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LOEB COMPLETION OF INTERNAL VECTOR-VALUED MEASURES

RADE T. ŽIVALJEVIĆ

0. Introduction.

Peter Loeb introduced in [8] a construction which to any internal measure space $(X, \mathscr{A}, \mu)$ with finitely additive measure μ associates a standard space $(X, \sigma(\mathscr{A}), \tilde{\mu})$, where $\sigma(\mathscr{A})$ is the σ-algebra generated by $\mathscr{A}$ and $\tilde{\mu}$ the unique σ-additive extension of $\mathrm{st}(\mu)$ on $\sigma(\mathscr{A})$. The completion of this space $(X, L(\mathscr{A}), L(\mu))$ is called the Loeb space associated with $(X, \mathscr{A}, \mu)$. This construction turned out to be very useful so, as C. W. Henson and L. C. Moore, Jr. suggest in their paper [5 (Problem 17)], it is potentially useful to extend this construction to the case of measures μ with values in internal Banach spaces. We shall give in this paper a few results in this direction and it will turn out, as it is natural to expect, that the Loeb completion of an internal Banach space valued measure takes values in the nonstandard hull of the Banach space (the last notion has been defined by W. A. J. Luxemburg in [10] and this definition is repeated below). Horst Osswald in his forthcoming paper [11] independently treated the same problem, but from a somewhat different point of view. He starts with a hyperfinite "weighting" function with values in ${}^*B$, where B is a reflexive Banach space, and ends up with a measure on B, Having in mind the continuous, linear map $p\colon \hat{B} \to B$ which sends $[x] \in \hat{B}$ to $\mathrm{st}_w(x) \in B$, i.e. the map induced by the weak standard part map, it can be checked that the B-valued measure, defined by H. Osswald, can be obtained from the Loeb completion of the ${}^*B$-valued internal measure which is determined by the "weighting" function above. Let us note that a natural projection $q\colon \hat{B} \to B$ can be also defined for some non-reflexive Banach spaces, e.g. for $C(K)$ the space of continuous real functions on a compact K. Indeed by the result of W. Henson [4], $C(K)\hat{}$ can be identified with $C(\hat{K})$ for some compact superspace $\hat{K}$ of K and q is simply the restriction map.

1. The main construction.

The general reference for all definitions and facts concerning vector-valued measures will be N. Dunford, J. T. Schwartz [3]. Facts about Loeb completion

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of spaces with positive measure μ can be found in Loeb's original paper [8], K. Stroyan's and J. Bayod's book [12] or elsewhere. Nevertheless, for the reader's convenience, we shall repeat the main definition.

DEFINITION 1.1. Let $(X, \mathscr{A}, \mu)$ be an internal measure space with internal, finitely-additive, finite measure μ. A subset $A \subset X$ is Loeb measurable if for every standard $\varepsilon > 0$ there exist $B, C \in \mathscr{A}$ such that $B \subset A \subset C$ and $\mathrm{st}(\mu((C \setminus B) < \varepsilon$. The family of all Loeb measurable sets is denoted by $L(\mathscr{A})$, whereas $L(\mu)$ denotes the natural extension of $\tilde{\mu} = \mathrm{st}\,\mu$ on $L(\mathscr{A})$.

PROPOSITION 1.2. *$(X, L(\mathscr{A}), L(\mu))$ is a complete measure space with the σ-additive measure $L(\mu)$.*

PROOF. See [8] or [12].

Let B be a Banach space and $(X, \mathscr{A}, \mu, {}^*B)$ an internal space with finitely additive ${}^*B$-valued measure μ. Let us suppose that the total variation $v(\mu, \cdot)$ (Dunford, Schwartz [3]), defined by

$$v(\mu, A) := {}^*\sup_{\mathscr{P}} \sum \{{}^*\|\mu(D)\| \mid D \in \mathscr{P}\}$$

where $\mathscr{P}$ ranges over set of all *-finite, $\mathscr{A}$-measurable partitions of X, is a finite internal positive measure on X. Recall that an element $x \in {}^*B$ (particularly $x \in {}^*\mathrm{R}$) is finite ($x \in \mathrm{fin}\,({}^*B)$) if ${}^*\|x\| \leq m$ for some $m \in \mathrm{R}$. The measure $|\mu| := \mathrm{var}\,(\mu, \cdot)$ is positive, hence let $(X, L(\mathscr{A}), L(|\mu|))$ be the completion of $(X, \mathscr{A}, |\mu|)$. Our aim is to define Loeb completion $(X, L(\mathscr{A}), L(\mu), ?)$ of the measure space $(X, \mathscr{A}, \mu, {}^*B)$. The natural candidate for ? is of course $\hat{B}$, the nonstandard hull of the Banach space B. Recall that

$$\hat{B} := \mathrm{fin}\,({}^*B)/\approx\ ,$$

where $x \approx y$ means that $x - y$ is of infinitesimal norm (see W. A. J. Luxemburg [10] or C. W. Henson, L. C. Moore, Jr. [5]). To simplify notation, $\|x\|$ will denote the norm of x for both $x \in B$ and $x \in \hat{B}$. Analogously to the case $B = \mathrm{R}$, the quotient map $\mathrm{fin}\,({}^*B) \to \hat{B}$ will be called the standard part map and denoted by st. We need the following well-known proposition.

PROPOSITION 1.3. *Let $(X, L(\mathscr{A}), L(\mu))$ be the Loeb completion of the space $(X, \mathscr{A}, \mu)$ with finitely-additive, positive, finite measure μ. Then, if $A \in L(\mathscr{A})$, then there exists $B \in \mathscr{A}$ such that $L(\mu)(A \triangle B) = 0$.*

PROOF. By the Definition 1.1, there exist sequences $(B_n \mid n \in \mathrm{N})$ and $(C_n \mid n \in \mathrm{N})$ of sets in $\mathscr{A}$ such that $\mathrm{st}\,(\mu)(C_n \setminus B_n) < 1/n$ and

$$B_0 \subset B_1 \subset \ldots \subset A \subset \ldots \subset C_1 \subset C_0 .$$

Using saturation, these sequences can be extended to internal sequences and then using the Internal Definition Principle we can select $\omega \in {}^*\mathbf{N} \setminus \mathbf{N}$ such that $B_\omega, C_\omega \in \mathscr{A}$, $B_\omega \subset C_\omega$ and $(\forall i \leqq \omega)$ $(B_i \subset B_\omega \wedge C_\omega \subset C_i)$. Hence,

$$A \triangle B_\omega = (A \setminus B_\omega) \cup (B_\omega \setminus A) \subset (C_n \setminus B_\omega) \cup (B_\omega \setminus B_n) \subset C_n \setminus B_n ,$$

for all $n \in \mathbf{N}$, and $L(\mu)(A \triangle B_\omega) = 0$.

Applying the last proposition to the measure space $(X, \mathscr{A}, |\mu|)$, where

$$|\mu| := \operatorname{var}(\mu, \cdot) ,$$

we get that $A \in L(\mathscr{A})$, iff there exists $B \in \mathscr{A}$ such that $L(|\mu|)(A \triangle B) = 0$. Also, if $B_1, B_2 \in \mathscr{A}$ are two sets which, in this sence, approximate A, then

$$\operatorname{st} |\mu|(B_1 \triangle B_2) \leqq L(|\mu|)(A \triangle B_1) + L(|\mu|)(A \triangle B_2) = 0 ,$$

which means that ${}^*\|\mu(B_1) - \mu(B_2)\| \in m(0)$, because

$${}^*\|\mu(B_1) - \mu(B_2)\| \leqq |\mu|(B_1 \triangle B_2) .$$

This allows us to give the following definition.

Definition 1.4. Let $(X, \mathscr{A}, \mu, {}^*B)$ be an internal measure space with a finitely-additive ${}^*B$-valued measure μ so that the total variation $|\mu| := \operatorname{var}(\mu, \cdot)$ is finite. Let $(X, L(\mathscr{A}), L(|\mu|))$ be the Loeb space associated with the internal space $(X, \mathscr{A}, |\mu|)$. For $A \in L(\mathscr{A})$ and $B \in \mathscr{A}$ with the property $L(|\mu|)(A \triangle B) = 0$, let

$$L(\mu)(A) := \operatorname{st}(\mu(B)) ,$$

where

$$\operatorname{st} : \operatorname{fin}({}^*B) \to \hat{B}$$

is the mapping defined above.

$L(\mu)$ is obviously an additive $\hat{B}$-valued measure. To prove its σ-additivity we need the following simple inequality.

Lemma 1.5. *For all* $A \in L(\mathscr{A})$ *holds* $\|L(\mu)(A)\| \leqq L(|\mu|)(A)$.

Proof. Let $B \in \mathscr{A}$ approximate A, that is $L(|\mu|)(A \triangle B) = 0$. Then

$$\|L(\mu)(A)\| = \|\operatorname{st} \mu(B)\| = \operatorname{st}{}^* \|\mu(B)\| \leqq \operatorname{st} |\mu|(B) \leqq L(|\mu|)(A) .$$

Proposition 1.6. *$L(\mu)$ is a σ-additive measure.*

PROOF. Let $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ be an increasing sequence of sets in $L(\mathscr{A})$ and $A = \bigcup_{k=1}^{\infty} A_k$. Let us show that

$$L(\mu)(A_n) \to L(\mu)(A)$$

i.e.

$$\|L(\mu)(A) - L(\mu)(A_n)\| \to 0, \quad n \to \infty .$$

Indeed,

$$\|L(\mu)(A \setminus A_n)\| \leqq L(|\mu|)(A \setminus A_n) \to 0, \quad n \to \infty ,$$

because $L(|\mu|)$ is a σ-additive measure.

We are ready now to give the first application of above defined notions. The well known result of Liapounoff states that the range of any R^n-valued σ-additive nonatomic measure is compact and convex. The following proposition is attributed to D. Brown in the paper of T. Armstrong and K. Prikry [2]. In this paper T. Armstrong and K. Prikry establish the fact that the range of a bounded finitely additive nonatomic measure is not only dense in a convex set but actually convex.

PROPOSITIOM 1.7. *Let $(X, \mathscr{B}, v, \mathsf{R}^n)$ be a standard space with a finitely additive R^n-valued, nonatomic bounded measure v. Then the range of v, $v(\mathscr{B})$, is dense in some compact, convex set in R^n.*

PROOF. Let us show first that the total variation $|v|$ of v is also a bounded measure. It is enough to show that the variation $|l \circ v|$ of $l \circ v$, where $l\colon \mathsf{R}^n \to \mathsf{R}$ is any linear function, is bounded. This is clear because if $\|v(C)\| \leqq m$ for all $C \in \mathscr{B}$, then

$$|l \circ v|(A) = \sup \{|l \circ v(B)| + |l \circ v(A \setminus B)| : B \subset A, B \in \mathscr{B}\} \leqq 2m \cdot \|l\| .$$

By The Transfer Principle $|{}^*v| = {}^*|v|$ hence, the total variation of ${}^*v$ is also finite. Let us consider the measure space $({}^*X, {}^*\mathscr{B}, {}^*v, {}^*\mathsf{R}^n)$. Since

$$\mathrm{st}\,({}^*v({}^*\mathscr{B})) = \mathrm{st}\,({}^*v(\mathscr{B})) = \mathrm{cl}\,(v(\mathscr{B}))$$

it is enough to show that $\mathrm{st}\,({}^*v({}^*\mathscr{B}))$ is convex and compact. This follows immediately from Liapounoff's theorem if one knows that

$$\mathrm{st}\,({}^*v({}^*\mathscr{B})) = L({}^*v)(L({}^*\mathscr{B}))$$

which is a consequence of the fact that every element of $L({}^*\mathscr{B})$ is approximated, in the sense of Definition 1.4, by an element of ${}^*\mathscr{B}$.

2. Lifting and pushing down.

A natural question to ask about the measure $L(\mu)$ is whether there exists a measurable function $f \in \mathcal{M}(X, L(\mathcal{A}), L(|\mu|), \hat{B})$ such that for each

$$A \in L(\mathcal{A}), \qquad L(\mu)(A) = \int_A f\, dL(|\mu|).$$

After that we can ask whether f has a lifting $g\colon X \to {}^*B$, because we would like to think of integral as about hyperfinite sums. Recall that a function $f\colon X \to \hat{B}$ is measurable (because $L(|\mu|)$ is finite), iff there exists a sequence j_n of simple measurable functions such that $j_n \to f$ almost surely. The proof of the following theorem is very close to the proof of R. Anderson's lifting theorem in [1]. Nevertheless, it is worth mentioning that the new ingredient which makes the proof go through is the fact that every measurable vector function is essentially separably valued.

THEOREM 2.1. *Let $(X, \mathcal{A}, \nu)$ be an internal space with a finitely additive, positive measure ν such that $\operatorname{st}(\nu)(X) < +\infty$ and $(X, L(\mathcal{A}), L(\nu))$ the corresponding Loeb space. If $g\colon X \to \hat{B}$ is $L(\nu)$-measurable function then there exists an internal function $h\colon X \to {}^*B$ which satisfies the following conditions*:

(1) *The range of h, $\operatorname{ran}(h) \subset {}^*B$, is a hyperfinite set and for every $t \in \operatorname{ran}(h)$, $h^{-1}(t) \in \mathcal{A}$; shortly, h is a $*$-simple function.*
(2) *For almost all $x \in X$, $\operatorname{st}(h)(x) = g(x)$.*

PROOF. According to Lemma III.6.9 from [3] which gives a characterisation of measurable functions we can assume the following:

(3) $\operatorname{ran}(g) \subset \hat{B}$ is separable.
(4) $g^{-1}(V(a, \varepsilon)) \in L(\mathcal{A})$ for every open ball $V(a, \varepsilon) \subset \hat{B}$.

Let $(a_n \mid n \in \mathbf{N})$ be a sequence in $\hat{B}$ which is dense in $\operatorname{ran}(g)$ and $(b_n \mid n \in \mathbf{N})$ a sequence in ${}^*B$ such that $(\forall n \in \mathbf{N})\ \operatorname{st}(b_n) = a_n$. It is known that the last sequence can be extended to an internal sequence $(b_n \mid n \in {}^*\mathbf{N})$. The family

$$\left\{ V\left(a_n, \frac{1}{m}\right) \cap \operatorname{ran}(g) \right\}_{n, m \in \mathbf{N}}$$

is a base for the space $\operatorname{ran}(g)$. Let $(U_k \mid k \in \mathbf{N})$ be a sequence of internal sets such that $U_0 = {}^*B$ and $(U_k \mid k \geq 1)$ is any ordering of open balls $V(b_n, 1/m) \subset {}^*B$. Let $(V_k \mid k \in \mathbf{N})$ be defined in a similar way that is $V_0 = \hat{B}$ and $(V_k \mid k \geq 1)$ is the ordering of $\{V(a_n, 1/m) \mid n \geq 1, m \geq 1\}$ such that $U_k = V(b_n, 1/m)$, iff $V_k = V(a_n, 1/m)$. We can assume that $(U_k \mid k \in \mathbf{N})$ is extended to an internal sequence $(U_k \mid k \in {}^*\mathrm{N})$. Let us construct a sequence $(A_n \mid n \in \mathrm{N})$ of elements in $\mathscr{A}$ with the following properties:

$$A_0 = X,$$
$$L(v)(A_n \triangle g^{-1}(V_n)) = 0, \quad \text{and}$$
$$(\forall n \in \mathrm{N})(\exists f_n\colon X \to {}^*B \text{ internal})(\forall S \subset \{1,\ldots,n\}) f_n\Big(\bigcap_{i \in S} A_i\Big) \subset \bigcap_{i \in S} U_i.$$

If $A_0 = X$, and all A_i, for $i < n$, are defined let $A'_n \in \mathscr{A}$ be such that

$$L(v)(A'_n \triangle g^{-1}(V_n)) = 0.$$

Now,

$$A_n = A'_n \setminus \bigcup \Big\{\bigcap_{i \in S} A_i \;\Big|\; S \subset \{1,\ldots,n\} \wedge \bigcap_{i \in S} U_i = \varnothing\Big\}$$

is an internal set for which a corresponding function f_n can be defined. Let us extend sequences $(f_n \mid n \in \mathrm{N})$, $(A_n \mid n \in \mathrm{N})$ to internal sequences $(f_n \mid n \in {}^*\mathrm{N})$ and $(A_n \mid n \in {}^*\mathrm{N})$. Let

$$D = \{n \in {}^*\mathrm{N} \mid f_n \text{ is a } {}^*\text{-simple, } {}^*\text{-measurable function and}$$
$$\text{for all } k \leqq n,\ f_k(A_k) \subset U_k\}.$$

Obviously $\mathrm{N} \subset D$, hence there exists $m \in D \cap ({}^*\mathrm{N} \setminus \mathrm{N})$. Let us show that the function f_m satisfies both conditions (1) and (2) in the theorem. It is enough to check (2). Let

$$X' = X \setminus \bigcup_{n=1}^{\infty} (A_n \triangle g^{-1} V_n);$$

then $X' \in L(\mathscr{A})$ and $L(v)(X \setminus X') = 0$. Let $x \in X'$. But from the condition $g(x) \in V_i$ follows $x \in A_i$ and $f_m(x) \in U_i$. Hence, $\mathrm{st}(f_m(x)) = g(x)$.

Let us note that the range of the function h whose existence was proved in the Theorem 2.1 is almost S-separable in the sense of the following definition.

Definition 2.2. Let B be a Banach space. A set $A \subset {}^*B$ is called S-separable if there exists a countable set $C \subset \mathrm{fin}({}^*B)$ such that for every $x \in A$ and standard $\varepsilon > 0$ there exists $y \in C$ so that ${}^*\|x - y\| < \varepsilon$ holds. The range of a function $h\colon X \to {}^*B$, defined on a measure space $(X, \mathscr{B}, v)$, is said to be v-almost S-separable, if there exists a null set D such that $h(X \setminus D)$ is S-separable.

Theorem 2.3. *As before, let $(X, \mathscr{A}, v)$ be an internal space with a finitely additive positive measure v such that* $(\mathrm{st}\, v)(X) < +\infty$ *and let* $(X, L(\mathscr{A}), L(v))$ *be* *the corresponding Loeb space. Let $g: X \to {}^*B$ be an internal function which is internally measurable with respect to the measure space $(X, \mathscr{A}, \nu)$. In other words g satisfies the *-transform of the usual definition of measurability of vector functions* ([3]). *Assume that the range of g is $L(\nu)$-almost S-separable. Under these conditions $h = \operatorname{st}(g): X \to \hat{B}$ is a $L(\nu)$-measurable function.*

PROOF. Again, we shall use Lemma III.6.9 from [3] which guarantees that $h: X \to \hat{B}$ is measurable if the following conditions are satisfied:

(1) There exists $X' \subset X$ of measure zero such that $h(X \setminus X')$ is a separable subset of $\hat{B}$,
(2) $h^{-1}(V(a,r)) \in L(\mathscr{A})$ for each open ball $V(a,r) \subset \hat{B}$.

Since the function g is internally $(X, \mathscr{A}, \nu)$-measurable there exist an internal set $A \subset X$ of infinitesimal measure and a *-simple function g' such that $(\forall\, x \in X \setminus A)\ \operatorname{st}(g')(x) = \operatorname{st}(g)(x)$. Let us work with the function g'. The first condition is fulfilled because of the assumption on the range of g. Now, let $V(a,r)$, $a \in \hat{B}$ and $r > 0$, be an open ball in $\hat{B}$. Let $b \in {}^*B$ such that $\operatorname{st}(b) = a$ and $x \in X \setminus A$.

$$x \in h^{-1}(V(a,r)) \Leftrightarrow \|h(x) - a\| < r \Leftrightarrow \exists n \in \mathbf{N} \left({}^*\|g'(x) - b\| < r - \frac{1}{n} \right)$$

$$\Leftrightarrow x \in \bigcup_{n \in \mathbf{N}} (g')^{-1}\left({}^*V\left(b, r - \frac{1}{n} \right) \right).$$

The function g' is *-simple, hence $(g')^{-1}({}^*V(b, r - 1/n)) \in \mathscr{A}$, therefore $h^{-1}(V(a,r)) \in L(\mathscr{A})$ and the second condition is also fulfilled.

Theorem 2.1 permits us to obtain more information about the Loeb completion of the measure space $(X, \mathscr{A}, \mu, {}^*B)$ in case $\hat{B}$ has the Radon-Nikodym property (RN). Recall that a Banach space B has the RN-property if for any σ-additive, B-valued measure μ defined on a σ-algebra $\mathscr{B}$ of subsets of a given set S, with finite total variation $|\mu|$, there exists a Bochner-integrable function f such that

$$\mu(A) = \int_A f\, d(|\mu|) \qquad \text{for all } \mathscr{A} \in \mathscr{B}.$$

It is known that all reflexive Banach spaces, particularly all Hilbert spaces, have the RN-property.

PROPOSITION 2.4. *Let $(X, \mathscr{A}, \mu, {}^*B)$ be as before. If $\hat{B}$ has the* RN-*property, then there exists a *-simple function $f: X \to {}^*B$ such that*

$$L(\mu)(A) = \mathrm{st}\left(\int_A f\, d|\mu|\right).$$

In other words if $f=\sum_{i=1}^{H} a_i\varphi_{A_i}$, *where* $\{A_i\}_{i=1}^{H}$ *is a hyperfinite* $\mathscr{A}$*-partition of* X, *and* φ_{A_i} *the characteristic function of* A_i, *then*

$$L(\mu)(A) = \mathrm{st}\sum_{i=1}^{H} |\mu|(A\cap A_i)\cdot a_i\,.$$

Proof. From $\hat{B}\in \mathrm{RN}$ follows the existence of a bounded $L(|\mu|)$-measurable function $g\colon X\to \hat{B}$ such that for all $B\in L(\mathscr{A})$

$$\int_B g\, dL(|\mu|) = L(\mu)(B)\,.$$

Note that we used here unproved but easily checkable fact

$$\mathrm{var}\,(L(\mu), A) = L(|\mu|)(A)\,.$$

Let $f\colon X\to {}^*B$ be a function which (Theorem 2.1) satisfies the following conditions:

(1) $f=\sum_{i=1}^{H}\varphi_{A_i}\cdot b_i$, where $(b_i\mid 1\leqq i\leqq H)$ is a hyperfinite sequence of elements in ${}^*B$ and $\{A_i\}_{i=1}^{H}$ a hyperfinite $\mathscr{A}$-measurable partition of X,

(2) for almost all $x\in X$, $\mathrm{st}\,(f(x))=g(x)$.

Obviously, it is enough to check the equality

$$\int_A g\, dL(|\mu|) = \mathrm{st}\int_A f\, d|\mu|\,.$$

Choose $m\in \mathbf{R}$ such that

$$\forall x\in X\quad (\|g(x)\|\leqq m \wedge {}^*\|f(x)\|\leqq m)$$

and $\varepsilon>0$ be a standard real number. Then there exists $Z\in\mathscr{A}$ and $j\colon X\to\hat{B}$ a simple function of the form

$$j = \sum_{i=1}^{K}\varphi_{D_i}\cdot c_i$$

such that $L(|\mu|)(Z)<\varepsilon$, $D_i\in\mathscr{A}$ and $\forall x\in X\setminus Z$, $(\|g(x)-j(x)\|<\varepsilon)$. If $d_i\in {}^*B$ are chosen so that $\mathrm{st}\,(d_i)=c_i$, then the function

$$\hat{j} = \sum_{i=1}^{K}\varphi_{D_i}\cdot d_i$$

satisfies $\forall x\in X\setminus Z$, $({}^*\|f(x)-\hat{j}(x)\|<\varepsilon)$. Without loss of generality we can assume that both $\|c_i\|\leqq m$ and ${}^*\|d_i\|\leqq m$. Hence,

$$\left\|\int_X g\,dL(|\mu|)-\sum_{i=1}^{K} c_i\cdot L(|\mu|)(D_i)\right\| \leqq \varepsilon\cdot L(|\mu|)(X)+2m\varepsilon$$

$$^*\left\|\int_X f\,d|\mu|-\sum_{i=1}^{K} d_i\cdot|\mu|(D_i)\right\| \leqq \varepsilon\cdot|\mu|(X)+2m\varepsilon\,.$$

Hence,

$$\left\|\int_X g\,dL(|\mu|)-\mathrm{st}\int_X f\,d|\mu|\right\| \leqq 2\varepsilon L(|\mu|)(X)+4m\varepsilon$$

which proves the desired equality for $A=X$. For any $A\in L(\mathscr{A})$ the proof goes analogously.

3. The Riesz Representation Theorem.

The proof of the Riesz Representation Theorem given below does not really depend on the results in the previous sections. We should note that the basic facts about integration of complex-valued functions w.r.t. complex-valued measures, required in the proof, can be easily derived from P. Loeb's results from [8]. Nevertheless, we take this opportunity to give a short proof of the general (complex-valued) case of the Riesz theorem which is much better than the clumsy proof of the author given in [13]. Note that P. Loeb has given a proof of the case of a positive linear functional, which is based on the first principles, in [9].

As in the first proof, [13], we assume basic facts about representations of Radon spaces by suitable Loeb spaces as given in R. Anderson [1].

THEOREM 3.1. *Let X be a compact Hausdorff space and $C(X,\mathbf{C})$ the complex Banach space of all continuous complex functions on X. If $L\colon C(X,\mathbf{C})\to\mathbf{C}$ is any bounded linear functional on $C(X,\mathbf{C})$, then there exists a complex Radon measure ν on X such that*

$$L(f) = \int_X f\,d\nu \quad \textit{for } f\in C(X,\mathbf{C})\,.$$

PROOF. Let $F=\{f_i \mid 1\leqq i\leqq D\}$ be a hyperfinite partition of unity, that is $0\leqq f_i\leqq 1$ for all $1\leqq i\leqq D$ and $\sum_{i=1}^{D} f_i=1$, such that $(\forall i)(\exists x\in X)(\operatorname{supp}(f_i)\subset m(x))$. Let $\{y_i \mid 1\leqq i\leqq D\}$ be an internal set such that $y_i\in\operatorname{supp}(f_i)$, if the last set is nonempty, and μ the measure defined by

$$\mu(B) = \sum\{{}^*L(f_i) \mid y_i\in B\}$$

for internal $B\subset{}^*X$. Let $L(\mu)$ be the Loeb completion of μ, defined on ${}^*X$, and

$\nu = \mathrm{st}(L(\mu))$ the projection of $L(\mu)$ which is by [1] a Radon measure. Let us prove that ν is the desired measure. Let $f \in C(X, \mathbf{C})$.

$$L(f) = {}^*L({}^*f) = \sum_{i=1}^{D} {}^*L({}^*f \cdot f_i),$$

hence

$$\left|L(f) - \int_X f\, d\nu\right| = \left|L(f) - \int_{{}^*X} \mathrm{st}\,({}^*f)\, dL(\mu)\right|$$
$$= \left|\sum_{i=1}^{D} {}^*L({}^*f \cdot f_i) - \mathrm{st} \sum_{i=1}^{D} {}^*f(y_i)\, {}^*L(f_i)\right|$$

The function f is uniformly continuous on X, hence the oscilation of ${}^*f$ on any of the sets $\mathrm{supp}\,(f_i)$, $1 \leqq i \leqq D$, is smaller than a fixed infinitesimal η. Hence,

$$|[{}^*f(x) - {}^*f(y_i)]f_i(x)| \leqq \eta \cdot f_i(x)$$

and therefore

$${}^*\left\|\sum_{i=1}^{D} [{}^*f - {}^*f(y_i)]f_i\right\| = {}^*\sup_{x \in {}^*X} \left|\sum_{i=1}^{D} [{}^*f(x) - {}^*f(y_i)]f_i(x)\right|$$
$$\leqq {}^*\sup_{x \in {}^*X} \sum_{i=1}^{D} |f(x) - f(y_i)| f_i(x) \leqq {}^*\sup_{x \in {}^*X} \sum_{i=1}^{D} \eta \cdot f_i(x) = \eta\,.$$

Since $\|L\| = {}^*\|{}^*L\| < +\infty$,

$$\left|\sum_{i=1}^{D} {}^*L({}^*f \cdot f_i) - \sum_{i=1}^{D} {}^*L({}^*f(y_i) \cdot f_i)\right| = \left|{}^*L\left(\sum_{i=1}^{D} \left[{}^*f - {}^*f(y_i)\right] f_i\right)\right|$$
$$\leqq \|L\| \cdot {}^*\left\|\sum_{i=1}^{D} [{}^*f - {}^*f(y_i)]f_i\right\| \leqq \eta \cdot \|L\| \approx 0\,.$$

Therefore, $|L(f) - \int_X f\, d\nu| = 0$ and we are done.

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MATEMATIČKI INSTITUT
KNEZ MIHAILOVA 35/1
11000 BEOGRAD
JUGOSLAVIJA