partial isometry in C/J with $\bar{u}^*\bar{u}=\bar{p}$, $\bar{u}\bar{u}^*=\bar{q}$. Let p and q be projections in C with $\pi(p)=\bar{p}$, $\pi(q)=\bar{q}$. Then for sufficiently large n there is a partial isometry $u \in C/J_n$ with $\pi(u)=\bar{u}$ and $u^*u=\pi_n(p)$, $uu^*=\pi_n(q)$.*

PROOF. Let $x \in C$ with $\pi(x)=\bar{u}$. Then for large n we have

$$\|\pi_n(x)^*\pi_n(x)-\pi_n(p)\| \quad \text{and} \quad \|\pi_n(x)\pi_n(x)^*-\pi_n(q)\|$$

small. If f is a continuous function which is identically zero near 0 and for which $f(\lambda)=\lambda^{-1/2}$ for λ near 1, then $w=\pi_n(x)f(\pi_n(x^*x))$ is a partial isometry in C/J_n with $\|p'-\pi_n(p)\|$ and $\|q'-\pi_n(q)\|$ small, where $p'=w^*w$, $q'=ww^*$. Then if

$$v_i = z_i(z_i^*z_i)^{-1/2} \ (i=1,2) ,$$

where

$$z_1 = (2p'-1)(2\pi_n(p)-1)+1, \qquad z_2 = (2\pi_n(q)-1)(2q'-1)+1$$

[11; § 6, Lemma 4], then the v_i are unitaries in $(C/J_n)^\sim$ which conjugate $\pi_n(p)$ and q' to p' and $\pi_n(q)$ respectively, and $\pi(v_i)=1 \in (C/J)^\sim$; so $u=v_2wv_1$ is the desired partial isometry.

COROLLARY 2.24. *$\mathcal{O}_A$ is semiprojective for any matrix A.*

PROPOSITION 2.25. *$M_2(\mathbf{C})$ is semiprojective.*

PROOF. Let $e_{11}, e_{12}, e_{21}, e_{22}$ be matrix units in C/J. Lift e_{11} to a projection p in C/J_k, and e_{12} to a partial isometry u in C/J_n for some $n \geqq k$ with $u^*u=\pi_n(p)$, $uu^*=1-\pi_n(p)$. Then $\{\pi_n(p), u, u^*, 1-\pi_n(p)\}$ is a system of matrix units in C/J_n which lift the e_{ij}.

PROPOSITION 2.26. *If $\varphi\colon A \to B$ is semiprojective in $\mathscr{S}_1$, then $\varphi_2\colon M_2(A) \to M_2(B)$ is semiprojective in $\mathscr{S}_1$. So if A unital and semiprojective, then $M_2(A)$ is semiprojective.*

PROOF. Let $\{\bar{e}_{ij}\}$ be the images of the matrix units $\{f_{ij}\}$ of $M_2(\mathbf{C}) \subseteqq M_2(A)$ in C/J. Lift $\{\bar{e}_{ij}\}$ to matrix units $\{e_{ij}\} \subseteqq C/J_k$. Replace C by $e_{11}(C/J_k)e_{11}$, J_n by $e_{11}(J_n/J_k)e_{11}$ for $n \geqq k$, and J by $e_{11}(J/J_k)e_{11}$. Then $\varphi_2 \mid f_{11}M_2(A)f_{11}$ looks like φ, so $\sigma \circ \varphi_2 \mid f_{11}M_2(A)f_{11}$ lifts to a homomorphism

$$\psi\colon f_{11}M_2(A)f_{11} \to \pi_n(e_{11})[(C/J_k)/(J_n/J_k)]\pi_n(e_{11})$$

for some large n. If $x \in M_2(A)$, write

$$x = \sum_{i,j=1}^{2} f_{i1}x_{ij}f_{j1} ,$$

where $x_{ij} \in f_{11}M_2(A)f_{11}$; set

$$\psi_2(x) = \sum_{i,j=1}^{2} \pi_n(e_{i1})\psi(x_{ij})\pi_n(e_{j1}) .$$

ψ_2 is a lift of $\sigma \circ \varphi_2$.

PROPOSITION 2.27. *Let A be unital and semiprojective, and p a full projection in A. Then pAp is semiprojective.*

PROOF. Since p is full, for sufficiently large r we can find projections p' and q and a partial isometry v in $M_r(pAp)$ such that $qM_r(pAp)q \cong A$, $p' \leq q$, $v^*v = \operatorname{diag}(p, 0, \ldots, 0)$, $vv^* = p'$. Let $\sigma: pAp \to C/J$. Extend to

$$\sigma_r : M_r(pAp) \to M_r(C/J) .$$

Let $\bar{q} = \sigma_r(q)$. Lift $\bar{q}$ to a projection $q' \in M_r(C/J_k)$. Let ω be a lift of $\sigma_r | qM_r(pAp)q$ to $\pi_n(q')M_r(C/J_n)\pi_n(q')$ for some $n \geq k$. Set $\bar{u} = \sigma_r(v)$; by increasing n if necessary, we may find a partial isometry $u \in M_r(C/J_n)$ which lifts $\bar{u}$, for which $u^*u = \operatorname{diag}(1, 0, \ldots, 0)$ and $uu^* = \omega(p')$. Identify pAp and C/J_n with the upper left-hand corners in $M_r(pAp)$ and $M_r(C/J_n)$, respectively. If $\psi(x) = u^*\omega(vxv^*)u$, then ψ is a lift of σ to C/J_n.

COROLLARY 2.28. *If A is unital and semiprojective, then $M_n(A)$ is semiprojective for all n. In particular, $M_n(\mathbf{C})$ is semiprojective for all n.*

PROOF. $M_{2^k}(A)$ is semiprojective for all k by induction from 2.26, and $M_n(A)$ is a full corner in $M_{2^k}(A)$ for large k.

COROLLARY 2.29. *If A and B are unital* C*-*algebras which are strongly Morita equivalent, then A is semiprojective if and only if B is semiprojective.*

PROOF. A and B are each isomorphic to full corners in matrix algebras over the other.

This can be false if A or B is nonunital: it follows from 3.1 that $\mathcal{K}$ is not semiprojective.

COROLLARY 2.30. *If A is unital and semiprojective and F is a finite-dimensional* C*-*algebra, then $A \otimes F$ is semiprojective. In particular, F and $C(S^1) \otimes F$ are semiprojective.*

PROOF. Follows from 2.28 and 2.19.

PROPOSITION 2.31. *If* $A_1,\ldots,A_r$ *are semiprojective in* $\mathscr{S}$ *(respectively* $\mathscr{S}_1$*), then* $A_1 * \ldots * A_r$ *is semiprojective in* $\mathscr{S}$ *(respectively* $A_1 *_{\mathbf{C}} \ldots *_{\mathbf{C}} A_r$ *is semiprojective in* $\mathscr{S}_1$*).*

PROPOSITION 2.32. *If* $A_1,\ldots,A_r$ *are semiprojective and* F *is a finite-dimensional* C*-*subalgebra of* $A_1,\ldots,A_r$, *then* $A_1 *_F \ldots *_F A_r$ *is semiprojective.*

PROOF. Almost identical to [9; 3.11].

Note that an infinite free product of semiprojective C*-algebras will not in general be semiprojective unless all but finitely many are projective.

EXAMPLE 2.33. $C([0,1]^2)$ is semiprojective (in fact projective) in $\mathscr{S}\mathscr{C}_1$; but it is not semiprojective in $\mathscr{S}_1$. For let u be the unilateral shift, C the C*-algebra of all sequences in $C^*(u)$ converging to a scalar multiple of the identity,

$$J_n = \{(x_1,x_2,\ldots):\ x_i \in \mathscr{K} \subseteq C^*(u) \text{ for all } i,\ x_i=0 \text{ for } i>n\}\ ,$$

$$J = \{(x_1,x_2,\ldots):\ x_i \in \mathscr{K} \text{ for all } i,\ x_i \to 0\}\ .$$

Then $J=\overline{\bigcup J_n}$. C/J is isomorphic to the C*-algebra of all sequences in $\pi(C^*(u))$ converging to a scalar multiple of the identity, where

$$\pi : C^*(u) \to C^*(u)/\mathscr{K} \cong C(S^1)\ .$$

Let $x=(x_1,x_2,\ldots)$ and $y=(y_1,y_2,\ldots)$ with $x_n=\operatorname{Re}(\pi(u))/n$, $y_n=\operatorname{Im}(\pi(u))/n$. Then x and y are commuting self-adjoint contractions in C/J, so there is a homomorphism σ of $C([0,1]^2)$ onto $C^*(x,y)\subseteq C/J$. But it is easily seen that σ cannot lift to C/J_n for any n. One can also define a homomorphism from $C(S^1\times S^1)$ into C/J which cannot be lifted by sending the two generators to e^{ix} and e^{iy}; thus $C(S^1\times S^1)$ is not semiprojective in $\mathscr{S}_1$. A similar argument shows that the irrational rotation algebras are not semiprojective.

2.34. Example 2.33 shows that even a universal C*-algebra on a finite set of generators and relations need not be semiprojective. For such a C*-algebra to be semiprojective, the relations must be partially liftable in the sense that if $x_1,\ldots,x_n \in C/J$ satisfy the relations, then suitable preimages in C/J_k for sufficiently large k also satisfy the relations. The propositions in this section and elementary C*-theory show that many types of relations are partially liftable, such as $(\|x_\alpha\| \leq \eta)$, $(x_\alpha = x_\alpha^*)$, $(x_\alpha = x_\alpha^* = x_\alpha^2)$, $(x_\alpha^* x_\alpha = 1)$, $(x_\alpha^* x_\alpha = x_\alpha x_\alpha^* = 1)$, (the matrix (x_{ij}) is a projection or unitary in $M_n(A)$), etc. But 2.33 shows that commutation relations among the generators are not partially liftable. It would be very interesting to know whether every C*-algebra is the universal C*-algebra on a set of generators and compatible partially liftable relations. This would imply that every C*-algebra is an inductive limit of semiprojectives. A more extensive list (or, preferably, a characterization) of partially liftable relations is needed.

There is a weakened form in which any finite set of relations is partially liftable, which will be crucial in the development of shape theory:

PROPOSITION 2.35. *Let* $G=\{x_1,\ldots,x_n\}$,

$$\mathscr{R} = \{\|p_1(\cdot)\| \leqq \eta_1, \ldots, \|p_k(\cdot)\| \leqq \eta_k\},$$

with all $\eta_i > 0$, *and let* $A = C^*(\mathscr{G},\mathscr{R})$. *If* $\varphi\colon A \to B$ *satisfies*

$$\|p_i(\varphi(x_1),\ldots,\varphi(x_n),\varphi(x_1^*),\ldots,\varphi(x_n^*))\| < \eta_i$$

for each i, then φ *is semiprojective.*

PROOF. Follows easily from 2.13.

Finally, we prove a generalization of the theorem from topology that a compact retract of an open set in an ANR is an ANR [10; III.7.7].

DEFINITION 2.36. If A and B are C*-algebras with A unital, A is retract of B if there is a unital homomorphism $\omega\colon A \to M(B)$, the multiplier algebra of B, and a surjective homomorphism $\varrho\colon B \to A$, such that $\tilde{\varrho}\circ\omega = \mathrm{id}_A$, where $\tilde{\varrho}$ is the canonical extension of ϱ to a homomorphism from $M(B)$ to A [13; 3.12.10].

THEOREM 2.37. *Let D be semiprojective in* $\mathscr{S}$, *I an ideal in D, and A a unital C*-algebra which is a retract of I. Then A is semiprojective.*

PROOF. Let $\omega\colon A \to M(I)$ and $\varrho\colon I \to A$ be as above. Let C be unital, and let $\sigma\colon A \to C/J$ be unital, with $J = \overline{\bigcup J_n}$. The argument of [13; 3.12.8] gives a homomorphism $\theta\colon D \to M(I)$ which is the identity on I. If $\alpha = \sigma\circ\varrho\circ\theta$, then α lifts to a map $\beta\colon D \to C/J_k$ for some k. $\pi\circ\beta(I) = \sigma(A)$ contains the identity of C/J, so for sufficiently large n, $B_n = \pi_n\circ\beta(I)$ contains the identity of C/J_n. For such an n, $\pi_n\circ\beta$ extends to a unital homomorphism $\gamma\colon M(I) \to B_n \subseteq C/J_n$ which lifts $\sigma\circ\tilde{\varrho}$, and so $\psi = \gamma\circ\omega$ gives a lift of $\sigma\circ\tilde{\varrho}\circ\omega = \sigma$.

The identical statement and proof, restricted to $\mathscr{S}\mathscr{C}_1$, gives [10; III.7.7].

2.38. There is a strong converse to Theorem 2.37 in $\mathscr{S}\mathscr{C}_1$: every compact ANR is a retract of an open set in a compact AR; in fact, a metrizable compact ANR is a retract of an open set in the Hilbert cube [10; III.6.3]. It is not known whether every unital semiprojective C*-algebra is a retract of an ideal in a projective C*-algebra. It may be true that every (separable) unital semiprojective C*-algebra is a retract of an ideal in the unital free product of a countable number of copies of $C([0,1])$. The difficulty in trying to prove this is that the primitive ideal space of such a C*-algebra is not Hausdorff. (In a separable commutative C*-algebra A, every closed subset of Prim (A) is a G_δ.)

3. Properties of semiprojectives.

We retain the notation of section 2. We require that $\mathscr{C}$ be closed under quotients and countable inductive limits, and also that $C \in \mathscr{C} \Rightarrow C \otimes C([0,1]) \in \mathscr{C}$.

We will need to consider inductive limits of the form

$$D = \varinjlim (D_n, \gamma_{n,n+1}),$$

where $\gamma_{n,n+1}\colon D_n \to D_{n+1}$ is *not* assumed to be injective. To construct such an inductive limit, set $I_{nm} = \ker \gamma_{n,m} \subseteq D_n$, where

$$\gamma_{n,m}\colon D_n \to D_m \quad \text{for } m > n .$$

We have $I_{nm} \subseteq I_{n,m+1}$ for all $m > n$, and if $I_n = \overline{\bigcup_m I_{nm}}$, $\gamma_{n,n+1}$ drops to an injective map

$$\bar{\gamma}_{n,n+1} : D_n/I_n \to D_{n+1}/I_{n+1} ,$$

and $D = \varinjlim (D_n/I_n, \bar{\gamma}_{n,n+1})$, where the connecting maps are injective. In fact, the C*-algebras C/J considered in section 2 are actually inductive limits with surjective connecting maps: $C/J = \varinjlim (C/J_n, \pi_{n+1})$. An inductive limit with injective connecting maps will be called a faithful inductive limit. If

$$D = \varinjlim (D_n, \gamma_{n,n+1}),$$

we will denote by γ_n the canonical homomorphism from D_n into D. If D is unital, then D_n is unital for sufficiently large n.

THEOREM 3.1. *Let $\varphi\colon A \to B$ be semiprojective in $\mathscr{C}$, $D = \varinjlim (D_n, \gamma_{n,n+1})$, and $\beta\colon B \to D$ a $\mathscr{C}$-morphism. Then for sufficiently large n, there are homomorphisms $\alpha_n\colon A \to D_n$ such that $\gamma_n \circ \alpha_n \simeq \beta \circ \varphi$, and such that $\gamma_n \circ \alpha_n \to \beta \circ \varphi$ pointwise.*

PROOF. Let C be the C*-subalgebra of $\prod_n C([n, n+1], D_n)$ consisting of all sequences (f_n) for which

$$f_{n+1}(n+1) = \gamma_{n,n+1}(f_n(n+1)) \quad \text{for all } n ,$$

and for which

$$\lim_{\substack{s,t\to\infty \\ s\leqq t}} \|f_n(t)-\gamma_{m,n}(f_m(s))\| = 0, \qquad \text{where } m\leqq s\leqq m+1,\ n\leqq t\leqq n+1,\ m\leqq n\,.$$

(C is Brown's "mapping telescope" [16; 5.2].) Let

$$J_k = \{(f_n)\in C \mid f_n\equiv 0 \text{ for } n>k\}\,,$$

$$J = \{(f_n)\in C \mid \lim \|f_n\|_\infty = 0\} = \overline{\bigcup J_k}\,.$$

Then $C/J\cong D$. Let σ be β, regarded as a map from B to C/J. Lift $\sigma\circ\varphi$ to ψ: $A \to C/J_k$, and let α_n be the composition of ψ with evaluation at n ($n\leqq k$). The homotopy is given by composing ψ with evaluation at t ($t\geqq n$) and then with γ_r ($r\leqq t\leqq r+1$).

COROLLARY 3.2. *If A is semiprojective in $\mathscr{S}_1$, then A is semiprojective in the sense of* [9].

PROOF. The condition in [9] is exactly the conclusion of 3.1 for faithful unital inductive limits. (In [9], no assumption of separability is made, but one may restrict to the image of A without loss of generality.)

THEOREM 3.3. (cf. [9; 3.2]) *Let φ: $A \to B$ be semiprojective in $\mathscr{C}$,*

$$D = \varinjlim\,(D_n, \gamma_{n,n+1})\,,$$

and let β_0, β_1: $B \to D_k$ for some k with $\gamma_k\circ\beta_0 \simeq \gamma_k\circ\beta_1$. Then for sufficiently large $n\geqq k$, $\gamma_{k,n}\circ\beta_0\circ\varphi \simeq \gamma_{k,n}\circ\beta_1\circ\varphi$.

PROOF. Let

$$E = \{(x,f,y)\in D_k\oplus C([0,1],D)\oplus D_k \mid f(0)=\gamma_k(x), f(1)=\gamma_k(y)\}\,,$$

and for $n\geqq k$ let

$$E_n = \{(x,f,y)\in D_k\oplus C([0,1],D_n)\oplus D_k \mid f(0)=\gamma_{k,n}(x), f(1)=\gamma_{k,n}(y)\}\,.$$

Then $E=\varinjlim\,(E_n,\theta_{n,n+1})$ for obvious maps $\theta_{n,n+1}$. If ϱ_t is a path of homomorphisms from B to D with $\varrho_t=\gamma_k\circ\beta_t$ for $t=0,1$, define σ: $B\to E$ by

$$\sigma(x) = (\beta_0(x), f, \beta_1(x))\,,$$

where $f(t)=\varrho_t(x)$. Lift σ to a map α: $A\to E_n$ with $\theta_n\circ\alpha\simeq\sigma\circ\varphi$. Thus, if π_n^0 (respectively π^0) is the projection of E_n (respectively E) onto its first coordinate D_k, we have

$$\delta_0 = \pi_n^0\circ\theta_n\circ\alpha \simeq \pi^0\circ\sigma\circ\varphi = \beta_0\circ\varphi\,.$$

Similarly, if π_n^1 and π^1 are projections onto the third coordinates, we have

$\delta_1 \simeq \beta_1 \circ \varphi$. The map α gives a homotopy from $\gamma_{k,n} \circ \delta_0$ to $\gamma_{k,n} \circ \delta_1$. Thus we have

$$\gamma_{k,n} \circ \beta_0 \circ \varphi \simeq \gamma_{k,n} \circ \delta_0 \simeq \gamma_{k,n} \circ \delta_1 \simeq \gamma_{k,n} \circ \beta_1 \circ \varphi$$

as maps from A to D_n.

Of course, 3.1 and 3.3 may be rephrased in terms of semiprojective algebras by taking $\varphi = \mathrm{id}$.

3.4. Theorem 3.3 suggests that a generalization of 2.32 to allow F to be an arbitrary semiprojective C*-algebra may be possible. Part of the proof of [9; 3.11] carries over to this case—homomorphisms φ_i: $A_i \to C/J$ can be lifted to ψ_i: $A_i \to C/J_n$ such that $\psi_i|F \simeq \psi_j|F$ for all i,j. The problem is then to extend the homotopy to homotopies of the homomorphisms of the A_i.

3.5. In $\mathscr{S}\mathscr{C}_1$, semiprojectives (ANR's) are "locally projective" [10; III.7.1, 7.2, 7.9]. As a result, if X is a compact ANR, there is a finite open cover $\mathscr{U}$ of X such that, whenever φ,ψ: $Y \to X$ are continuous functions which are $\mathscr{U}$-close, then $\varphi \simeq \psi$ [10; IV.1.1]. Since there are simple semiprojective C*-algebras which are not projective, the "local projectivity" result does not carry over to the commutative case. There are several ways to rephrase the homotopy result which make sense for an arbitrary unital C*-algebra A (e.g. considering open covers of the state space of A or writing A as a sum of ideals or left ideals), but I have been unable to prove any direct generalization for semiprojective C*-algebras in $\mathscr{S}_1$.

We do, however, have a weaker version of the homotopy result:

THEOREM 3.6. *If φ: $A \to B$ is semiprojective in $\mathscr{C}$, and β_n, β are $\mathscr{C}$-morphisms from B to D with $\beta_n \to \beta$ pointwise, then for sufficiently large n, $\beta_n \circ \varphi \simeq \beta \circ \varphi$.*

PROOF. Let $C = C([0,1], D)$, $J_k = \{f \in C \mid f(1/n) = 0$ for all n, $f \equiv 0$ on $[0, 1/k]\}$,

$$J = \{f \in C \mid f(1/n) = 0 \text{ for all } n\} = \overline{\bigcup J_k}\,.$$

C/J is isomorphic to the C*-algebra of all convergent sequences of elements of D. Let σ: $B \to C/J$ be defined by $\sigma(x) = (\beta_n(x))$. Lift $\sigma \circ \varphi$ to C/J_k.

4. Shape systems and shape equivalence.

We retain the notation of section 3.

DEFINITION 4.1. Let $A \in \mathscr{C}$. A shape system for A in $\mathscr{C}$ is an inductive system $(A_n, \gamma_{n,n+1})$ in $\mathscr{C}$ with

$$A \cong \varinjlim (A_n, \gamma_{n,n+1})$$

and $\gamma_{n,n+1}\colon A_n \to A_{n+1}$ semiprojective in $\mathscr{C}$. A strong shape system for A is a shape system in which each A_n is semiprojective. A faithful shape system (respectively faithful strong shape system) is a shape system (respectively strong shape system) for which each $\gamma_{n,n+1}$ is injective.

PROPOSITION 4.2. *Let* $(A_n, \gamma_{n,n+1})$ *be a shape system (respectively strong shape system) for* A *in* $\mathscr{S}$ *(respectively* $\mathscr{S}_1$*), and let* $\bar{A}$, $\bar{A}_n$, $\bar{\gamma}_{n,n+1}$ *be the abelianizations of* A, A_n, $\gamma_{n,n+1}$ *respectively. Then* $(\bar{A}_n, \bar{\gamma}_{n,n+1})$ *is a shape system (respectively strong shape system) for* $\bar{A}$ *in* $\mathscr{S}\mathscr{C}$ *(respectively* $\mathscr{S}\mathscr{C}_1$*).*

PROOF. Follows immediately from 2.12.

THEOREM 4.3. *Every (separable)* $\mathscr{C}$**-algebra has a shape system in* $\mathscr{S}$. *A unital* C**-algebra has a shape system in* $\mathscr{S}_1$.

PROOF. Write $A = C^*(\mathscr{G}, \mathscr{R})$ for a countable set of generators $\mathscr{G} = \{x_1, x_2, \ldots\}$ and relations

$$\mathscr{R} = \{(\|p_1(\cdot)\| \leqq \eta_1), (\|p_2(\cdot)\| \leqq \eta_2), \ldots\}$$

as in 1.3(b). Let $\mathscr{G}_n = \{x_1, \ldots, x_n\}$,

$$\mathscr{R}_n = \{(\|p_i(\cdot)\| \leqq \eta_i + 1/n),\ (\|x_i\| \leqq \|x_i\|_A + 1/n) \mid 1 \leqq i \leqq n,$$
$$p_i \text{ involves only } x_1, \ldots, x_n\}.$$

Set $A_n = C^*(\mathscr{G}_n, \mathscr{R}_n)$. There is a natural map $\gamma_{n,n+1}\colon A_n \to A_{n+1}$, since $\mathscr{G}_n \subseteqq \mathscr{G}_{n+1}$ and the relations in $\mathscr{R}_{n+1}$ include stronger forms of all of the relations in $\mathscr{R}_n$. $\gamma_{n,n+1}$ is semiprojective by 2.35, and it is clear that $A \cong \varinjlim (A_n, \gamma_{n,n+1})$.

4.4. It is not clear whether every (separable) C*-algebra has a strong shape system in $\mathscr{S}$ (cf. 2.34.) This is true in $\mathscr{S}\mathscr{C}_1$: every compact Hausdorff space is a projective limit of ANR's (in fact of polyhedra) [4; IX.1.4]. It appears highly unlikely that a general separable C*-algebra has a faithful shape system in $\mathscr{S}$—$C([0,1]^2)$ is probably a counterexample (it is easily seen that $C([0,1]^2)$ has no faithful strong shape system in $\mathscr{S}$.) Even in $\mathscr{S}\mathscr{C}_1$ it is doubtful whether every algebra has a faithful shape system. Thus in the definition of shape systems it seems to be necessary to allow nonfaithful inductive limits to obtain a universally applicable theory.

While Theorem 4.3 shows the existence of shape systems in general, the algebras that arise from the construction are usually rather ugly and difficult to analyze. However, many C*-algebras which arise in practice have natural and very tractable shape systems. For example, AF algebras, inductive limits of algebras of the form $C(S^1)\otimes F$, and such algebras as $\mathcal{O}_\infty$ [6; 3.1] have natural faithful strong shape systems.

QUESTION 4.5. Does every separable nuclear C*-algebra have a shape system (strong shape system) in $\mathcal{S}$ of nuclear C*-algebras?

It is not even clear that commutative C*-algebras have nuclear shape systems.

DEFINITION 4.6. Let $(A_n, \gamma_{n,n+1})$ and $(B_n, \theta_{n,n+1})$ be inductive systems in $\mathcal{C}$. $(A_n, \gamma_{n,n+1}) \sim_{\mathcal{C}} (B_n, \theta_{n,n+1})$ if there are sequences of homomorphisms $\alpha_i\colon A_{k_i} \to B_{n_i}$ and $\beta_i\colon B_{n_i} \to A_{k_{i+1}}$ with $k_i < n_i < k_{i+1}$, such that

$$\beta_i \circ \alpha_i \simeq \gamma_{k_i, k_{i+1}} \quad \text{and} \quad \alpha_{i+1} \circ \beta_i \simeq \theta_{k_i, k_{i+1}} \quad \text{for each } i\,.$$

If we have such α_i and β_i with $\beta_i \circ \alpha_i \simeq \gamma_{k_i, k_{i+1}}$, we say $(A_n, \gamma_{n,n+1}) \precsim_{\mathcal{C}} (B_n, \theta_{n,n+1})$.

It is clear that $\precsim_{\mathcal{C}}$ is transitive and that $\sim_{\mathcal{C}}$ is an equivalence relation. If

$$(A_n, \gamma_{n,n+1}) \sim_{\mathcal{S}} (B_n, \theta_{n,n+1})$$

then $(A_n, \gamma_{n,n+1}) \precsim_{\mathcal{C}} (B_n, \theta_{n,n+1})$ and $(B_n, \theta_{n,n+1}) \precsim_{\mathcal{C}} (A_n, \gamma_{n,n+1})$, but the converse is not true.

PROPOSITION 4.7. *Let $(A_n, \gamma_{n,n+1})$ and $(B_n, \theta_{n,n+1})$ be inductive systems of* C*-*algebras, with abelianizations $(\bar{A}_n, \bar{\gamma}_{n,n+1})$ and $(\bar{B}_n, \bar{\theta}_{n,n+1})$. If*

$$(A_n, \gamma^{n,n+1}) \sim_{\mathcal{S}} (B_n, \theta_{n,n+1})$$

(respectively $(A_n, \gamma_{n,n+1}) \precsim_{\mathcal{S}} (B_n, \theta_{n,n+1})$), then

$$(\bar{A}_n, \bar{\gamma}_{n,n+1}) \sim_{\mathcal{S}\mathcal{C}} (\bar{B}_n, \bar{\theta}_{n,n+1})$$

(respectively $(\bar{A}_n, \bar{\gamma}_{n,n+1}) \precsim_{\mathcal{S}\mathcal{C}} (\bar{B}_n, \bar{\theta}_{n,n+1})$). A similar statement holds for $\mathcal{S}_1$ and $\mathcal{S}\mathcal{C}_1$.

PROOF. Abelianize the maps α_i and β_i and the homotopies.

The following may be regarded as the fundamental theorem in the shape theory of C*-algebras.

THEOREM 4.8. *Let A and B be* C*-*algebras in $\mathcal{C}$ with shape systems $(A_n, \gamma_{n,n+1})$ and $(B_n, \theta_{n,n+1})$ respectively* (in $\mathcal{C}$). *If there exist inductive systems $(C_n, \omega_{n,n+1})$*

and $(D_n, \delta_{n,n+1})$ *in* $\mathscr{C}$ *(not necessarily shape system) which are equivalent in* $\mathscr{C}$ *and with*

$$A \cong \varinjlim (C_n, \omega_{n,n+1}) \quad \textit{and} \quad B \cong \varinjlim (B_n, \delta_{n,n+1}),$$

then $(A_n, \gamma_{n,n+1}) \sim_{\mathscr{C}} (B_n, \theta_{n,n+1})$. *An analogous statement holds if* $(C_n) \precsim_{\mathscr{C}} (D_n)$.

PROOF. Suppose we have $\varrho_j: C_{p_j} \to D_{q_j}$ and $\sigma_j: D_{q_j} \to C_{p_{j+1}}$ with $p_j < q_j < p_{j+1}$ and $\sigma_j \circ \varrho_j \simeq \omega_{p_j, p_{j+1}}$, $\varrho_{j+1} \circ \sigma_j \cong \delta_{q_j, q_{j+1}}$. We will construct the maps α_i and β_i inductively. Suppose $\alpha_1, \beta_1, \ldots, \alpha_{r-1}, \beta_{r-1}$ have been chosen, with the following properties:

(1) $\beta_{r-1}: B_{n_{r-1}} \to A_{k_r}$ satisfies $\beta_{r-1} = \tilde{\beta} \circ \theta_{n_{r-1}, n_{r-1}+2}$ for some $\tilde{\beta}: B_{n_{r-1}+2} \to A_{k_r}$;

(2) there are numbers q_{j-1} and p_j with $n_{r-1} + 2 < q_{j-1} < p_j < k_r$;

(3) identifying A with $\varinjlim A_n$ and with $\varinjlim C_n$, there is a map

$$\xi: B_{n_{r-1}+2} \to D_{q_{j-1}}$$

such that $\gamma_{k_r} \circ \tilde{\beta} \simeq \omega_{p_j} \circ \sigma_{j-1} \circ \xi$ as maps from $B_{n_{r-1}+2}$ to A, and $\delta_{q_{j-1}} \circ \xi \simeq \theta_{n_{r-1}+2}$ as maps from $B_{n_{r-1}+2}$ to

$$B = \varinjlim B_n = \varinjlim D_n .$$

If $r=1$ we take the conditions to be vacuous. We will construct α_r with analogous properties to (1)–(3), such that $\alpha_r \circ \beta_{r-1} \simeq \theta_{n_{r-1}, n_r}$. The construction can then be repeated inductively to yield the equivalence. The map α_r is constructed in several steps. First, regarding

$$\gamma_{k_r+3} = \gamma_{k_r+4} \circ \gamma_{k_r+3, k_r+4}: A_{k_r+3} \to A_{k_r+4} \to A$$

as a map into $\varinjlim C_n$, by semiprojectivity of γ_{k_r+3, k_r+4} and Theorem 3.1 there is a map $\psi: A_{k_r+3} \to C_{p_s}$ for sufficiently large s with $\omega_{p_s} \circ \psi \simeq \gamma_{k_r+3}$. Then

$$\omega_{p_s} \circ \psi \circ \gamma_{k_r, k_r+3} \circ \tilde{\beta} \simeq \gamma_{k_r} \circ \tilde{\beta} \simeq \omega_{p_s} \circ \omega_{p_j, p_s} \circ \sigma_{j-1} \circ \xi$$

as maps from $B_{n_{r-1}+2}$ to $A = \varinjlim C_n$; so by Theorem 3.3, the semiprojectivity of $\theta_{n_{r-1}+1, n_{r-1}+2}$ implies that by increasing s we may obtain

$$f = \psi \circ \gamma_{k_r, k_r+3} \circ \tilde{\beta} \circ \theta_{n_{r-1}+1, n_{r-1}+2} \simeq \omega_{p_j, p_s} \circ \sigma_{j-1} \circ \xi \circ \theta_{n_{r-1}+1, n_{r-1}+2} = g .$$

Now regard

$$h = \delta_{q_s} \circ \varrho_s \circ \psi \circ \gamma_{k_r+3}: A_{k_r+2} \to A_{k_r+3} \to B = \varinjlim B_n .$$

By semiprojectivity of γ_{k_r+2, k_r+3} and 3.1 there is a map $\tilde{\alpha}: A_{k_r+2} \to B_l$ for sufficiently large $l > q_s$ with $\theta_l \circ \tilde{\alpha} \simeq h$. So we have

$$\theta_l \circ \tilde{\alpha} \circ \gamma_{k_r, k_r+2} \circ \tilde{\beta} \circ \theta_{n_{r-1}+1, n_{r-1}+2} \simeq \delta_{q_s} \circ \varrho_s \circ f \simeq \delta_{q_s} \circ \varrho_s \circ g \simeq \delta_{q_{j-1}} \circ \xi \circ \theta_{n_{r-1}+1, n_{r-1}+2}$$

(since $\delta_{q_s} \circ \varrho_s \circ \omega_{p_j, p_s} \circ \sigma_{j-1} \simeq \delta_{q_{j-1}}$ by assumption) $\simeq \theta_{n_{r-1}+1}$. Again by 3.3 we can increase l so that $\alpha_r \circ \beta_{r-1} \simeq \theta_{n_{r-1}, n_r}$, where $k_r = l$ and $\alpha_r = \tilde{\alpha} \circ \gamma_{k_r, k_r+2}$. For the next stage in the induction, the analog of ξ is $\psi \circ \gamma_{k_r+2, k_r+3}$. The construction in the proof is summarized in the diagram of Fig. 1.

COROLLARY 4.9. *Any two shape systems for a C*-algebra in $\mathscr{C}$ are equivalent.*

DEFINITION 4.10. A and B have the same shape in $\mathscr{C}$, or are shape equivalent in $\mathscr{C}$, written $\mathrm{Sh}_{\mathscr{C}}(A) = \mathrm{Sh}_{\mathscr{C}}(B)$, if $(A_n, \gamma_{n,n+1}) \sim_{\mathscr{C}} (B_n, \theta_{n,n+1})$ for some (hence any) shape systems for A and B in $\mathscr{C}$. The shape of B dominates the shape of A, written $\mathrm{Sh}_{\mathscr{C}}(A) \leqq \mathrm{Sh}_{\mathscr{C}}(B)$, if $(A_n, \gamma_{n,n+1}) \precsim_{\mathscr{C}} (B_n, \theta_{n,n+1})$.

By 4.8, $\mathrm{Sh}_{\mathscr{C}}(A) = \mathrm{Sh}_{\mathscr{C}}(B)$ if and only if A and B have equivalent inductive systems in $\mathscr{C}$. If $\mathrm{Sh}_{\mathscr{C}}(A) = \mathrm{Sh}_{\mathscr{C}}(B)$, then $\mathrm{Sh}_{\mathscr{C}'}(A) = \mathrm{Sh}_{\mathscr{C}'}(B)$ for any category $\mathscr{C}' \supseteq \mathscr{C}$.

This definition agrees with the topological definition: if X and Y are compact metrizable spaces, then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ (respectively $\mathrm{Sh}(X) \leqq \mathrm{Sh}(Y)$) if and only if

$$\mathrm{Sh}_{\mathscr{S}\mathscr{C}_1}(C(X)) = \mathrm{Sh}_{\mathscr{S}\mathscr{C}_1}(C(Y))$$

(respectively $\mathrm{Sh}_{\mathscr{S}\mathscr{C}_1}(C(X)) \leqq \mathrm{Sh}_{\mathscr{S}\mathscr{C}_1}(C(Y))$). There are spaces X and Y for which $\mathrm{Sh}(X) \leqq \mathrm{Sh}(Y)$ and $\mathrm{Sh}(Y) \leqq \mathrm{Sh}(X)$ but $\mathrm{Sh}(X) \neq \mathrm{Sh}(Y)$. In fact, the spaces $X(t)$ constructed in [4; VII.8.2] satisfy

$$\mathrm{Sh}(X(t)) \leqq \mathrm{Sh}(X(t')) \quad \text{for all } t, t',$$

since $X(t)$ is homeomorphic to a retract of $X(t')$ even when $t \nleqq t'$; but

$$\mathrm{Sh}(X(t)) \neq \mathrm{Sh}(X(t')) \quad \text{for } t \neq t'.$$

If A is unital, then there is no distinction between $\mathrm{Sh}_{\mathscr{S}}(A)$ and $\mathrm{Sh}_{\mathscr{S}_1}(A)$ (except formally).

We now obtain some more corollaries of Theorem 4.8.

COROLLARY 4.11. *If A and B are homotopy equivalent in $\mathscr{C}$* [9; § 2], *then* $\mathrm{Sh}_{\mathscr{C}}(A) = \mathrm{Sh}_{\mathscr{C}}(B)$. *If B homotopy dominates A, then* $\mathrm{Sh}_{\mathscr{C}}(A) \leqq \mathrm{Sh}_{\mathscr{C}}(B)$.

PROOF. A homotopy equivalence between A and B induces an equivalence between the systems (A, id_A) and (B, id_B).

Of course, the converse is not generally true, as shown by the circle and the "Warsaw circle" [9; § 5]. So shape equivalence is a strictly weaker notion than homotopy equivalence.

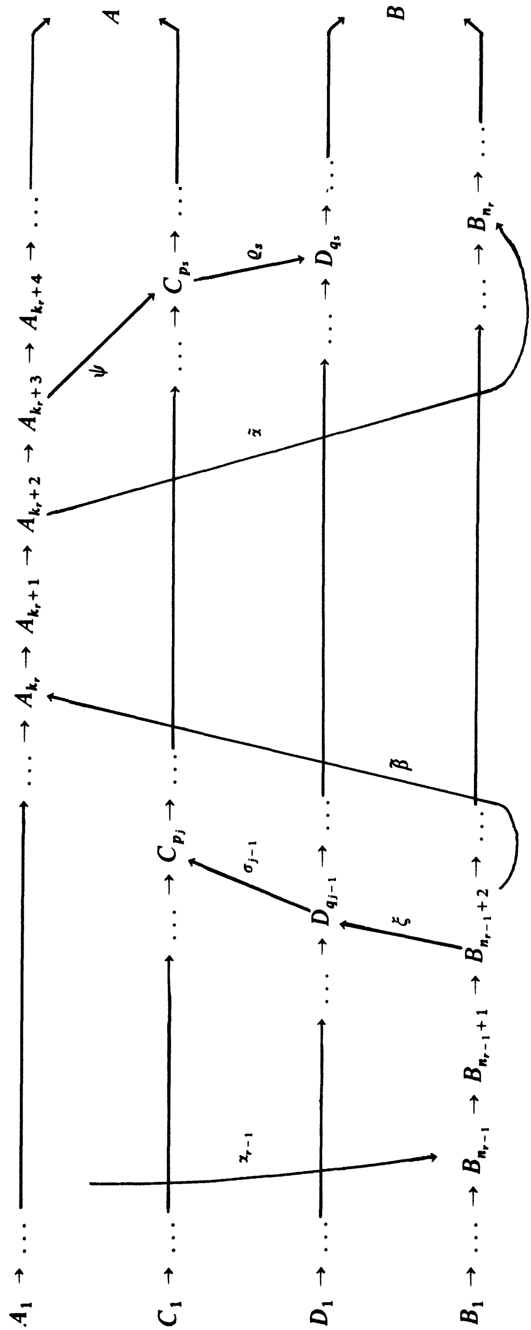

A
B
$A_1 \to \cdots \to A_{k_r} \to A_{k_r+1} \to A_{k_r+2} \to A_{k_r+3} \to A_{k_r+4} \to \cdots$
$C_1 \to \cdots \to C_{p_j} \to \cdots \to C_{p_s} \to \cdots$
$D_1 \to \cdots \to D_{q_{j-1}} \to \cdots \to D_{q_s} \to \cdots$
$B_1 \to \cdots \to B_{n_{r-1}} \to B_{n_{r-1}+1} \to B_{n_{r-1}+2} \to \cdots \to B_{n_r} \to \cdots$
α_{r-1}
σ_{j-1}
ξ
$\bar{\beta}$
$\bar{\alpha}$
ψ
ϱ_s

Figure 1.

COROLLARY 4.12. *If A and B have faithful inductive systems which are semiprojective in the sense of* [9], *and A and B have the same shape in the sense of* [9], *then* $\mathrm{Sh}_{\mathcal{S}_1}(A)=\mathrm{Sh}_{\mathcal{S}_1}(B)$.

The converse is not clear, since it is not obvious that such inductive systems are shape systems in our sense (and are in fact probably not in general.) However, among unital C*-algebras which have faithful strong shape systems, such as inductive limits of algebras of the form $C(S^1)\otimes F$, the two notions of shape equivalence coincide.

COROLLARY 4.13. *If X and Y are locally compact metrizable spaces, then*

$$\mathrm{Sh}_{\mathcal{S}}(C_0(X)) = \mathrm{Sh}_{\mathcal{S}}(C_0(Y))$$

if and only if $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$.

PROOF. One direction follows from 4.8, and the other from 4.2, 4.7, and the remarks after 4.10.

COROLLARY 4.14. *Let A, B, C, D be (separable)* C**-algebras with* $\mathrm{Sh}_{\mathcal{S}}(A) =\mathrm{Sh}_{\mathcal{S}}(C)$ *and* $\mathrm{Sh}_{\mathcal{S}}(B)=\mathrm{Sh}_{\mathcal{S}}(D)$. *Then*

$$\mathrm{Sh}_{\mathcal{S}}(A\otimes_{\max}B) = \mathrm{Sh}_{\mathcal{S}}(C\otimes_{\max}D),$$

$$\mathrm{Sh}_{\mathcal{S}}(A\otimes_{\min}B) = \mathrm{Sh}_{\mathcal{S}}(C\otimes_{\min}D),$$

and

$$\mathrm{Sh}_{\mathcal{S}}(A*B) = \mathrm{Sh}_{\mathcal{S}}(C*D);$$

if the algebras are unital, then

$$\mathrm{Sh}_{\mathcal{S}}(A*_{\mathrm{C}}B)=\mathrm{Sh}_{\mathcal{S}}(C*_{\mathrm{C}}D).$$

PROOF. Let $(A_n,\gamma_{n,n+1})$, $(B_n,\theta_{n,n+1})$, $(C_n,\omega_{n,n+1})$, $(D_n,\delta_{n,n+1})$ be shape systems for A,B,C,D. Then $(A_n\otimes_{\max}B_n,\gamma_{n,n+1}\otimes_{\max}\theta_{n,n+1})$ and $(C_n\otimes_{\max}D_n, \omega_{n,n+1}\otimes_{\max}\delta_{n,n+1})$ are equivalent systems for $A\otimes_{\max}B$ and $C\otimes_{\max}D$. The other parts are similar.

Finally, as in [9], if A and B are AF algebras, then $\mathrm{Sh}_{\mathcal{S}}(A)=\mathrm{Sh}_{\mathcal{S}}(B)$ if and only if $A\cong B$.

5. Shape theory and K-theory.

One of the main features of topological shape theory is that standard cohomology and K-theory are shape invariants. We prove analogous results for the non-commutative case.

If A is a C*-algebra, we denote by $H(A)$ the semigroup of equivalence classes of projections in $A \otimes \mathscr{K}$, with orthogonal addition, and $K_0(A)$ and $K_1(A)$ the K-groups of A. There is a canonical homomorphism from $H(A)$ into $K_0(A)$; denote the image by $K_0(A)_+$. $(K_0(A), K_0(A)_+)$ is a "pre-ordered" group; in general, it fails to have the two properties necessary for an ordering, namely

(1) $K_0(A)_+ - K_0(A)_+ = K_0(A)$ and
(2) $K_0(A)_+ \cap (-K_0(A)_+) = \{0\}$.

If $A \otimes \mathscr{K}$ has an approximate identity of projections (i.e. A is "stably unital"), then $K_0(A)$ can be identified with the Grothendieck group of $H(A)$, so $(K_0(A), K_0(A)_+)$ satisfies (1). If, in addition, $A \otimes \mathscr{K}$ contains no infinite projections (i.e. A is "stably finite"), then $(K_0(A), K_0(A)_+)$ satisfies (2) also. Thus if A is stably unital and stably finite, then $(K_0(A), K_0(A)_+)$ is an ordered group.

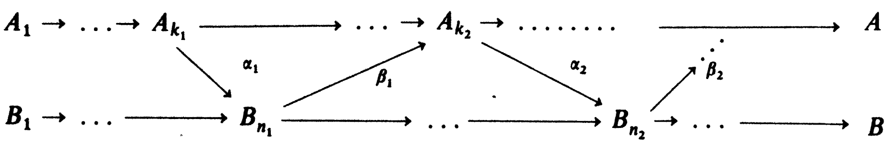
We denote by $\Sigma(A)$ the subset of $H(A)$ (or its image in $K_0(A)$) corresponding to the projections of A. $\Sigma(A)$ is called the scale of A. Even if A is simple, stably unital, and stably finite, we can have $\Sigma(A) = \{0\}$ [3; 5.1(3)]. $\Sigma(A)$ does not in general generate $K_0(A)$ even if A is simple, unital, and stably finite [3; 4.12]. $(H(A), \Sigma(A))$ is called the scaled semigroup of A, and $(K_0(A), K_0(A)_+, \Sigma(A))$ is called the scaled pre-ordered K_0 group of A.

PROPOSITION 5.1. *If* $A = \varinjlim (A_n, \gamma_{n,n+1})$, *then* $(H(A), \Sigma(A))$ *is the algebraic direct limit of* $((H(A_n), \Sigma(A_n)), (\gamma_{n,n+1})_*)$ *in the obvious sense, and similarly for* $K_0(A)$ *and* $K_1(A)$.

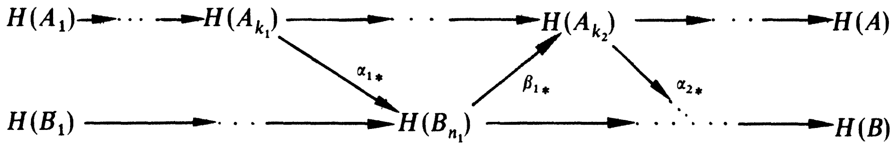
PROOF. This follows in a manner similar to the case of faithful inductive limits, but one needs to use 2.14 and 2.23 to handle the noninjectivity of the connecting maps. Details are left to the reader.

PROPOSITION 5.2. *If* $\mathrm{Sh}_{\mathscr{S}}(A) = \mathrm{Sh}_{\mathscr{S}}(B)$, *then* $(H(A), \Sigma(A)) \cong (H(B), \Sigma(B))$ *as scaled semigroups,* $(K_0(A), K_0(A)_+, \Sigma(A)) \cong (K_0(B), K_0(B)_+, \Sigma(B))$ *as scaled pre-ordered groups, and* $K_1(A) \cong K_1(B)$. *If* $\mathrm{Sh}_{\mathscr{S}}(A) \leqq \mathrm{Sh}_{\mathscr{S}}(B)$, *then* $H(A)$, $K_0(A)$, $K_1(A)$ *are direct summands of* $H(B)$, $K_0(B)$, $K_1(B)$, *with the induced order and scale.*

PROOF. Let $(A_n, \gamma_{n,n+1})$ and $(B_n, \theta_{n,n+1})$ be shape systems for A and B. An equivalence between the systems gives a diagram



where the triangles commute up to homotopy. This diagram induces diagrams



(and similarly for K_0 and K_1) which actually commute, so one obtains an isomorphism between the direct limits. All the homomorphisms preserve order and scale, and the order and scale of the direct limits is the direct limit order and scale. In the case that $\mathrm{Sh}_{\mathscr{S}}(A) \leqq \mathrm{Sh}_{\mathscr{S}}(B)$, only the odd triangles commute, so one obtains scaled homomorphisms $\alpha_*: H(A) \to H(B)$ and $\beta_*: H(B) \to H(A)$ with $\beta_* \circ \alpha_* = \mathrm{id}_{H(A)}$, and similarly for K_0 and K_1.

COROLLARY 5.3. *If* $\mathrm{Sh}_{\mathscr{S}}(A) = \mathrm{Sh}_{\mathscr{S}}(B)$ *and* $\mathrm{Sh}_{\mathscr{S}}(C) = \mathrm{Sh}_{\mathscr{S}}(D)$, *then*

$$K_0(A \otimes_{\max} B) \cong K_0(C \otimes_{\max} D)\,,$$

$$K_0(A \otimes_{\min} B) \cong K_0(C \otimes_{\min} D)\,,$$

$$K_0(A * B) \cong K_0(C * D)$$

as scaled preordered groups. If A and B are stably shape equivalent (i.e. $\mathrm{Sh}_{\mathscr{S}}(A \otimes \mathscr{K}) = \mathrm{Sh}_{\mathscr{S}}(B \otimes \mathscr{K})$*), then* $K_0(A) \cong K_0(B)$ *as preordered groups.*

In connection with the order structure on K_0, the following proposition is of interest.

PROPOSITION 5.4. *Let A and B be stably unital* C*-*algebras, with* $\mathrm{Sh}_{\mathscr{S}}(A \otimes \mathscr{K}) \leqq \mathrm{Sh}_{\mathscr{S}}(B \otimes \mathscr{K})$. *If B is stably finite, then A is stably finite.*

PROOF. Let $(A_n, \gamma_{n,n+1})$ and $(B_n, \theta_{n,n+1})$ be shape systems for $A \otimes \mathscr{K}$ and $B \otimes \mathscr{K}$ respectively. We may assume A_n and B_n are unital for each n (although the connecting maps will not be unital.) Then

$$(A \otimes \mathscr{K})^1 = \varinjlim (A_n^1, \gamma_{n,n+1}^1)$$

and

$$(B \otimes \mathscr{K})^1 = \varinjlim (B_n^1, \theta_{n,n+1}^1)$$

with unital connecting maps, and $(A_n^1, \gamma_{n,n+1}^1) \precsim_{\mathscr{S}_1} (B_n^1, \theta_{n,n+1}^1)$. Let α_i and β_i be the maps of 4.6. By assumption, $B \otimes \mathscr{K}$ contains no infinite projections; hence $\theta_n(B_n)$ contains no infinite projections; the same is true for

$$\theta_n^1(B_n^1) \cong \theta_n(B_n)\oplus \mathbf{C},$$

and hence also for

$$(B\otimes\mathscr{K})^1 \cong \varinjlim \theta_n^1(B_n^1).$$

Thus $(B\otimes\mathscr{K})^1$ contains no nonunitary isometries. Note that a unital inductive limit

$$D = \varinjlim (D_n, \delta_{n,n+1})$$

contains no nonunitary isometries if and only if, for any k and any isometry $u \in D_k$, there is an $n>k$ such that $\delta_{k,n}(u)$ is unitary in D_n. If u is an isometry in A_m^1, choose i with $k_i>m$; then $\alpha_i \circ \gamma_{m,k_i}^1(u)$ is an isometry v in $B_{n_i}^1$, so for sufficiently large j we have $\theta_{n_i,n_j}^1(v)$ unitary in $B_{n_j}^1$. Then $\beta_j \circ \theta_{n_i,n_j}^1(v)$ is a unitary in $A_{k_{j+1}}^1$ which is connected by a path of isometries to $\gamma_{m,k_{j+1}}^1(u)$. This implies that $\gamma_{m,k_{j+1}}^1(u)$ is unitary, so $(A\otimes\mathscr{K})^1$ contains no nonunitary isometries.

5.5. It follows easily from the results of [15] that if A and B are stably shape equivalent. and if they have shape systems in a suitably nice class of C*-algebras, then

$$KK(A\otimes C, D) \cong KK(B\otimes C, D) \quad \text{and} \quad KK(C, A\otimes D) \cong KK(C, B\otimes D)$$

for all suitably nice C*-algebras C and D. It would be much more satisfactory to explicitly construct an invertible element in $KK(A,B)$, and it should be possible to do so even in cases where the results of [15] do not apply. However, even when A and B are AF or when $A=C(WS^1)$ (the Warsaw circle) and $B=C(S^1)$ it is difficult to write down an explicit invertible element of $KK(A,B)$.

Note that Kasparov equivalence (existence of an invertible element in $KK(A,B)$) is much weaker than stable shape equivalence. For example, if A and B are AF algebras, then $KK(A,B)\cong \mathrm{Hom}\,(K_0(A), K_0(B))$, and the intersection product exactly corresponds to composition of homomorphisms, so A and B are Kasparov equivalent if and only if $K_0(A)\cong K_0(B)$ as groups (ignoring the order structure completely). But A and B are stably shape equivalent if and only if they are stably isomorphic. It appears to be possible to build some kind of order structure into the Kasparov groups, which would be preserved under shape equivalence.

Questions 5.6. (a) Is the stable shape of a C*-algebra A completely determined by the K-theory or Kasparov theory of A and other C*-algebras constructed in standard ways from A, including the order structure? In particular, If A and B are (separable) C*-algebras and $K_0(A\otimes D)\cong K_0(B\otimes D)$ as preordered groups for all D, is $\mathrm{Sh}_{\mathscr{S}}(A\otimes\mathscr{K})=\mathrm{Sh}_{\mathscr{S}}(B\otimes\mathscr{K})$?

(b) Is there a noncommutative analog of the generalized Whitehead theorem of topological shape theory? The appropriate notion of "dimension" for a C*-algebra might be Rieffel's topological stable rank [14].

I plan to investigate some of these questions in a future paper.

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