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# DEFINABILITY AND FORCING IN E-RECURSION*)

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**Abstract.**

Forcing methods in the setting of E-recursion are reviewed from the point of view of computations. The effects of forcing on definability classes associated with E-recursion at levels of the pure type structure are studied. The "mildest" possible forcing extensions of these definability classes are determined. Finally, it is shown that the RE-degree structure of E-closed sets is unchanged on certain forcing extension via "effective" posets.

## 0. Introduction.

This paper will present an index-free version of forcing over E-closed sets. The definition of the forcing relation follows the schematic definition of computations in E-recursion (see Normann [1]). Within the context of forcing in generalized recursion the fundamentally new tool, the Moschkovakis Phenomenon (MP), was first isolated by Sacks [11] where he showed that set-generic extensions via countably closed posets preserve the E-closure of many E-closed sets.

The forcing definition and its properties appear in Sections 1–3. Section 4 discusses the role of selection and definability in Cohen extensions and in Section 5 we show the independence of the well-foundedness of the E-degrees of reals (here we use the absoluteness results of Lévy [8] for forcing extensions via semi-homogeneous posets).

Sections 6–11 address the problem of extending $k$-sections of ${}^{k+2}\boldsymbol{E}$ non-trivially. Finally, in Section 12, we use the implicit uniformity in the inductive definition of the forcing relation to show that the structure of the RE-degrees of the ground model is unaffected in certain set-forcing extensions.

## 1. The forcing technology.

We say that a set $D \subsetneq \boldsymbol{P}$ is *dense* in $\boldsymbol{P}$ if for all $r \in \boldsymbol{P}$ there exists a $d \in D$ such that $r$ and $d$ are compatible (i.e. have a common extension in $\boldsymbol{P}$). A set $G \subsetneq \boldsymbol{P}$ is $\boldsymbol{P}$*-generic over* $A$ ($\boldsymbol{P}$-generic/$A$) if

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(i) $G$ is a directed set;
(ii) $g \in G$ and $P \leqq_P g$, then $p \in G$; and
(iii) every dense $D \subsetneq P$ which is first order definable over $\langle A, \varepsilon \rangle$ with parameters from $A$ satisfies $G \cap D \neq \emptyset$.

Let $A[G]$ be the collection of sets E-recursive in $G$, $a_1, \ldots, a_n$ (where $a_i \in A$, $i = 1, \ldots, n$) computed via a computation of height less than the supremum of order-types in $A$. If $A[G]$ in E-closed; then it is the least E-closed set $x$ containing $A$ as a subset and $G$ as an element (set forcing).

The ramified language will be given with an eye to questions of effectiveness: $\mathscr{L}^*$ is defined effectively in $A$. The terms of $\mathscr{L}^*$ are built using parameters from $A$ such that those involving only $b \in A$ are present in $E(b)$.

SYMBOLS. $\varepsilon$, $=$; unranked variables $x, y, \ldots$; ranked variables $x^\lambda, y^\lambda \ldots$ for $\lambda < \varkappa$: logical connectives $\wedge$, $\neg$; and the quantifier $\exists$.

Formulas are built up using these symbols and a class of terms $C$, defined by induction, i.e. we will name all elements of $A[G]$ in $A$. For $x \in A$ we define $C^x$ by an induction of length $\varkappa = \mathrm{OR} \cap A$.

DEFINITION.

$$C_0^x = \{\underline{b} \mid b \in \mathrm{TC}(x) \vee b = X\} \cup \{\underline{G}\};$$

$C_{\alpha+1}^x$ satisfies: $C_\alpha^x \subsetneq C_{\alpha+1}^x$ and if $\varphi(v_0, \ldots, v_n)$ is in $\mathscr{L}^*$ with free variables in $v_0, \ldots, v_n$ and quantified variables of the form $x^\beta$, $\beta \leqq \alpha$, then

$$\hat{x}^\alpha \varphi(x^\alpha, c_1, \ldots, c_n) \in C_{\alpha+1}^x, \text{ if } c_1, \ldots, c_n \in C^x;$$

$$C_\lambda^x = \bigcup_{\alpha < \lambda} C_\alpha^x, \text{ if } \lim(\lambda) \text{ and } \lambda < \varkappa$$

$$C^x = \bigcup_{\alpha < \varkappa} C_\alpha^x \text{ and } C = \bigcup_{x \in A} C^x \text{ and each } c \in C \text{ is a symbol in } \mathscr{L}^*.$$

We say that a formula $\varphi \in \mathscr{L}^*$ is *ranked*, if all bound variables in $\varphi$ are ranked and assign an ordinal (rank ($\varphi$)) to each $\varphi \in \mathscr{L}^*$ as follows (in decreasing order of importance):

(i) the number of unranked quantifiers;
(ii) ordials associated with ranked quantifiers and constant terms;
(iii) logical complexity.

The *forcing relation* $p \Vdash \varphi$ is defined by induction on rank ($\varphi$). Apart from the clauses given by the schemata of E-recursion, all clauses are standard. The symbol $\underline{x}$ denotes a term. We consider the bounding scheme and composition.

First suppose

$$\{e\}^G(x,\vec{y}) = \bigcup_{z\in x}\{\{e_0\}^G(z,\vec{y})\},$$

then

$$p \Vdash^* \{e\}^G(\underline{x},\underline{\vec{y}})| = \lambda \text{ iff}$$

(a) $p \Vdash \forall z \in \underline{x}\, \exists\gamma < \lambda\, [|\{e_0\}^G(z,\underline{\vec{y}})| = \gamma]$; and
(b) $p \Vdash \forall\sigma < \lambda\, \exists z \in \underline{x}[|\{e_0\}^G(z,\underline{\vec{y}})| \geqq \sigma]$.

If we have

$$\{e\}^G(x,\vec{y}) = \{e_o\}^G(\{e_1\}^G(x,\vec{y}),x,\vec{y}),$$

then

$$p \Vdash \{e\}^G(\underline{x},\underline{\vec{y}})| = \lambda \text{ iff}$$

(a) $p \Vdash \forall z \in \underline{x}\, \exists\gamma < \lambda\, [|\{e_0\}^G(z,\underline{\vec{y}})| = \gamma]$; and
(b) $p \Vdash \forall\sigma < \lambda\, \exists z \in \underline{x}[|\{e_0\}^G(z,\underline{\vec{y}})| \geqq \sigma]$.

If we have

$$\{e\}^G(x,\vec{y}) = \{e_o\}^G(\{e_1\}^G(x,\vec{y}),x,\vec{y}),$$

then

$$p \Vdash |\{e\}^G(\underline{x},\underline{\vec{y}})| = \sigma$$

iff there exists $\sigma_1, \sigma_2 \leqq \sigma$ such that

$$p \Vdash |\{e_1\}^G(\underline{x},\underline{\vec{y}})| = \sigma_1 \{e_1\}^G(\underline{x},\underline{\vec{y}}) = \underline{z}$$

and

$$p \Vdash |\{e_0\}^G(\underline{z},\underline{x},\underline{\vec{y}})| = \sigma_2, \text{ where } \sigma = \max(\sigma_1,\sigma_2)+1.$$

REMARK. We have not explicitly defined what it means to say $p \Vdash \{e_1\}^G(\underline{x},\underline{\vec{y}}) = \underline{z}$, however for such a computation which converges there is an index which gives the characteristic function of the set which is its value. Proceeding inductively this is the same as forcing that these functions values are the same as those of the term $\underline{z}$ on all appropriate arguments (i.e. terms of lower rank).

Applications are often simplified by considering the 'weak' forcing relation $\Vdash^*$ defined by

$$P \Vdash^* \varphi \text{ iff } p \Vdash \neg\neg\varphi.$$

We shall assume the standard result that if $G \subsetneq \boldsymbol{P}$ is $\boldsymbol{P}$-generic/$A$, then

$$A[G] \models \varphi \text{ iff } \exists p \in G[p \Vdash^* \varphi].$$

**2. Preserving E-closure: Closure conditions.**

Now assume that $A$ is E-closed and $\boldsymbol{P} \in A$. To show that E-closure is preserved by a generic extension of $A$ ($A[G]$ is E-closed), Sacks [11] shows that for $x \in A$, $\vec{y} \in A^n$ (for some $n \in \omega$), the relation

$$p \Vdash^* \{e\}^G(x,\vec{y})\downarrow \text{ is } \underline{\mathrm{RE}}.$$

Lemma 2.0 (Sacks [11]). *Suppose $\gamma \in \mathrm{OR} \cap A$, then the relation $p \Vdash^* \varphi$ restricted to $\varphi$'s of ordinal* rank $\leqq \gamma$ *and quantifiers restricted to $E(z)$ for $z \in A$ is recursive in $\gamma$, $z$, $\boldsymbol{P}$.*

Proof. Sack's proof proceeds by induction on the definition of the forcing relation. Consider only the cases $\neg$ and $\exists x^\beta$. Let $\psi = \neg\varphi$ and suppose $p \Vdash \psi$, then by Definition (iii):

$$\forall q \leqq_{\boldsymbol{P}} p(q \nVdash^* \varphi).$$

By induction hypothesis and the bounding principle we have the desired conclusion.

Now let $\varphi \equiv \exists x^\beta \psi$ and suppose $p \Vdash \varphi$, then by definition $p \Vdash \psi(c)$ for some $c \in C^x_\beta$, where $x$ is the parameter from $A$ in $\psi$. By induction hypothesis $p \Vdash \psi(c)$ is recursive in $\gamma$, $z$, $\boldsymbol{P}$. $C^x_\beta$ is recursive in $x, \beta$ and by the bounding principle applied to that procedure $p \Vdash^* \varphi$ is recursive in $\gamma, z, \boldsymbol{P}$. The remaining cases are routine.

Definition. Let $\langle p,a \rangle$ and $\langle q,b \rangle \in \boldsymbol{P} \times C$ and let $\langle p,a \rangle >_S \langle q,b \rangle$ if $q \leqq_{\boldsymbol{P}} p$ and $q \Vdash^*$ '$b$ is a subcomputation of $a$'.

Lemma 2.1 (Sacks [11]). *Suppose $\boldsymbol{P} \in A$ and that $<_S$ is well-founded below $\langle p,a \rangle$, then $\exists q \in \boldsymbol{P} \exists \gamma < \varkappa$, $q$ and $\gamma$ uniformly recursive in $p$, $a$, $\boldsymbol{P}$ such that*

$$q \Vdash^* |\{a_0\}^G(a_1)| = \gamma, \text{ where } a = \langle a_0, a_1 \rangle.$$

Remark. The fundamentals required for the proof of Lemma 2.1 were established for $E(2^\omega)$ by Sacks [11]. The lemma in its present form appeared in Sacks–Slaman [12].

Corollary 2.2. *If $\boldsymbol{P} \in A$ and $\forall p \subsetneq \boldsymbol{P} \forall a \in C$ [$<_S$ is well-founded below*

$$\langle p,a \rangle \Leftrightarrow p \Vdash^* \{a_0\}^G(a_1)\downarrow],$$

*then the relation $p \Vdash^* \{e\}(\underline{x},\underline{\vec{y}})\downarrow$ in* RE *in $\boldsymbol{P}$.*

The procedure defined in the lemma allows one to reduce the forcing of an apparently $\Sigma_1(A)$ formula (i.e. there exists a well-founded computation tree with values) effectively to a ranked formula. What has been shown here is that the $<_S$ height of $\langle q_0, a\rangle$ is recursive in $p$, $a$ and bounds the value of $|\{e\}^G(x,\vec{y})|$, where $G$ is $\boldsymbol{P}$-generic/$A$ extending $q_0$.

Countable closure of $\boldsymbol{P}$ is one way of insuring the closure of $A[G]$. The virtue of countable closure is its ability to exploit the MP. Consider a procedure applied to a pair $\langle p,\tau\rangle$, where $p \in \boldsymbol{P}$ is a forcing condition and $\tau$ is a term in the associated forcing language:

(i) if $P \Vdash \tau\downarrow$, then we produce by induction a bound less than $\varkappa$ on $\|\tau\|$;
(ii) if $p \Vdash \tau\downarrow$, then we build a sequence $\langle p_n, \tau_n\rangle_{n\in\omega}$ such that $\forall n[p_{n+1} \leqq p_n)$ and $p_n \Vdash$"$\tau_n$ is a subcomputation of $\tau_{n-1}$"].

By countable closure we take $p_\infty$ such that $\forall n[p_\infty \leqq_P p_n]$, then $p_\infty \Vdash \langle\tau_n\rangle_{n\in\omega}$ is a Moschkovakis witness (MW) for $\tau''$.

LEMMA 2.4. *Suppose $\boldsymbol{P}$ is countably closed in $A$, $A \models$MP and $<_S$ is not well-founded below $\langle p,a\rangle$, then there exists a term $t$ and a condition $q$ such that $q \Vdash^*$ '$t$ is a* MW *for $a'$.*

REMARK. This fundamentally new feature of forcing in the setting of generalized recursion was first isolated by Sacks [11] in the setting of $E(2^\omega)$. Lemma 2.4 in its present form first appeared in Sacks–Slaman [12].

Sacks' theorem on countable closure is now immediate.

THEOREM 2.5 (Sacks–Slaman). *Suppose $A$ is* E-*closed, $A \models$*MP*, $\boldsymbol{P} \in A$ such that*

$$A \models \text{`}\boldsymbol{P} \text{ is countably closed'}.$$

*If $G$ is $\boldsymbol{P}$-generic/$A$, then $A[G]$ is* E-*closed and satisfies* MP.

The existence of $\boldsymbol{P}$-generics over E-closed $A$ is not probable in general for uncountable $A$. We say that $G \leqq \boldsymbol{P}$ is *$\boldsymbol{P}$-bounded generic/$A$*, if $G$ is generic with respect to all sentences of bounded rank in $\mathscr{L}^*$ (that is $G$ meets the associated dense subsets of $\boldsymbol{P}$). Sacks [11] first noticed that such a generic is often sufficient for applications.

LEMMA 2.6 (Sacks [11]). *Suppose $A$ and $\boldsymbol{P}$ satisfy the conditions of the above theorem and that for some transitive set $X$*

$$A = E(X) \text{ (\textit{the} E-\textit{closure of} } X)$$

*and that $X$ is well-orderable in $A$. If $\gamma(<\varkappa = \mathrm{OR} \cap A)$ is the height of the shortest such well-ordering of $X$ in $A$ and $A \models$ "$\gamma$ is regular", then a $\boldsymbol{P}$-bounded generic over $A$ exists, where*

$$\boldsymbol{P} = \{f: \gamma \to \{0,1)f^{=A} < \gamma\}.$$

PROOF. (Sketch). Since $A = E(X)$, every set $z \in A$ is recursive in some $\tau < \gamma$ (modulo the parameter giving the well-ordering of $X$ in type $\gamma$).

The sentences of bounded rank in $\mathscr{L}^*$ can be recursively enumerated by $\gamma$ such that the enumeration restricted to an initial segment of $\gamma$ is bounded below $\varkappa$.

This can be seen, for example, by appeal to Griffor–Normann Selection [3] which in this case gives that for $r < \gamma_S$ we have uniformly in $\tau, \gamma$ selection over RE subsets of $\tau$.

The forcing relation for these sentences (essentially those giving computation tuples) is RE in $\boldsymbol{P}$. Using the well-ordering of $\boldsymbol{P}$ define by transfinite recursion $p: \gamma \to \gamma$ by $\tau < \gamma$:

For $\tau = \alpha + 1$: $p(\tau)$ is the least $p \in \boldsymbol{P}$ such that $p \leqq_{\boldsymbol{P}} p(\alpha)$ and $p$ decides $\varphi_{\|\tau\|}$, if $\tau \downarrow$ and $p(\tau)$ is $p(\alpha)$ otherwise, where $\varphi_{\|\tau\|}$ is the $\|\tau\|^{\text{th}}$ sentence of $\mathscr{L}^*$ of bounded rank.

For limit $(\tau)$:

$$p(\tau) = \bigcup_{\gamma < \tau} p(\gamma).$$

CLAIM. *For all $\sigma < \gamma$, $p''\sigma$ is bounded below $\gamma$.*

PROOF (Claim). Given $\sigma < \gamma$ we have that

$$G_\sigma = \{\tau < \sigma \mid \tau \text{ codes a convergent computation}\}$$

is an element of $A$ (we have identified $X$ with $\gamma$ via the well-ordering). Using $G_\sigma$, $p''\sigma$ is an element of $A$ and by the assumption that $\gamma$ is regular in $A$, $p''\sigma$ is bounded below $\gamma$.

The first application of forcing in the setting of E-recursion was due to Sacks [11], where he made use of the above result concerning forcing with countably closed posets. Sacks showed that if there exists a recursively regular well-ordering of $2^\omega$ recursive in ${}^3\boldsymbol{E}$ and a real, then the $2 - \mathrm{sc}\,({}^3\boldsymbol{E})$ is not RE in any real.

### 3. Antichain conditions and E-closure.

Antichain conditions on $\boldsymbol{P}$ are yet another way of preserving E-closure. For the sake of completeness we mention the results of Sacks in this direction.

DEFINITION. Let $A$ be E-closed and $\boldsymbol{P} \in A$ be a poset, then

(i) $x \subsetneq \boldsymbol{P}$ is an *antichain* if all elements of $x$ are incompatible via $<_{\boldsymbol{P}}$;
(ii) an antichain $x$ is *maximal* if every element of $\boldsymbol{P}$ is compatible via $<_{\boldsymbol{P}}$ with some element of $x$;
(iii) $\boldsymbol{P}$ satisfies the *$\sigma$-chain condition* ($\sigma$ – cc) in $A$, if every $\boldsymbol{P}$-antichain in $A$ has $A$-cardinality *less than* $\sigma$.

For example, if $\boldsymbol{P}$ has the $\beta^+$ – cc in $A$, then every $\boldsymbol{P}$-antichain in $A$ has $A$-cardinality less than or equal to $\beta$. As a consequence any effective phenomenon in $A[G]$ can be restricted to at most $\beta$ many possibilities in $A$.

THEOREM 3.0 (Sacks [11]). *Let $A$ be* E-*closed*, $\boldsymbol{P} \in A$, $\gamma \in A$ *such that*

(i) *$\boldsymbol{P}$ has the $\gamma^+$* – cc *in $A$*;
(ii) *there is an $a \in A$ such that $\langle a,x\rangle$ selects from $\gamma$ for all $x \in A$*;
(iii) *each $x \in A$ is well-orderable in $A$.*

*Then if $G$ is $\boldsymbol{P}$-generic/$A$ we have that $A[G]$ is* E-*closed.*

Recall that if $X$ is a set, then an ordinal $\alpha$ is $X$-reflecting if

$$(z = \mathrm{TC}(\{x\} \cup X)) : L_\alpha[Z] = \{e\}(X)t \text{ implies that } L_{K_0^x}[Z] \Vdash \{e\}(x)\downarrow,$$

where $K_0^x$ is the supremum of all ordinals E-recursive in $X$. Then

$$\underline{K_r^x} = \operatorname{supp}\{\alpha \mid \alpha \text{ is } X\text{-reflecting}\}.$$

In the theorem $\langle a,x\rangle$ *selects from* $\gamma$ if we can E-recursively compute an element of any non-empty RE subset of $\gamma$ uniformly in $\langle a,x\rangle$ and an index for the RE subset of $\gamma$.

REMARK. (a) Sacks' argument proceeds by approximating computations in $A[G]$ by building antichains in $A$. The reader is directed to Slaman [13] for the proof.

(b) Slaman notices that Sacks' proof actually yields that for $X \subsetneq \mathrm{OR}$ such that

(i) $\mathrm{gc}(x)$ is the greatest cardinal in $E(X)$;
(ii) $\boldsymbol{P} \in E(X)$ and has the $\gamma^+$ – cc in $E(X)$ and $\gamma \leqq \mathrm{gc}(x)$ ($\boldsymbol{P} \in \mathrm{OR}$ and $\boldsymbol{P} \leqq \mathrm{gc}(x)$ and $\boldsymbol{P} \leqq_e \boldsymbol{P}, X, \mathrm{gc}(X)$);
(iii) $\exists a \in E(X)\ y \in E(X)$, $\langle a,y\rangle$ selects on $\gamma$ (fix this $a$), then if $G$ is $\boldsymbol{P}$-generic over $E(X)$ and $b \subsetneq \mathrm{gc}(X)$ with $b \in E(X)$:

$$K_r^{b,b,\langle a,\boldsymbol{P},\gamma,x,\mathrm{gc}(x)\rangle} = K_r^{b,\langle a,\boldsymbol{P},\gamma,x,\mathrm{gc}(x)\rangle}.$$

COROLLARY 3.1 (Sacks): C.c.c. ($\aleph_1 -$cc) *set forcing* (*with* (iii) *of the theorem*) *preserves* E-*closure*.

PROOF. Use Gandy Selection.

## 4. Cohen reals.

In this section we consider the result of adding Cohen reals to $E(X)$. First we address the question posed in the previous section concerning the preservation of E-closure.

Let $X \in V$ be infinite and transitive and consider $E(X)$. Let the poset

$$\boldsymbol{P} = \{f:\omega \to \{0,1\} \mid f \text{ is a partial function and dom}(f) \text{ is finite}\}$$

and for $p,q \in \boldsymbol{P}$, let $p \leqq_{\boldsymbol{P}} q$ iff $p$ extends $q$ set-theoretically. $\boldsymbol{P}$ is just the Cohen poset for adding a new real.

LEMMA 4 (Sacks). *With* $\boldsymbol{P}$ *as above let* $G \subsetneq \boldsymbol{P}$ *be* $\boldsymbol{P}$-*generic*/$E(\cdot)$ *then*

(i) $\bigcup G = f:\omega \to \{0,1\}$;
(ii) $E(X)[f]$ *is* E-*closed*; *and*
(iii) $K_0^{X,f} = K_0^X$.

(ii) follows immediately from (iii), while (i) is a standard density argument. Using the fact that the forcing relation is RE: assume $\{e\}(f)\downarrow$in $E(X)[f]$, then letting $G$ be the term for $f$ in $\mathscr{L}^*$ we have that there exists a $p \in G$ such that $p \Vdash \{e\}(G)\downarrow$.

The set of integers (under some standard coding of $\boldsymbol{P}$ as integers): $\{p \in \boldsymbol{P} \mid p \Vdash \{e\}(G)\downarrow\}$ is RE and, by Gandy Selection, we can effectively select such a $p$. (The reader should verify that this set of conditions is RE – see Sacks [11].

Now consider the case of Kleene recursion in ${}^3\boldsymbol{E}$. Harrington [4] showed that

$$E(2^\omega) = L_{\aleph_1}{}^{{}^3\boldsymbol{E}}(2^\omega).$$

Let the 1-*section of* ${}^3\boldsymbol{E}$ be defined by:

$$1-\mathrm{sc}({}^3\boldsymbol{E}) = \{a \subsetneq \omega \mid a \leqq_{{}^3\boldsymbol{E}} 2^\omega\}.$$

If every real is constructable, then

$$L_{\aleph_1}{}^{3}\boldsymbol{E}(2^\omega) = L_{\aleph_1}{}^{{}^3\boldsymbol{E}}$$

and a natural question is whether a real $b$ Cohen-generic/$L_{\aleph_1}{}^{{}^3\boldsymbol{E}}$ satisfies: $b \in 1-\mathrm{sc}({}^3\boldsymbol{E})$ in $L_{\aleph_1}{}^{{}^3\boldsymbol{E}}[b]$.

Sacks showed that such a real computes no more ordinals than $\varnothing$ in the ground model. A result of Lévy [8] will allow us to answer this question negatively in a strong sense.

DEFINITION. If $\boldsymbol{P}$ is a poset we say that $\boldsymbol{P}$ is *semi-homogeneous* iff $\forall p, p' \in \boldsymbol{P}$ there exists an automorphism of $\boldsymbol{P}$ $\pi:\boldsymbol{P}\to\boldsymbol{P}$ such that $\pi(p)$ and $p'$ are compatible (i.e. $Eq\in\boldsymbol{P}$ such that $q \leqq \pi(p)$ and $q \leqq p'$).

Using this condition on $\boldsymbol{P}$, Lévy shows the following remarkable result about generic extensions via $\boldsymbol{P}$.

THEOREM 4.2 (Lévy [8]). *Assume $\boldsymbol{P}$ is semi-homogeneous and let $M$ be a countable model of* ZF *with $\boldsymbol{P}\in M$. Let $G \subsetneq \boldsymbol{P}$ be $\boldsymbol{P}$-generic/$M$ and $N = M[G]$. Then we have that for every $x\in N$ and $y\in M$*:

$$x\in[\mathrm{HOD}(y)]^N \to x\in M.$$

REMARK. HOD $(y)$ are those sets hereditarily ordinal definable from $y$. A closer look at Lévy's proof reveals that the same ordinal parameters suffice to define $x$ in $M$ as did in $N$. The proofs of Lévy's result is a transfinite induction on rank (see Lévy [8]).

LEMMA 4.3. *Let $\boldsymbol{P}$ be the Cohen poset for adding a real, then $\boldsymbol{P}$ is semi-homogeneous.*

PROOF. $\boldsymbol{P} = \{f:\omega\to\{0,1\} \mid f \text{ partial with finite domain}\}$ so given $p,p'\in\boldsymbol{P}$: if $p$ and $p'$ are compatible, the identity automorphism will suffice. Otherwise let

$$B = \{n\in\omega \mid n\in\mathrm{dom}\,(p)\cap\mathrm{dom}\,(p') \text{ and } p(n)\neq p'(n)\}$$

and consider the case where $B=\{n_0\}$ (the general case is similar). Let $m = \max\,(\mathrm{dom}\,(p), \mathrm{dom}\,(p'))$ and define a permutation $\rho:\omega\Leftrightarrow\omega$ by $z\in\omega$

$$\rho(z) = \begin{cases} m+1, & \text{if } z = n_0 \\ n_0, & \text{if } z = m+1 \\ z, & \text{otherwise.} \end{cases}$$

Then $\rho$ induces an automorphism $\pi:\boldsymbol{P}\to\boldsymbol{P}$ given by: $q\in\boldsymbol{P}$

$$\mathrm{dom}\,(\pi(q)) = \{\rho(n) \mid n\in\mathrm{dom}\,(q)\}$$

and for $z\in\mathrm{dom}\,(\pi(q))$ we let $\pi(q)(z) = q(\rho^{-1}(z))$. Then, if we consider $n_0$ above, we have

$$n_0 \notin \operatorname{dom}(\pi(p))$$

and

$$\pi(p)(\rho(n_0)) = p(\rho^{-1}(\rho(n_0))) = p(n_0)$$

and so $\pi(p)$ and $p'$ are compatible with extension $q = \pi(p) \cup p'$.

Thus if we force with this $\boldsymbol{P}$ over $L$, the following fact shows that there is no hope of extending $1 - \mathrm{sc}\,({}^3\boldsymbol{E})$.

FACT 4.4. Let $M$ be a transitive model of ZF and let $X \in (k - \mathrm{sc}\,({}^{k+1}\boldsymbol{E}))$, then $X \in \mathrm{HOD}^M$. To see this notice that for any $n$, type$(n)$ is definable in $M$.

Combining these results we can now show

THEOREM 4.5. *Let $\boldsymbol{P}$ be the Cohen poset for adding a real to L and let $a \leqq \omega$ be $\boldsymbol{P}$-generic/$L$, then*

$$(1 - \mathrm{sc}\,({}^3\boldsymbol{E}))^{L[a]} = (1 - \mathrm{sc}\,({}^3\boldsymbol{E}))^L.$$

PROOF. Assume that $b \in (2^\omega)^{L[a]}$ and suppose that $b \leqq_{{}^3\boldsymbol{E}} 2^\omega$ in $L[a]$, then $b \in \mathrm{OD}^{L[a]}$ and since $b \subsetneq \omega$, we have that $b \in \mathrm{HOD}^{L[a]}$. By Theorem 4.2, $b \in L$, contradicting the choice of $b$. If $b \in L$ such that

$$b \in (1 - \mathrm{sc}({}^3\boldsymbol{E}))^{L[a]},$$

then by Lemma 4.0, $b \leqq_{{}^3\boldsymbol{E}} 2^\omega$, for some $\gamma < (\aleph_0^{2^\omega})^L$ and by the remark following Theorem 4.2, we have $b \leqq_{{}^3\boldsymbol{E}} 2^\omega, \gamma$ in $L$, as desired.

## 5. ${}^3\boldsymbol{E}$-degrees of reals.

We will use Lévy's result to show that the well-foundedness of the set of degrees of reals modulo ${}^3\boldsymbol{E}$ under the induced ordering is independent of ZF. This answers a question of Normann and also one of Sacks concerning the relative computability of mutually Cohen generic reals.

DEFINITION. If $a \subsetneq \omega$, then the *degree of a* mod ${}^3\boldsymbol{E}$ is

$$[a]_{{}^3\boldsymbol{E}} = \{b \subsetneq \omega \mid a \leqq_{{}^3\boldsymbol{E}} b,\ b \leqq_{{}^3\boldsymbol{E}} a\}$$

and $\mathscr{D}({}^3\boldsymbol{E}) = \{[a]_{{}^3\boldsymbol{E}} \mid a \subsetneq \omega\}$. Therefore $[a]_{{}^3\boldsymbol{E}}[b]_{{}^3\boldsymbol{E}} \in \mathscr{D}({}^3\boldsymbol{E})$:

$$[a]_{{}^3\boldsymbol{E}} \leqq [b]_{{}^3\boldsymbol{E}} \text{ iff } \exists a_0 \in [a]_{{}^3\boldsymbol{E}} \exists b_0 \in [b]_{{}^3\boldsymbol{E}}$$

such that $a_0 \leqq_{{}^3\boldsymbol{E}} b_0$.

PROPOSITION 5.0. $(V = L)$ *$\langle \mathscr{D}({}^3\boldsymbol{E}), \leqq \rangle$ is well-founded.*

PROOF. Let $\leqq_L$ denote the well-ordering of $L$, then $\leqq_L \restriction (2^\omega)^L$ is recursive in ${}^3\boldsymbol{E}$, $(2^\omega)^L$. Given $a \in (2^\omega)^L$ we can effectively compute $|a|_{\leqq_L}$, the height of a in the well-ordering, and a counting of $|a|_{\leqq_L}$. Thus for every $b \in (2^\omega)^L$ with $b \leqq_L a$, $b$ is recursive in $^3E$, $(2^\omega)^L$ and some integer ($b$'s place in the counting of $|a|_{\leqq_L}$). This shows that in $L$, the degree ordering follows $\leqq_L$ and is therefore well-founded.

COROLLARY 5.1. Con (ZF) $\rightarrow$ Con (ZF) + $\langle \mathscr{D}(^3E), \leqq \rangle$ *is well-founded*).

We will now show that the mildest possible extension of $L$ adding reals, namely adding a single Cohen real, yields an infinite descending path through this ordering.

THEOREM 5.2. *Let $M$ be a countable, transitive model of* ZF + $V = L$ *and let $a \subsetneq \omega$ be Cohen-generic/$M$, then*

$$M[a] \models \text{“}\langle \mathscr{D}(^3E), \leqq \rangle \textit{ is not well-founded”}.$$

PROOF. $M$ fulfils the condition of Lévy's theorem and the Cohen poset for adding a real is semi-homogeneous as we have shown. Define the following splitting of the Cohen real $a$:

$$a_{0,0} = \text{even part of } a$$
$$a_{0,1} = \text{odd part of } a$$

and in general at stage $n$:

$$a_{n+1,0} = \text{even part of } a_{n,0}$$
$$a_{n+1,/} = \text{odd part of } a_{n,0}.$$

A standard argument shows that $\forall n\,[a_{n,0}$ and $a_{n,1}$ are mutually Cohen generic]. By Lévy's result we have in $L[a]: \forall n \in \omega$

$$a_{n,0} \nleqq_{^3E} a_{n,1}$$

and

$$a_{n,1} \nleqq_{^3E} a_{n,0}.$$

As a result $\forall i \in \omega\,[a_{0,i+1} \nleqq_{^3E} a_{0,i}]$ and $a_{0,0} \nleqq_{^3E} a$. The sequence $\{a_{0,i} | i \in \omega] \in N$ and hence

$$N = \text{“}\langle \mathscr{D}(^3E), \leqq \rangle \text{ is not well-founded”}.$$

**6. Extending the $1 - \mathrm{sc}(^3E)$.**

Recall that the extension via a Cohen real $a$ in the previous section satisfies $(\aleph_1^{^3E})^L = (\aleph_1^{^3E})^{L[a]}$. If we are willing to give up this constraint we can extend the $1 - \mathrm{sc}(^3E)$ by forcing over a well-known partially ordered set.

THEOREM 6.0. *Let $M$ be a countable, transitive model of* ZF + $V = L$ *and let $a \subsetneq \omega$ be* Col $(\omega, \aleph_1)$ *is the Lévy poset for collapsing $\aleph_1$ to $\omega$.) Then*

$$(1 - \mathrm{sc}(^3\boldsymbol{E})^{M[a]}.$$

PROOF. Define the complete set of integers relative to $^3\boldsymbol{E}$ by

$$C = \{\langle e,m\rangle \mid \{e\}(^3\boldsymbol{E},m)\downarrow\},$$

then $C \in L_{\varkappa_1{}^3\boldsymbol{E}}$ but $C \notin 1 - \mathrm{sc}(^3\boldsymbol{E})$ in $L$. In $M[a]$, $q_1^L$ is recursive in $^3\boldsymbol{E}, 2^\omega$ and therefore $(\varkappa_1{}^3\boldsymbol{E})^L \leqq_{^3\boldsymbol{E}} 2^\omega$ in $M[a]$. Thus, if we denote by $C^M$ the interpretation of $C$ in $M$, then using $(\varkappa_1{}^3\boldsymbol{E})^L$, $C^M$ is recursive in $^3\boldsymbol{E}, 2^\omega$ in $M[a]$, i.e.

$$C^M \in (1 - \mathrm{sc}(^3\boldsymbol{E}))^{M[a]},$$

as desired.

A reasonable question is whether we can extend the $1 - \mathrm{sc}(^3\boldsymbol{E})$ as above without violating $\varkappa_1{}^3\boldsymbol{E}$ of the ground model. In the next section we provide such an example.

## 7. Jensen-Johnsbråten reals and $1 - \mathrm{sc}(^3E)$.

Here we consider a forcing extension preserving $\varkappa_1{}^3\boldsymbol{E}$; but extending the $1 - \mathrm{sc}(^3\boldsymbol{E})$.

The relevant theorem is an improvement of Solovay's result [14] (that it is consistent with ZF to assume that there is non-constructable $\Delta_3^1$ subset of $\omega$ by Jensen-Johnsbråten [7].

THEOREM 7.0 (Jensen-Johnsbråten [7]). *There exists a $\pi_2^1$ formula $\varphi$ such that the following are provable in* ZF:

(a) $\varphi(x) \to x \subsetneq \omega$;
(b) $V = L \to \exists x \varphi(x)$
(c) $\omega_1^L = \omega_1 \to (\exists \leqq 1) x \varnothing(x)$
(d) $\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZF} + \mathrm{GCH} + \omega_1^L = \omega_1 + \exists a(\varphi(a) V = L[a]))$;
(e) *If* $M \models \mathrm{ZFC} + \omega_1^L = \omega_1 + \varphi(a)$ *and $N$ is a cardinal preserving extension of $M$, then* $N \models \varphi(a)$.

If $\{a\} \in \pi_2^1$ (i.e. $a$ is implicitly $\pi_2^1$-definable), then $a \in \Delta_2^1$. It is this definability ($a \in \Delta_2^1$ clearly implies that $a \leqq_{^3\boldsymbol{E}} \varnothing$) and the chain condition on the necessary iterated forcing that gives the desired result. For the proof of Theorem 7.0, consult Jensen-Johnsbråten [7] or Devlin-Johnsbråten [1].

THEOREM 7.1. *There is a countable chain condition* (c.c.c) *iterated forcing (set forcing) $\boldsymbol{P}_\omega$ such that if $G$ is $\boldsymbol{P}_\omega$-generic$/L_{\varkappa_1{}^3\boldsymbol{E}}$, then*

(i) $1-\mathrm{sc}^{L}({}^{3}E) \subsetneqq 1-\mathrm{sc}^{L[G]}({}^{3}E)$; *and*
(ii) $L_{\aleph_1}{}^{3}{}_{E}[G]$ *is* E-*closed.*

PROOF. Jensen-Johnsbråten show that the necessary trees are $\Sigma_1(L_{\omega_1}L)$ and are hence recursive in ${}^{3}E$ in $L$. The real coding $\langle b_n | n\in\omega\rangle$ the sequence of branches through these $\omega$-many trees is $\Delta^1_3$ and also recursive in ${}^{3}E$, which gives (i).

(ii) follows from Theorem 3.0 and each stage in the iteration is c.c.c.. The iteration is given by

$$P_0 = T_0 \quad \text{(under the reverse ordering)}$$
$$P_{n+1} = T^*_{n+1} \text{ over } M_{n+1} = L[\langle b_0, \dots, b_{n+1}\rangle],$$

then

$$P_\omega = \varinjlim \langle P_n | n\in\omega\rangle.$$

Each $P_n$ is c.c.c. and hence the direct limit is also c.c.c.. The desired model is the result of forcing with the direct limit iterated forcing.

## 8. Almost disjoint codes and $1-\mathrm{sc}({}^{k+2}E)$.

We consider here the effect of adding reals which are almost disjoint codes for subsets of $\aleph_1$ upon the $1-\mathrm{sc}({}^{3}E)$ as a characteristic case. First we give a brief outline of this notion of forcing.

Let $\mathfrak{A} = \{A_\alpha | \alpha < \omega_1\}$ be a family of almost disjoint subsets of $\omega$ and let $X \subsetneq \omega_1$. Define $P_{\mathfrak{A},X}$ as follows:

A *condition* is a function from a subset of $\omega$ into $\{0,1\}$ such that

(i) $\mathrm{dom}(p) \cap A_\alpha$ is finite for every $\alpha\in X$;
(ii) $\{n\, p(n) = 1\}$ is finite.

The set $P_{\mathfrak{A},X}$ is partially ordered by inverse inclusion: $P \leqq q$ iff $p$ extends $q$. If $p$ and $q$ are incompatible, then

$$\{n | p(n) = 1\} \neq \{n | q(n) = 1\}$$

and so $P_{\mathfrak{A},X}$ satisfies the c.c.c. Thus if $P_{\mathfrak{A},X}\in L_{\aleph_1}{}^{3}E$ and $f\colon \omega \to \{0,1\}$ is $P_{\mathfrak{A},X}$-generic/$L$, then $L_{\aleph_1}{}^{3}Ef$ is E-closed by Sacks (see Slaman [13]).

This example of a generic cannot extend $1-\mathrm{sc}({}^{k+2}E)$, $k \geqq 1$.

THEOREM 8.0. *Suppose* $P_{\mathfrak{A},X}\in L_{\aleph_1}{}^{k+2}E$ *and* $f$ *is* $P_{\mathfrak{A},X}$-generic/$L_{\aleph_1}{}^{k+2}E$, *then*

$$f \notin (1-\mathrm{sc}({}^{k+2}E))^{L[f]}.$$

PROOF. We consider the case $k = 1$ and $X \subsetneq \aleph_1^L$ for simplicity. As before we use the result of Lévy and Fact 4.4.

Suppose that $f \leqq_{^3E} \varnothing$, then $f \in \mathrm{OD}^N$. Since $f \subsetneq \omega$, $f$ is an element of $\mathrm{HOD}^N$. All that remains is to show that $\boldsymbol{P}_{\mathfrak{A},X}$ satisfies the hypothesis of Lévy's theorem.

LEMMA 8.1. *The poset* $\boldsymbol{P}_{\mathfrak{A},X}$ *for almost disjoint coding is semi-homogeneous.*

PROOF. We can view two conditions as

$$p = \langle k, (A_1, \ldots, A_n)\rangle,\ p' = \langle h, (B_1, \ldots, B_m)\rangle,$$

where $k$ and $h$ are finite subsets of $\omega$ and the $A_i$ and $B_j$ ($i \leqq n$, $j \leqq m$) are finite subsets of $\{A_\alpha \mid \alpha \in X\}$.

We find a permutation $p : \mathsf{N} \to \mathsf{N}$ as follows: let

$$A = \bigcup_{i \leqq n} A_i \quad \text{and} \quad B = \bigcup_{j \leqq m} B_j,$$

then

$$\begin{aligned} &x \in k \to p(x) \in h \ \vee \ p(x) \notin B \\ &x \in H \to p^{-1}(x) \in k \ \vee \ p^{-1}(x) \notin A. \end{aligned}$$

Let $s_0 < s_1 < s_2$ be integers such that

(i) $x \in k \cup h \Rightarrow x < s_0$;
(ii) $\overline{\overline{[s_0,s_1) \setminus B}} \geqq \overline{\overline{k}}$
(iii) $\overline{\overline{[s_1,s_2) \setminus A}} \geqq \overline{\overline{h}}$.

Define as follows: $x \geqq s_2$, let $p(x) = x$ thus $\rho$ will be a permutation on $[O,s_2)$:

$$\left.\begin{aligned} &x \in k \cap h, \ \text{let } p(x) = x \\ &x \in k \setminus h, \ \text{let } p(x) \in [s_0,s_1) \setminus B \\ &x \in h \setminus k, \ \text{let } p^{-1}(x) \in [s_1,s_2) \setminus A \end{aligned}\right\} pH.$$

By taking $=$ in (ii) and (iii) above $p$ gives a permutation.

To define the automorphism $\pi : \boldsymbol{P} \to \boldsymbol{P}$ take $\pi(p)$ for $p = \langle k,A\rangle$ to be $\langle \rho(k), \rho(A)\rangle$ where $\rho(k) = \{\rho(n) \mid n \in k\}$ and $\rho(A) = \{\{\rho(n) \mid n \in b\} \mid b \in A\}$. Thus by Lemma 2, $f \in M$ which is absurd, since $f$was taken $\boldsymbol{P}_{\mathfrak{A},X}$-generic/$M$.

COROLLARY 8.2. *If we take $\boldsymbol{P}_{\mathfrak{A},X}$ to be the generalization of almost disjoint codes to regular $\varkappa$ over $L$ by taking the appropriate family $\mathfrak{A}$ where $\varkappa = \aleph_n$, $\exists n \in \omega$ where $X \subsetneq$ type $(n)$ and $G$ is $\boldsymbol{P}_{\mathfrak{A},X}$-generic/$L$ with $\boldsymbol{P}_{\mathfrak{A},X} \in L_{\varkappa_1}{}^{n+2}\boldsymbol{E}$ then*

$$G \notin (n - \mathrm{sc}({}^{n+2}\boldsymbol{E}))^{L[G]}.$$

PROOF. Using the fact that $L$ is the ground model and every element of $L$ is $\mathrm{HOD}^{L[G]}$, so if $G \in \mathrm{OD}^{L[G]}$, then $G \in \mathrm{HOD}^{L[G]}$.

The argument that this $\boldsymbol{P}_{\mathfrak{A},X}$ satisfies semi-homogeneity is suitably altered to handle the limit ordinals involved. The argument that *G preserves* E-closure uses Theorem 3.0 and selection over type $(n-1)$.

Until now we have been primarily concerned with 1-sections. In the next section we study $n$-sections for $n > 1$ for the Kleene functionals ${}^{k+2}\boldsymbol{E}$ for $k \geqq 2$. The $2 - \mathrm{sc}({}^3\boldsymbol{E})$ is determined completely by the reals and thus cannot be extended without adding new reals.

## 9. Extending the $2 - \mathrm{sc}({}^4E)$.

We shall argue here that we can by forcing add an element of the

$$2 - \mathrm{sc}({}^4\boldsymbol{E}) = \{X \subsetneq 2^\omega \mid X \text{ is recursive in } {}^4\boldsymbol{E}\}$$

over $L$ without violating $\varkappa_1\ {}^4\boldsymbol{E}$. The techniques involved had to confront the obstacle posed by Lévy's result concerning posets satisfying semi-homogeneity which states that forcing with such a poset cannot add new elements of HOD($x$) for any ground model set $x$.

The natural solution here is to resort to a poset $\boldsymbol{P}$ which has the identity as its only automorphism. We force over the rigid Souslin tree constructed by Jensen [5] in $L$ and using his methods for showing that the resulting $\omega_1$-tree is Souslin we show that the only $\omega_1$-path in the extension is the generic path. This yields the definability required for arguing that this path (viewed as a subset of $(2^\omega)^L$ via $<_L$) is recursive in ${}^4\boldsymbol{E}$.

If we work over $L$, then if we force with a semi-homogeneous poset $\boldsymbol{P}$, Lévy's result and the lemma show that there is no hope of extending the $2 - \mathrm{sc}({}^4\boldsymbol{E})$ without adding new reals (and hence having done so trivially). To see this suppose $N$ is such a generic extension of $L$ and

$$X \in (2 - \mathrm{sc}({}^4\boldsymbol{E}))^N.$$

Then $X \in \mathrm{OD}^N$ and if no new reals were added in forcing over $L$, we would have that $X \in \mathrm{HOD}^N$. By Lévy and semi-homogeneity $X \in L$ and definable in the same ordinal parameters, hence

$$X \in (2 - \mathrm{sc}(^4\boldsymbol{E}))^L.$$

FACT. If $\boldsymbol{P}$ is a notion of forcing such that the only automorphism is the identity, then $\boldsymbol{P}$ does not satisfy semi-homogeneity (just take $p$ and $q$ in $\boldsymbol{P}$ incompatible).

The following theorem of Jensen (see Devlin-Johnsbråten [1] gives us the required notion of forcing for extending the $2 - \mathrm{sc}(^4\boldsymbol{E})$.

DEFINITION. A partially ordered set $X = \langle X, \leqq \rangle$ is rigid, if id $\restriction X$ is the only automorphism on $X$.

THEOREM 9.0 (Jensen [6]). *Assume* $\diamondsuit$. *Then there exists a rigid Souslin tree.*

For our purposes work in $L$, then $\diamondsuit$ holds and there exists a rigid Souslin tree $T$, which is in fact $\Sigma_1(L_{\omega_1})$ and hence recursive in $^4\boldsymbol{E}, 2^{2^\omega}$ in $L$. Viewing $T$ as its coding

$$T \in (2 - \mathrm{sc}^4\boldsymbol{E}))^L,$$

so let us consider the result of forcing with the poset corresponding to $T$ over $L$ (we also use $T$ to refer to the Souslin algebra derived from $T$). $T$ satisfies the c.c.c. so if $G$ is $T$-generic/$L$, then $L[G]$ is a cardinal and cofinality preserving extension of $L$. By the following lemma we have a bit more.

LEMMA 9.1. *If $G$ is $T$-generic/$L$, then*

$$(2^\omega)^{L[G]} = (2^\omega)^L.$$

PROOF. Suppose not and let $\dot{f}: \omega \to \omega$ be a term for a real $f \in (2^\omega)^{L[G]} \setminus (2^\omega)^L$. In $L[G]$ consider the following map defined by induction on $\omega: n \to p_n$ given by $p_0 =$ least $p \in G$ such that $\exists m \in \omega$ with $p \Vdash \dot{f}(0) = m$; given $p_0, \dots, p_n$ let $p_{n+1} =$ least $y \in G$ such that $q \leqq p_n$ and $\exists m\ q \Vdash \dot{f}(n+1) = m$.

CLAIM. $F: \omega \to \omega_1$ defined by $F(n) = \bigcup \mathrm{dom}(p)$ is unbounded in $\omega_1$.

PROOF. Otherwise $\exists \delta < \omega_1$ such that $\bigcup_{n \in \omega} F(n) \leqq \delta$. But then $f: \omega \to \omega$ is definable from $G \restriction \delta + 1 \in L$ contradicting the choice of $f$.

Clean $F$ up by taking $F': \omega \to \omega_1$ and let $F'(n) = \alpha_n$. Each $\alpha_n$ is countable via some $a_n \in \mathrm{WO}$ and letting a code the family $\{a_n\}_{n \in \omega}$ in a standard way we get in $L[G]$ $g: \omega \leftrightarrow \omega_1$ contradicting the fact that $L[G]$ was a cardinal preserving extension of $L$.

REMARK. Thus no new reals are added and if we can show that $\bigcup G$ is definable from $T$ in $L[G]$, then the following theorem, giving the uniqueness of $\cup G$ as a path, will yield the desired non-trivial extension of $2-\mathrm{sc}({}^4\boldsymbol{E})$ in $L[G]$.

THEOREM 9.2. *Let $G$ be $T$-generic/$L$, then $\cup G$ is the only branch through $T$ in $L[G]$.*

PROOF. Suppose not and let $b \in L[G]$ be a branch through $T$ such that $b \neq \cup G$. Then there exists an $\alpha < \omega_1$ such that $b(\alpha) \neq \cup G(\alpha)$, take the least such $\alpha_0$. Let $\tau$ be a term in LST such that for $\vec{\alpha}$ a finite vector of ordinals: $\tau^{L[G]}(\vec{\alpha}, G) = b$ (take the least such in the sense of $<_L$). By the same argument showing that no new reals are added we have that $(\forall \alpha < \omega_1)(b \restriction \alpha \in L)$ and $\exists \beta < \omega_1$ such that $b \restriction \alpha_0 + 1 \in L_\beta$.

The term $t \in L_{\omega_2}$, so proceed now as in the proof of rigidity including $\tau$ and $\alpha_0 + 1$ in the chain of elementary substructures used in Devlin-Johnsbråten [7].

COROLLARY 9.3. *If we denote by $\{a_\gamma | \gamma < \omega_1\}$ the well-ordering of $(2^\omega)^L$ and $G^* = \{a_\gamma | \gamma \in G\}$ and $G$ is $T$-generic/$L$, then*

$$G^* \in 2 - \mathrm{sc}({}^4\boldsymbol{E}).$$

PROOF. The predicate

$$\varphi(T,x) \equiv x \text{ is a path through } T$$

is recursive in ${}^4\boldsymbol{E}$ (using $\omega_1 \leqq_{{}^4\boldsymbol{E}} \varnothing$) and hence, so is the set

$$\{x | (T,x)\} = \{G\}$$

by the above theorem. Again using the well-ordering of $(2^\omega)^L$ recursive in ${}^4\boldsymbol{E}$ we compute $G^*$ from $G$.

## 10. Extending the $k - \mathrm{sc}({}^{k+2}\boldsymbol{E})$.

In this section we generalize the methods used to extend the $2 - \mathrm{sc}({}^4\boldsymbol{E})$ to all finite types. We modify the proof of Jensen [6] that there exists a rigid Souslin tree in $L$ to prove the existence of a rigid $\varkappa$-tree which is $\varkappa$-Souslin in $L$. We then force over that tree preserving $\varkappa_1$ ${}^{k+2}\boldsymbol{E}$ for the appropriate $k$. Using the definability of the resulting $\varkappa$-branch (actually its uniqueness in the extension) we conclude that it is recursive in ${}^{k+2}\boldsymbol{E}$, $\varnothing$ and hence clearly extends the $k - \mathrm{sc}({}^{k+2}\boldsymbol{E})$. Throughout we consider the case of the $3 - \mathrm{sc}({}^5\boldsymbol{E})$. The generalization to all finite types is straightforward. We show that the extension of the section is non-trivial by showing that we add no new sets of lower type.

*$\omega_2$-trees which are $\omega_2$-Souslin.* In Jensen [6] one constructs $\omega_2$-trees which are $\omega_2$-Souslin, but the resulting tree is not obviously rigid. We modify that construction here using the main idea of the proof as presented in Devlin-Johnsbråten [1] to produce an $\omega_2$-Souslin tree which is rigid and later use the strategy for showing that the tree is rigid to argue that forcing over that tree yields a model in which there is only one branch. We include a proof for those uninterested in Souslin trees, but curious about the coding.

THEOREM 10.0 ($V = L$). *There exists an $\omega_2$-tree which is $\omega_2$-Souslin and rigid.*

PROOF. Let $\langle S_\alpha | \alpha < \omega_2 \rangle$ be the sequence given by $\diamondsuit$ in $L$. We wish to construct a Souslin tree $T$. The points of $T$ will be ordinals less than $\omega_2$. We shall construct $T$ in stages $T_\alpha$ ($q \leqq \alpha < \omega_2$), where $T_\alpha$ is to be the restriction of $T$ to points of rank $< \alpha$. Hence $T_\alpha$ will be a normal tree of length $\alpha$ and $T_\beta$ will be an end extension of $T_\alpha$ for $\beta > \alpha$. We define $T$ by induction on $\alpha$ as follows.

CASE 1. $\alpha = 1$, $T_1 = \{0\}$.

CASE 2. $T_{\alpha+1}$ is defined. Define $T_{\alpha+2}$ by appointing to immediate successors for each maximal point of $T_{\alpha+1}$.

CASE 3. lim ($\alpha$) and $T_\nu$ is defined for $\nu < \alpha$. Set $T_\alpha = \bigcup_{\nu > \alpha} T_\nu$.

CASE 4. lim ($\alpha$) and $T_\alpha$ is defined. We must define $T_{\alpha+1}$.

If cf ($\alpha$) $= \omega$ then define $T_{\alpha+1}$ by appointing a successor for each maximal point of $T_\alpha$. Our work is to be done at $\alpha$ such that cf ($\alpha$) $= \omega_1$. By induction on $\alpha < \omega_2$ let $\delta(\alpha)$ be the least ordinal $\delta > \alpha$ such that

(i) $L_\delta \prec L_{\omega_2}$, and
(ii) $\langle \delta(\nu) | \nu < \alpha \rangle \in L_\delta$, and

set $M_\alpha = L_\delta(\alpha)$. Then $M_\alpha$ has size $\leqq \aleph_1$ for $\alpha < \omega_2$. If $\alpha < \omega_2$ and lim ($\alpha$) and cf ($\alpha$) $= \omega_1$, assume that $T_\alpha \in M_\alpha$.

To define $T_{\alpha+1}$ we force over $M_\alpha$ with $\boldsymbol{P} = \langle \boldsymbol{P} \leqq_{\boldsymbol{P}} \rangle \in M_\alpha$ given by

$$\boldsymbol{P} = \{p | \exists a \leqq \omega_1 \wedge p : a \rightarrow T_\alpha)\}$$

with $p \leqq_{\boldsymbol{P}} q \leftrightarrow \operatorname{dom}(p) \supsetneqq \operatorname{dom}(q) \ \wedge \ \forall \alpha \in \operatorname{dom}(q)$.

Notice that $M_{\alpha+1} \models \bar{\bar{M}} = \aleph_1$ and also $M_\alpha \models$ '$\boldsymbol{P}$ is countably closed'. Let $G \subsetneqq \boldsymbol{P}$ be the $<_L$-least $\boldsymbol{P}$-generic/$M_\alpha$ set. Since

$$M_{\alpha+1} \dashv L_{\omega_2} \text{ and } M_\alpha \in M_{\alpha+1},$$

and $\exists f \in M_{\alpha+1} f: \omega_1 \leftrightarrow M_\alpha$ and since $\boldsymbol{P}$ is countably closed generics exist in $L_{\omega_2}$ and by elementarity also in $M_{\alpha+1}$. Hence $T_\beta \in M_\beta$ for $\lim(\beta)$ will be trivial.

For $\gamma < \omega_1$, let

$$b_\gamma = \{p_\gamma | p \in G\}.$$

CLAIM. (i) Each $b_\gamma$ is an $\alpha$-branch of $T_\alpha$;
(ii) each $b_\gamma$ is $T_\alpha$-generic/$M_\alpha$;
(iii) $b_\gamma \neq b_\delta$ for $\gamma \neq \delta$ less than $\omega_1$;
(iv) if $\alpha_1, \ldots, \alpha_n$ are distinct, then $b_{\alpha_1} \times \ldots \times b_{\alpha_n}$ is $(T_\alpha)^n$-generic/$M_\alpha$;
(v) $T_\alpha = \bigcup_{\alpha < \omega_1} b_\alpha$.

PROOF. (i), (ii), and (iii) follow easily from (iv): Let $\alpha_1, \ldots, \alpha_n$ be distinct ordinals less than $\omega_1$, and let $D \subsetneq (T_\alpha)^n$ be dense and closed under extensions. Let

$$D^* = \{p \in \boldsymbol{P} | \langle p_{\alpha_1}, \ldots, p_{\alpha_n} \rangle \in D\},$$

then $D^*$ is dense in $\boldsymbol{P}$ so let $p \in G \cap D^*$. By the choice of $p$

$$\langle p_{\alpha_1}, \ldots, p_{\alpha_n} \rangle \in b_{\alpha_1} \times \ldots \times b_{\alpha_n} \cap D$$

as desired. To see (v), let $\sigma \in T_\alpha$ and define

$$D' = \{p \in \boldsymbol{P} | \exists \gamma \in \operatorname{dom}(p)(p_\gamma \supsetneq \sigma)\},$$

then $D'$ is dense in $\boldsymbol{P}$ so let $p \in G \cap D'$.

Then $\exists \gamma \in \operatorname{dom}(p)$ such that $\sigma \subsetneq p_\gamma \in b_\gamma$ and so $\sigma \in b_\gamma$.

Now set $T_{\alpha+1} = \{\cup b_\alpha | \alpha < \omega_1\}$, then by (v), $T|(\alpha+1)$ is still normal and so $T = \bigcup_{\alpha < \omega_2} T_\alpha$ is a normal tree of length $\omega_2$.

CLAIM. $T$ is $\omega_2$-Souslin.

PROOF. It suffices to show that $T$ has no $\omega_2$-antichains so let $X \subsetneq T$ be a maximal antichain. We show $\overline{\overline{X}} \leq \aleph_1$. Let $A$ be the set of limit $\alpha < \omega_2$ such that $X \cap \alpha$ is a maximal antichain in $T_\alpha$. $A$ is club in $\omega_2$.

Now let $\varkappa_\gamma = \mathrm{OR} \cap M_\gamma$, for $\gamma < \omega_2$. $E = \{\alpha_\gamma | \gamma < \omega_2\}$ is also club in $\omega_2$, hence there exists $\alpha \in A \cap E$ such that $S_\alpha = X \cap \alpha$. By the construction of $T_{\alpha+1}$, then we have:

Every $X$ of level $\alpha$ lies above an element of $X \cap \alpha$. Hence $X \cap \alpha$ is a maximal antichain in $T$ and $X = X \cap \alpha$ has cardiality $<\aleph_2$.

The proof that $T$ is rigid proceeds as in Jensen's proof for the rigid $\omega_1$-Souslin tree.

REMARK. (i) Obvious modifications show that with $\diamondsuit_{\aleph_n}$ we construct a rigid $\aleph_n$-Souslin tree.

(ii) $T$ has the $\aleph_2$-c.c. by the above. By the construction at cf $(\alpha) = \omega$ stages and the fact that $\boldsymbol{P}$ at cf $(\alpha) = \omega_1$ stages was countably closed, $T$ itself is countably closed. For $\varkappa$ as in (i) equal to $\aleph_n$ for $n \geqq 2$ $T$ will have the $\varkappa$-c.c. and be $\aleph_{n-2}$-closed. This fact will prove indispensable.

(iii) It is an interesting question whether $\diamondsuit$ is enough to produce a $\varkappa$-Souslin tree for all $\varkappa$ not Mahlo. Jensen does so using $\square$.

**11. Forcing with rigid $\omega_2$-Souslin trees.**

We will work over $L_{\varkappa_1}{}^5\boldsymbol{E}$ and force with the $\omega_2$-Souslin tree constructed in the previous section to extend non-trivially the $3-\mathrm{sc}(^5\boldsymbol{E})$. The tree $T$ is recursive in $^5\boldsymbol{E}$, $\varnothing$ since $T \in \Sigma_1(L_{\omega_2})$. Let $G$ be $T$-generic, then the theorem guarantees that $\cup G$ preserves the E-closure of $L_{\varkappa_1}{}^5\boldsymbol{E}$ and more.

We shall argue that $G \leqq {}^5\boldsymbol{E}\ \varnothing$ on $L_{\varkappa_1}{}^5\boldsymbol{E}[G]$ by showing that $G$ is the only path through $T$ in $L_{\varkappa_1}{}^5\boldsymbol{E}[G]$.

THEOREM 11.0. *If $G$ is $T$-generic/$L$, then $\cup G$ is the only branch through $T$ in $L[G]$.*

PROOF. Suppose not and let $b \in [T]$ in $L[G]$ such that $n \neq \cup G$. Then as before there exists a term $\tau \in L_{\omega_3}$ such that $\tau^{L[G]} = b$, where $\tau$ depends on $G$ and finitely many ordinal parameters. There also exists a $p \in G$ such that

$$p \Vdash \text{`}\tau \text{ is a branch through } T \text{ different from } G\text{'}.$$

Now argue as in Jensen's proof of rigidity that, at some stage $\alpha < \omega_2$ in the construction, $\tau$ gives a branch through $T_\alpha$ different from $G \restriction_\alpha$ and that $\tau \in M_\alpha[G \restriction_\alpha]$ but as branches we extended through the $^\alpha$th stage $\tau \times G \restriction_\alpha$ is $(T_\alpha)^2$-generic/$M_\alpha$ and hence by the product lemma $\tau \notin M_\alpha[G \restriction_\alpha]$, a contradiction.

COROLLARY 11.1. $\cup G \leqq_{^5\boldsymbol{E}} \varnothing$ *in* $L_{\varkappa_1}{}^5\boldsymbol{E}[G]$.

PROOF. $\cup G$ is the unique branch through $T$, $T \leqq_{^5\boldsymbol{E}} \varnothing$ and we test all such candidates.

COROLLARY 11.2.

$$(3-\mathrm{sc}(^5\boldsymbol{E}))^{L_{\varkappa_1}{}^5\boldsymbol{E}[G]} \nsubseteq L_{\varkappa_1}{}^5\boldsymbol{E}$$

*and hence the extension of* $3-\mathrm{sc}(^5\boldsymbol{E})$ *is achieved.*

PROOF. Interpret $\cup G$ as a subset of $(2^{2^\omega})^L$.

In order to argue that the extension of $3-\mathrm{sc}({}^5E)$ in non-trivial, the following lemma suffices.

LEMMA 11.3. *In* $L_{\aleph_1}$ ${}^5E[G]$

(i) $\aleph_1^L$ *is preserved.*

(ii) $\aleph_2^L$ *is preserved.*

PROOF. (i) follows from the construction of $T$ at $\lim(\alpha)$ with $\mathrm{cf}(\alpha)=\omega$ where we extended all branches and the fact that $P$ at $\lim(\alpha)$ with $\mathrm{cf}(\alpha) = \omega$, was countably closed. Hence $\aleph_1^L$ is preserved.

(ii) follows from $\aleph_2 -$ c.c. which $T$ satisfies.

Countable closure of $T$ insures that, in addition, no new reals are added. Thus a new subset of the reals would be a new subset of $\aleph_1^L$. The following argument shows that no new subsets of the reals are added and hence that we have extended $3-\mathrm{sc}({}^5E)$ non-trivially.

LEMMA 11.4. $(2^{\aleph_1})^L = (2^{\aleph_1})^{L[G]}$.

PROOF. Suppose not and let $X \subsetneq \aleph_1$ satisfy $X \in (2^{\aleph_1})^{L[G]} \setminus (2^{\aleph_1})^L$. We will show that $\aleph_2^L$ is collapsed in $L[G]$, giving a contradiction. By recursion on $\aleph_1$ define $f:\aleph_1 \to \aleph_2$ from $G$ in $L[G]: f(\gamma) = \mu p_0 \in G$ such that $p_0 \Vdash \dot{X} \subsetneq \aleph_1$ and $p_0 | 0 \in X$

$$f(\tau+1) = \mu p_{\tau+1} \leqq p_\tau$$

such that

$$p_{\tau+1} \in G \begin{cases} p_{\tau+1} \Vdash \tau+1 \in \dot{X} \text{ or} \\ p_{\tau+1} \Vdash \tau+1 \notin \dot{X}. \end{cases}$$

If $\tau$ is limit ordered and $f(\gamma)$ has been defined $\forall \gamma < \tau$ let

$$f(\tau) = \mu p_\tau \leqq \bigcup_{\gamma<\tau} p.$$

(Since $\tau$ is countable and $T$ is countable closed $\bigcup_{\gamma<\tau} p_\gamma \in T$) such that $P_\tau \in G$ and $p_\tau | \tau \in \dot{X}$. Now define $F:\aleph_1 \to \aleph_2$ by taking

$$F(\gamma) = \cup\, \mathrm{dom}\,(p\gamma)$$

$\gamma F^{11} \leqq \lambda < \aleph_2$, then $X \in L$ were done. Otherwise define $F':\aleph_1 \to \aleph_2$ by recursion from $F$. Placing together the collapses of ordinals less than $\aleph_2$ to $\aleph_1$ in the range of $F'$ yields a collapse of $\aleph_2^L$ in $L[G]$. a contradiction.

As remarked above a straightforward generalization gives a way of non-trivially extending the $k - \mathrm{sc}({}^{1+2}\boldsymbol{E})$ and as a result the $n - \mathrm{sc}({}^{k+2}\boldsymbol{E})$, for $1 < n \leqq k$. This is best possible since the $k+1 - \mathrm{sc}({}^{k+2}\boldsymbol{E})$ cannot be altered without changing the set of objects of type $(k)$.

**12. Forcing and reduction procedures.**

The evolution of the fundamental questions associated with developing "degree theory" and the priority method, in analogy with the classical development for recursion theory on the integers, is rich and we make no effort here to summarize it. The interested reader is directed to Sacks [11] for a thorough foundational discussion of, for example, reduction procedure and "parity" between parameter, argument and the associated computation all with respect to a fixed universe for computation. Even formulating $A \leqq_E B$ ($A$ is E-recursive in $B$) for $A$ and $B$ subsets of a computational universe $M$ (which is itself closed under computation) is not without its problems under demands of parity, if $M$ is not closed under computation *relative to B*.

The question we address here is the effect upon degree structure of bounded generic extension via a poset $\mathscr{P} = \langle P, \leqq \rangle$ satisfying an effectiveness condition typically used to prove that the extension preserves closure under computation in ${}^{k+2}\boldsymbol{E}, tp(k)$. We shall work in the setting of recursion in ${}^{3}\boldsymbol{E}$, but the argument is quite general and the perceptive reader is invited to provide the "most general" result.

DEFINITION. Let $\mathscr{P} = \langle \boldsymbol{P}, \leqq \rangle$ be a forcing poset with $\mathscr{P} \in L_{\aleph_1}{}^{3}\boldsymbol{E}$, then $\mathscr{P}$ is *effective* if

$$\{\langle p,e,t\rangle \mid p \Vdash^{*} \{e\}(t, {}^{3}\boldsymbol{E})\downarrow\}$$

is RE on $L_{\aleph_1}{}^{3}\boldsymbol{E}$.

We prove then

THEOREM 12.0. *Let $\mathscr{P} = \langle p, \leqq \rangle \in L_{\aleph_1}{}^{3}\boldsymbol{E}$ be effective and suppose that $A,B \subsetneq L_{\aleph_1}{}^{3}\boldsymbol{E}$ be such that $B$ is regular and hyperregular. If there exist $p \in P$, $e \in \mathrm{N}$ and $\underset{\sim}{a}$ a closed term such that for all $G$ $\mathscr{P}$-generic/$L_{\aleph_1}{}^{3}\boldsymbol{E}$ with respect to ranked formula such that $p \in G$*

$$A \leqq_{{}^{3}\boldsymbol{E}} B \text{ on } L_{\aleph_1}{}^{3}\boldsymbol{E}[G] \text{ via } (\underset{\sim}{a})^{L_{\aleph_1}{}^{3}\boldsymbol{E}[G]},$$

*then*

$$A \leqq_{{}^{3}\boldsymbol{E}} B \text{ on } L_{\aleph_1}{}^{3}\boldsymbol{E} \text{ via } e,\underset{\sim}{a},\mathscr{P},p.$$

PROOF. We prove the result for $B = \varnothing$ (the general result is obtained by relativizing the argument to $B$ using the assumption that $B$ is regular and hyperregular).

Thus $p \Vdash^* \forall x \{e\}(\underset{\sim}{a},x)\downarrow$, that is for all closed terms $\underset{\sim}{i}$

$$p \Vdash^* \{e\}(\underset{\sim}{a},\underset{\sim}{i})\downarrow.$$

REMARK. Note that $\{e\}(\underset{\sim}{a},x)\downarrow$ is an abbreviation for:

$\exists T[T$ is a computation tree with values for the computation tuple $\langle e,\underset{\sim}{a},\underset{\sim}{i}\rangle$ and $T$ is well-founded].

Also $\{e\}(\underset{\sim}{a},\cdot)$ is taken to be $\{0,1\}$-valued giving $\chi_B$ on $L_{\varkappa_1}{}^3E[G]$ for any $G$ $\mathscr{P}$-generic/$L_{\varkappa_1}{}^3E$ with respect to ranked formulae such that $p \in G$.

By the reasoning in Sacks [11] if $\gamma < \varkappa_1{}^3E$, then

$$\{\langle p,\varphi\rangle \mid \varphi \text{ ranked and } p \Vdash^* \varphi \text{ and rank}(\varphi) \leqq \gamma\}$$

is ${}^3E$ precursive in $\mathscr{P},\gamma$. To compute $B(z)$ for $z \in L_{\varkappa_1}{}^3E$:

1) For each $q \leqq p$ we have

$$q \Vdash^* \{e\}(\underset{\sim}{a},\underset{\sim}{z})\downarrow$$

($\underset{\sim}{z}$ the canonical term for $z$) and by Sacks [11] there exist $q' \leqq q$ and $\gamma \in \mathrm{OR}$ *uniformly* ${}^3E$-recursive in $q,\mathscr{P},\underset{\sim}{a}$ such that

$$q' \Vdash^* |\{e\}(\underset{\sim}{a},\underset{\sim}{z})| \leqq \gamma.$$

Hence let $c(q,\underset{\sim}{a}) = q'$ and $h(q,\underset{\sim}{a}) = \gamma$ and compute first $\tau = \sup_{q \leqq p} h(q,\underset{\sim}{a})$;

2) By the above remark concerning the abbreviation $\{e\}(\underset{\sim}{a},\underset{\sim}{z})\downarrow$ and 1) there exists a term $\underset{\sim}{b} \in C_\tau^{\langle a,z\rangle}$ such that for $q \leqq p$:

$$q \Vdash^* \text{"}\underset{\sim}{b} \text{ is well-founded computation tree with values for } \langle e,\underset{\sim}{a},\underset{\sim}{z}\rangle\text{"}.$$

Thus $\{e\}(\underset{\sim}{a},\underset{\sim}{z}) = \underset{\sim}{0}$ and $\{e\}(\underset{\sim}{a},\underset{\sim}{z}) = \underset{\sim}{1}$ are ranked formulae of rank $\leqq \tau$.

Now $\{\langle p,\varphi\rangle \mid p \Vdash^* \varphi \text{ and rank}(\varphi) \leqq \tau\}$ is ${}^3E$-recursive in $\mathscr{P},\tau$ and there exists $q \leqq p$ such that

$$q \Vdash^* \{e\}(\underset{\sim}{a},\underset{\sim}{z}) = \underset{\sim}{0} \quad\text{or}\quad q \Vdash^* \{e\}(\underset{\sim}{a},\underset{\sim}{z}) = \underset{\sim}{1}$$

($\{e\}(\underset{\sim}{a},\underset{\sim}{z}) = \underset{\sim}{i}$ is an abbreviation for a statement about a term of rank $< \tau$ giving the corresponding computation tree with values). By our assumption on generics $G$ extending $p$, all such $q$ give the same value. If that value is $i$ ($\in \{0,1\}$), then set $B(z) = i$. This algorithm clearly computes $B$ ${}^3E$-recursively on $L_{\varkappa_1}{}^3E$.

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