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PYRAMIDS OF HIGHER ORDER COHOMOLOGY OPERATIONS IN THE p-TORSION-FREE CATEGORY

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1. Introduction.

It is our purpose here to consider an alternative, rather algebraic, approach to higher order cohomology operations. We shall construct pyramids of operations, in the sense of Maunder ([11] and [12]) that instead of being based upon relations in the mod p Steenrod algebra, find their source in the p-divisibility of certain "pseudo" primary operations. These operations are called "pseudo" because of their failure to exhibit the classically required naturality property. The nature of the source notwithstanding, the resulting higher order operations are quite proper and satisfy all the usual properties associated with cohomology operations of higher order. This paper is concerned with making these notions precise.

We further motivate our approach with the claim that the higher order operations which we shall define lend themselves particularly well to the task of making specific calculations. We shall illustrate this claim by providing, as a direct application of our operations, a simple proof of the following result:

1.1. Theorem. *Let X be a finite H-space such that $HZ_3^*(X)$ is isomorphic, as an algebra, to the Z_3-cohomology of the exceptional Lie group E_8. Then, X is not 3-equivalent to a product of non-trivial spaces.*

This result has been proved by R. Kane [9] by means of a rather complicated BP argument. Our approach provides both a rather simple proof as well as an obvious way to proceed with generalisations of (1.1), namely by making use of the "generalised Adem relations", defined below. We hope to return to this point in a future publication.

As further motivation, we present the calculation of several of our operations in the context of $HZ_2^*(CP^\infty)$. (See Theorem 6.11.)

These results are not presented as the main goal of this paper but rather as

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illustrations of the sort of theorems one can generate using the operations defined below.

Our presentation is organised in the following manner. In section 2, we shall define our pseudo primary operations and demonstrate several of their basic properties. We shall close this section with a proof of the fact that these pseudo operations coincide with well-known cohomology homomorphisms defined by J. R. Hubbuck ([6] and [7]). The present formulation, however, lends itself far better to certain calculations and to the definition of our higher order cohomology operations.

Next, in section 3, we shall define a similar family of pseudo primary operations, dual in the sense of [13] to those presented in section 2. Pyramids of higher order cohomology operations based upon these pseudo operations will be constructed in sections 4 and 5. We shall conclude, in section 6, with the applications mentioned above.

During the writing of [5], from which this paper is excerpted, the author has indebted himself to several institutions and individuals. It is with great pleasure that I exploit this forum to express my gratitude. Thanks are due, in the first instance, to Linacre College, Oxford and to the Catholic University of Nijmegen for their generous financial support. Secondly, professors I. M. James, H. O. Singh Varma, S. Gitler, and E. G. Rees are all owed thanks for a great deal of help and guidance that they have extended to the author. Thanks are also due to the referee for many helpful comments and criticisms. Lastly, and, perhaps, most importantly of all, the author's very deep gratitude is due to Prof. J. R. Hubbuck for the generous giving of his time, knowledge and perspective during the writing of [5] and thereafter.

2. The pseudo primary operations.

Let us begin by introducing some notation which we shall use throughout this paper. We shall be working in the category the objects of which will be topological spaces with the homotopy type of a CW complex of finite type for which the integral cohomology is free of p-torsion, for some fixed prime p. The morphisms of our category will be homotopy classes of continuous maps of such spaces. This category shall be denoted by $\mathscr{F}_p$.

Let $\mathbf{Q}_p$ indicate the subring of the rational numbers, where the denominators are all relatively prime to p. We will write $H^*(X)$ and $K(X)$ in place of $H\mathbf{Q}_p^*(X)$ and $K\mathbf{Q}_p^0(X)$, the cohomology and zero-graded unitary K-theory, respectively, of a space X, with coefficients in $\mathbf{Q}_p$. Let $\mathbf{Z}_p$ denote the integers modulo p, $\mathbf{Z}/p\mathbf{Z}$. The obvious homomorphisms: $\varrho\colon \mathbf{Z} \to \mathbf{Z}_p$, $\varrho'\colon \mathbf{Q}_p \to \mathbf{Z}_p$, $k\colon \mathbf{Z} \to \mathbf{Q}$, $k'\colon \mathbf{Z} \to \mathbf{Q}_p$, and $l\colon \mathbf{Q}_p \to \mathbf{Q}$ induce the coefficient homomorphisms in cohomology: ϱ_*, ϱ'_*, k_*, k'_* and l_*, respectively.

Given a space X in $\mathscr{F}_p$, we write the standard skeletal filtration as follows:

(2.1) $$K(X) = K_0(X) \supseteqq K_1(X) \supseteqq \dots \supseteqq K_n(X) \supseteqq \dots \supseteqq * .$$

Because we will be working frequently with the residue classes mod $(p-1)$, we shall fix the notation $m=p-1$. We will write ch_n for the component of the Chern character in dimension $2n$.

Before we can define our pseudo primary operations, we must make several observations about K-theory in our category $\mathscr{F}_p$. We know from [3] and [7] that p-localised unitary K-theory splits up into a direct sum of K-theories, one for each of the mod m residue classes. Thus, we have

(2.2) $$K(X) = \bigoplus_{i=0}^{m-1} K(X)^{(i)} .$$

Such a decomposition is respected by the action of the Adams operations ψ^k, and it induces a mod m splitting on the skeletal filtration (2.1). The following theorem which is due to Adams and Hubbuck ([3], [4], and [7]) makes this more precise.

2.3. THEOREM. *There is a canonical direct sum splitting given by* (2.2) *such that*:

(i) *each* $K(X)^{(i)}$ *is closed under the action of* ψ^k *for each* $k \in \mathbf{Z}$;
(ii) *the associated graded group is defined by*

$$K_{2n}(X)^{(i)}/K_{2n+1}(X)^{(i)} := G_{2n}K(X)^{(i)}$$

and it equals the usual associated graded group $G_{2n}K(X)$ (*which is naturally isomorphic to* $H^{2n}(X)$), *if and only if* $n \equiv i \bmod m$. *If* $n \not\equiv i \bmod m$, *then* $G_{2n}K(X)^{(i)} = 0$.

Now, given that the p-local K-theory breaks up into m summands, we may consider the associated split, local cohomology. This is related to the split K-theory as follows:

2.4. PROPOSITION. *There exists a* (*non-natural*) *isomorphism* $J\colon H^{\mathrm{ev}}(X) \to K(X)$, *for* $X \in \mathscr{F}_p$, *such that*:

(i) $J(H^{2n}(X)) \subseteqq K_{2n}(X)$;
(ii) *the composition of* J *with the quotient map*

$$I_{2n}: K_{2n}(X) \to K_{2n}(X)/K_{2n+1}(X) \cong H^{2n}(X) ,$$

is the identity map on $H^{2n}(X)$;
(iii) *we may decompose* J *into a direct sum*, $\oplus_{i=0}^{m-1} J^{(i)}$, *such that*

$$J^{(i)}: H^{2n}(X)^{(i)} \to K_{2n}(X)^{(i)},$$

where $H^{2n}(X)^{(i)}$ *is defined to be* $K_{2n}(X)^{(i)}/K_{2n+1}(X)^{(i)}$.

PROOF. Let $\{x_1, \ldots, x_t\}$ be a basis for $H^{2n}(X)^{(i)}$, for some fixed i, $0 \leqq i \leqq m-1$. For each x_j $(1 \leqq j \leqq t)$ one can choose elements, $u_j \in K_{2n}(X)^{(i)}$ of exact filtration $2n$ such that $I_{2n}^{(i)}(u_j) = x_j$. Let us define $J^{(i)}$ by $x_j \to u_j$, for $1 \leqq j \leqq t$. Now define $J = \oplus_{i=0}^{m-1} J^{(i)}$. This gives us, in view of (2.3), the desired results.

Note that $J: H^{ev}(X) \to K(X)$ is the isomorphism of filtered modules uniquely induced by a sequence of morphisms: $J_{2n}: H^{2n}(X) \to K_{2n}(X) \to K(X)$.

From this point onward, we shall only consider "splitting isomorphisms" of this form, namely those J's which satisfy (2.4).

2.5. DEFINITION. A cohomology class, $x \in HQ^n(X)$ is said to be integral mod p if it lies in the image of $l_*: H^n(X) \to HQ^n(X)$.

We have the following theorem of Adams [1]:

2.6. THEOREM. *Let* η *be a complex vector bundle over a* CW *complex* X, *such that* η *is trivial when restricted to the* $(2q-1)$*-skeleton of* X. *Then* $p^r \mathrm{ch}_{q+rm}\eta$ *is integral* mod p.

We are now in a position to define our pseduo primary cohomology operations. These will be homomorphisms on cohomology groups defined on and evaluated in $H^{ev}(\mathscr{F}_p)$, the subring of H^* with even grading and arguments taken from our category $\mathscr{F}_p$.

2.7. DEFINITION. Let J be a splitting (satisfying (2.4)) and let u be any element of $H^{2n}(X)$, for $X \in \mathscr{F}_p$ and $n \in \mathbf{Z}^+$, the non-negative integers. Then for each $q \geqq 0$, we define a pseudo primary cohomology operation of the first kind of degree q by:

$$l_*^{-1} p^q \mathrm{ch}_{n+qm} J(u).$$

We shall denote this "operator" by $\theta_J^q: H^{2n}(X) \to H^{2n+2qm}(X)$.

We set the convention that $\theta_J^q(u) = 0$ for $q < 0$.

Invoking Theorem 2 of [1] yields:

2.8. PROPOSITION. *Let* $X \in \mathscr{F}_p$. *Then the following diagram commutes:*

$$\begin{array}{ccc} H^{2n}(X) & \xrightarrow{\theta_J^q} & H^{2n+2qm}(X) \\ \varrho'_* \downarrow & & \downarrow \varrho'_* \\ H\mathbf{Z}_p^{2n}(X) & \xrightarrow{\chi \mathscr{P}^q} & H\mathbf{Z}_p^{2n+2qm}(X) \end{array}$$

Here and throughout this paper we shall use χ to denote the canonical anti-automorphism of the Steenrod Algebra. Moreover, we shall always take $\mathscr{P}^q$ to mean Sq^{2q} in the case $p=2$.

We note that $J(u)$ (in (2.7) above and elsewhere) can be considered to be an element of $K(X)$ despite the fact that X could well have been an infinite complex. This follows since u will always be taken from some finite cohomological grading and we shall only be dealing with finite skeleta on every occasion. This justifies our use of ordinary K-theory as opposed to the $\mathscr{K}$-theory of Atiyah, even though we are working in $\mathscr{F}_p$. This also accounts for our use of direct sum notation where one might expect direct products.

The property that will turn out to be very useful in the evaluation of higher order operations derived from θ_J^q is established in the following manner. Let X and Y be spaces in $\mathscr{F}_p$ and let f: $Y \to X$ be a morphism in this category. We may choose splittings:

$$J : H^{\mathrm{ev}}(X) \to K(X) \quad \text{and} \quad L : H^{\mathrm{ev}}(Y) \to K(Y) .$$

We define a homomorphism f_{JL} by requiring the commutativity of the following diagram:

(2.9)
$$\begin{array}{ccc} H^{\mathrm{ev}}(X) & \xrightarrow{J} & K(X) \\ \downarrow f_{JL} & & \downarrow f^! \\ H^{\mathrm{ev}}(Y) & \xrightarrow{L} & K(Y) \end{array}$$

Taking the Chern character of both sides of the equation derived from the commutativity of (2.9) yields:

(2.10)
$$\mathrm{ch}\, L f_{JL} = \mathrm{ch}\, f^! J = f^* \mathrm{ch}\, J .$$

Because J and L were chosen such that they satisfy (2.4), the diagram (2.9) must commute for each mod m residue class. Thus, for some fixed n and some $j \in [0, m-1]$ such that $n \equiv j(m)$, we get the following commutative diagram:

(2.11)
$$\begin{array}{ccc} H^{2n}(X)^{(j)} & \xrightarrow{J^{(j)}} & K_{2n}(X)^{(j)} \\ \downarrow f_{JL} & & \downarrow f^! \\ \bigoplus_{i \geqq 0} H^{2n+2im}(Y)^{(j)} & \xrightarrow{\oplus_{i \geqq 0} L^{(j)}_{2n+2im}} & K_{2n}(Y)^{(j)} \end{array}$$

Now by (2.3)

$$K_{2n+1}(Y)^{(j)} = K_{2n+2}(Y)^{(j)} = \ldots = K_{2n+2m}(Y)^{(j)} .$$

Thus f_{JL} can be written as a sum of linear maps:

$$f_{JL} = \sum_{i \geqq 0} f_i , \tag{2.12}$$

where each f_i raises degree by $2im$ and where $f_0 = f^*$.

Now for any fixed n and q we may consider the component of (2.10) in dimension $2n+2qm$:

$$\mathrm{ch}_{n+qm} L f_{JL} = f^* \mathrm{ch}_{n+qm} J . \tag{2.13}$$

Applying $l_*^{-1} p^q$ to (2.13) and incorporating the identification given by (2.12) yields the following important formula which measures the deviation from naturality of our pseudo primary operations:

2.14. THEOREM. *Under the above hypotheses*:

$$f^* \theta_J^q = \sum_{i=0}^{q} p^{q-i} \theta_L^i f_{q-i} : H^{2n}(X) \to H^{2n+2qm}(Y) .$$

We end this section with a proof of the correspondence between our pseudo operations and the cohomology homomorphisms of Hubbuck ([7] and [6]) which are defined as follows.

Given a splitting J and a $k \in \mathbf{N}$ we define a map Φ_J^k by requiring the commutativity of the following diagram:

$$\begin{array}{ccc} H^{\mathrm{ev}}(X) & \xrightarrow{\Phi_J^k} & H^{\mathrm{ev}}(X) \\ \downarrow J & & \downarrow J \\ K(X) & \xrightarrow{\psi^k} & K(X) \end{array} \tag{2.15}$$

Consequently, we have:

$$\mathrm{ch}\, \Psi^k J(u) = \mathrm{ch}\, J \Phi_J^k(u) . \tag{2.16}$$

In [7] and [6], Hubbuck has defined homomorphisms of evenly graded $\mathbf{Q}_p$-cohomology, Q_J^q and S_J^q, in terms of this map Φ_J^k. In [7] (see 2.9 and 2.8) it is shown that for a given $x_0 \in H^{2n}(X)$ and a given J and k, one has the existence of a unique finite set of elements, $x_j \in H^{2n+2jm}(X)$, $1 \leqq j \leqq t$, such that $x = \sum_{j=0}^{t} p^{-j} x_j$ satisfies $\Phi_J^k(x) = k^n(x)$. Hubbuck then defines the homomorphism S_J^q: $H^{2n}(X) \to H^{2n+2qm}(X)$ by $x_0 \to x_q$. The homomorphism Q_J^q with the same domain and range as S_J^q, is defined by the requirement that

$$S_J(t) = \sum_{q \geq 0} t^q S_J^q \quad \text{and} \quad Q_J(t) = \sum_{q \geq 0} t^q Q_J^q$$

be formal inverses. Moreover, it is shown (see 2.10 of [7]) that:

$$\Phi_J^k = k^n \sum_{q \geq 0} \sum_{r \geq 0} (1/p)^q (k^m/p)^r S_J^q Q_J^r .$$

2.17. THEOREM. *Let $X \in \mathscr{F}_p$ and let $u \in H^{2n}(X)$, for some $n \in \mathbf{N}$. Then $\theta_J^q(u) = Q_J^q(u)$.*

PROOF. Fix a q and restrict attention to dimension $2n+2qm$; the righthand side of (2.16) becomes:

$$\text{(2.18)} \qquad \mathrm{ch}_{n+qm} J\left\{k^n \sum_{a=0}^{q} (1/p)^{q-a}(k^m/p)^a S_J^{q-a} Q_J^a(u)\right\} .$$

This equals:

$$\text{(2.19)} \qquad (1/p)^q \,\mathrm{ch}_{n+qm} J\{k^n S_J^q(u) + k^{n+m} S_J^{q-1} Q_J^1(u) + \ldots + k^{n+qm} Q_J^q(u)\} .$$

On the other hand, the left side of (2.16) becomes ([2]):

$$\text{(2.20)} \qquad k^{n+qm}\, \mathrm{ch}_{n+qm} J(u) .$$

Now, multiplying both (2.19) and (2.20) by p^q and equating the coefficients of k^{n+qm} yields:

$$\text{(2.21)} \qquad \theta_J^q(u) = \mathrm{ch}_{n+qm} J Q_J^q(u) .$$

The theorem now follows from (2.21) and the fact that $Q_J^q(u)$ is a $(2n+2qm)$-dimensional $\mathbf{Q}_p$-cohomology class upon which $\mathrm{ch}_{n+qm} J$ acts as the identity.

3. Pseudo primary operations of the second kind.

In this section we shall define a family of pseudo primary operations, dual to those of section 2.

3.1. DEFINITION. Let J be a splitting and u be an element of $H^{2n}(X)$, $X \in \mathscr{F}_p$ and $n \in \mathbf{N}$. Then for each $q \in \mathbf{N}$ we may define a pseudo primary operation of the second kind, $\bar{\theta}_J^q$, by the formula:

$$\text{(3.2)} \qquad \sum_{i=0}^{q} \bar{\theta}_J^{q-i} \theta_J^i(u) = 0 \in H^{2n+2qm}(X) ,$$

where we define $\bar{\theta}_J^0(u) = u$ and $\bar{\theta}_J^q(u) = 0$, for $q < 0$. The result is a homomorphism of $\mathbf{Q}_p$-cohomology groups of spaces in $\mathscr{F}_p$:

$$\bar{\theta}^q_J : H^{2n}(X) \to H^{2n+2qm}(X) ,$$

the formal inverse of θ^q_J.

The proofs of the following two propositions are immediate.

3.3. PROPOSITION. *With u, X, J and q as above,*

$$\bar{\theta}^q_J(u) = S^q_J(u) .$$

3.4. PROPOSITION. *Let $X \in \mathscr{F}_p$. Then the following diagram commutes:*

$$\begin{array}{ccc} H^{2n}(X) & \xrightarrow{\bar{\theta}^q_J} & H^{2n+2qm}(X) \\ \downarrow \varrho'_* & & \downarrow \varrho'_* \\ HZ^{2n}_p(X) & \xrightarrow{\mathscr{P}^q} & HZ^{2n+2qm}_p(X) \end{array}$$

Now it turns out that the two pseudo operations θ^q_J and $\bar{\theta}^q_J$ are not only one another's formal inverses, but are also dual to each other in the sense of [13]. More precisely, let X and Y be two spaces in $\mathscr{F}_p$, Spanier dual to one another. We shall denote by $(\cdot, \cdot)$ the usual cohomology pairing induced by the duality ([8] and [14]) and choose elements u and v in $\tilde{H}^{\mathrm{ev}}(X)$ and $\tilde{H}^{\mathrm{ev}}(Y)$, respectively.

Let J be a splitting for $H^*(X)$ and define a splitting for $H^*(Y)$, $\tilde{J}$ as follows. Let w be an element in $K(Y)$ and denote by $(\cdot, \cdot)_K$ the pairing in K-theory corresponding to $(\cdot, \cdot)$ ([8]). Then we define $\tilde{J}$ by the relation:

$$(J(u), w)_K := (u, \tilde{J}^{-1}(w)) .$$

With these hypotheses we have:

3.5. THEOREM. $(\bar{\theta}^q_J u, w) = (u, \theta^q_{\tilde{J}} w)$.

PROOF. See either [8] or [5].

Dual to the deviation from naturality formula for θ^q_J, we have (by virtue of 3.3):

3.6. THEOREM. *With the notation of* (2.14), *we have:*

$$\bar{\theta}^q_L f^* = \sum_{i=0}^{q} p^{q-i} f_{q-i} \bar{\theta}^i_J : H^{2n}(X) \to H^{2n+2qm}(Y) .$$

A further result of the correspondences between our pseudo primary operations and the cohomology homomorphisms of Hubbuck is the following:

3.7. THEOREM ("GENERALISED ADEM RELATIONS"). *Let k be any positive integer prime to p. Then we have:*

$$T_J^q(k) := (1-k^{qm})\bar{\theta}_J^q + (k^m - k^{qm})\bar{\theta}_J^{q-1}\theta_J^1 + \dots$$
$$+ (k^{(q-1)m} - k^{qm})\bar{\theta}_J^1\theta_J^{q-1} \equiv 0, \bmod p^q .$$

(See [7].)

4. Pyramids of higher order operations.

We shall construct, in this section, an algebraic system of higher order operations acting on the evenly graded $\mathbf{Z}_p$-cohomology of spaces in $\mathscr{F}_p$. These operations, which will be based upon "sums" of pseudo primary operations of the form $\sum_{i=0}^{s} \theta_J^{q-i}$, will form pyramids in the sense of Maunder [11]. We begin by establishing some notation.

4.1. DEFINITION. (i) Let $\{u_i\}$ be a vector in the $\mathbf{Q}_p$-cohomology of some space $X \in \mathscr{F}_p$, where $u_i \in H^{2n+2im}(X)$, for i between 0 and some non-negative integer s, and where n is some fixed natural number. For a given $q > s \geq 0$, we shall denote a sum of pseudo primary cohomology operations of the first kind, of degree q and of type s by the expression $\sum_{i=0}^{s} \theta_J^{q-i}$, defined upon a vector $\{u_i\}$ and taking values in $H^{2n+2qm}(X)$.

(ii) Let $\{u_i\}$ be as above. Suppose that $\sum_{i=0}^{s} \theta_J^{q-i} u_i = y$ is divisible by p^{r-s-1}, for some integer $r > s$. Then we shall say that $[J, \{u_i\}]$ is a (q, r, s)-pair of the first kind.

Before we rigorously present the construction of a pyramid based upon the sum $\sum_{i=0}^{s} \theta_J^{q-i}$, we offer a somewhat more heuristic discussion of our method of procedure. The general idea is as follows. Assume we are given a vector $\{x_i\}$, $(0 \leq i \leq s)$ in the $\mathbf{Z}_p$-cohomology of some space $X \in \mathscr{F}_p$. Suppose, moreover, that there exists a splitting isomorphism, J, such that $[J, \{u_i\}]$ forms a (q, r, s)-pair of the first kind (as we have yet to define another kind of pair, we shall suppress explicit mention of kind until it becomes meaningful to do so) for some $\mathbf{Q}_p$-lifting $\{u_i\}$ of the given $\mathbf{Z}_p$-vector. That is to say, we assume the existence of a splitting J and a $\mathbf{Q}_p$-vector $\{u_i\}$ such that $\varrho'_* u_i = x_i$ (for $0 \leq i \leq s$) and such that $y = \sum_{i=0}^{s} \theta_J^{q-i} u_i$ is divisible by p^{r-s-1} in $H^{2n+2qm}(X)$. Consequently, dividing y by p^{r-s-1} is a valid operation in the context of the $\mathbf{Q}_p$-cohomology of X. Performing this division and then reducing to $\mathbf{Z}_p$-cohomology gives the coset value of our cohomology operation of order $(r-s)$ acting on the vector $\{x_i\}$. We shall denote this by:

$$\Phi_q^{r,s}\{x_i\} = \varrho'_*\left[\left(\sum_{i=0}^{s} \theta_J^{q-i} u_i\right)\Big/ p^{r-s-1}\right]\Big/ Q(\Phi_q^{r,s}) , \tag{4.2}$$

where J and $\{u_i\}$ run over all possible (q,r,s)-pairs associated with the vector $\{x_i\}$ in the sense described above. The symbol $Q(\Phi_q^{r,s})$ denotes the indeterminacy of the operation $\Phi_q^{r,s}$. This will be a subgroup of $H\mathbf{Z}_p^{2n+2qm}(X)$ generated by the choices of splitting isomorphism and $\mathbf{Q}_p$-lifting. It will be shown that the value of $Q(\Phi_q^{r,s})$ is the image of the operation $\Phi_q^{r,s+1}$. It will be demonstrated, furthermore, that every element in the indeterminacy can be explicitly realised in the sense of (4.2). As a result we shall be able to conclude that any element in the kernel of $\Phi_q^{r,s}$ can be represented by an explicit $(q,r+1,s)$-pair. In this way, we shall inductively construct a pyramid of higher order cohomology operations $\{\Phi_q^{r,s}\}$, $q \geqq N \geqq r > s \geqq 0$ (for some N), such that

$$\Phi_q^{r,s} : \operatorname{Ker} \Phi_q^{r-1,s} \to \operatorname{Cok} \Phi_q^{r,s+1} . \tag{4.3}$$

This will be a pyramid shaped collection of operations with a bottom row consisting of primary operations, a second row of secondary operations, a third row of tertiary operations, and so on, up to a peak of a single Nth order operation. Domain and range of any operations in the pyramid will be found in the kernel and cokernel, respectively of operations in rows lower in the pyramid. Before we proceed to make this precise, we establish some definitions and notation.

Firstly we fix for all that follows some natural number n.

4.4. Definitions. (i) We shall denote by V a countably infinite direct sum of free cyclic $\mathbf{Q}_p$-modules,

$$V := \bigoplus_{t,i \in \mathbf{Z}^+} V_{t,i} .$$

Each $V_{t,i}$ is filtered in the following trivial way:

$$V_{t,i} = V_{t,i}^0 = V_{t,i}^1 = \dots = V_{t,i}^{2(n+im)} \supseteqq V_{t,i}^{2(n+im)+1} = 0 .$$

We denote a generator in exact filtration $2(n+im)$ by $\zeta_{t,i}$.

(ii) Similarly, we shall write W for the countably infinite direct sum of the associated free cyclic graded $\mathbf{Q}$-modules,

$$W := \bigoplus_{t,i \in \mathbf{Z}^+} W_{t,i} ,$$

where

$$W_{t,i} := V_{t,i}^{2(n+im)} / V_{t,i}^{2(n+im)+1} \cong V_{t,i} .$$

We shall write $\xi_{t,i}$ for the image of $\zeta_{t,i}$ under the action of the quotient map.

(iii) We define a Chern character ch: $V \to W \otimes \mathbf{Q}$ by

$$\mathrm{ch}_{n+(i+s)m} \zeta_{t,i} = \begin{cases} \xi_{t,i} \otimes 1 & \text{if } s=0 \\ 0 & \text{if } s>0 . \end{cases}$$

We define ch universally by extending linearly.

(iv) We shall write W_i for $\bigoplus_{t \in \mathbf{Z}^+} W_{t,i}$ and $\tilde{W}_i$ for $W_i \otimes \mathbf{Z}_p$.

(v) We extend our notion of splitting isomorphism in the obvious way:

$$J : H^{\mathrm{ev}}(X) \otimes W \to K^0(X) \oplus V .$$

We restrict our attention, however, for all of what follows to the case where $J(\xi_{t,i}) = \zeta_{t,i}$, for "almost all" (i.e. with only finitely many exceptions) i and t.

(vi) Let $\xi_{t,i}$ be one of the finite number of exceptions in the above sense. We shall then say that J crosses $\xi_{t,i}$.

Before we proceed with the construction of our pyramid, we record the following lemma for later use:

4.5. LEMMA. *Let J be a splitting isomorphism for $X \in \mathscr{F}_p$. Suppose, further, that $\{g_i\}$, $i \geqq 0$, is a collection of linear maps,*

$$g_i : H^{2n}(X) \to H^{2n+2im}(X) ,$$

with g_0 = the identity map. Then the $\{g_i\}$ uniquely determine another splitting L for X such that $J(u) = L(g_i(u))$ in exact filtration $2n+2im$, for all non-zero $u \in H^{2n}(X)$, for all $i \geqq 0$.

PROOF. For each $i \geqq 0$, choose bases for $H^{2n+2im}(X)$ such that g_i injects onto its image. For each basis element l not in the image of g_i, define L by the condition that $L(l) \cap K_{2n+2im}(X) = 0$. For each basis element in the image of g_i, define L to take the value determined by Jg_i^{-1}. Extend linearly.

Let us now suppose that we are given integers N and q such that $q \geqq N > 0$. Suppose, moreover, that we are given a space $X \in \mathscr{F}_p$ for which at least one (q, r, s)-pair exists, for all r and s in the range $N \geqq r > s \geqq 0$. With this fixed notation we proceed with the inductive construction of the pyramid of cohomology operations, $\{\Phi_q^{r,s}\}$, which acts on the $\mathbf{Z}_p$-cohomology of X.

4.6.i CONSTRUCTION, THE FIRST ORDER. The "base" of the pyramid will consist of primary operations of the form $\Phi_q^{r,r-1}$ for $1 \leqq r \leqq N$. By (4.2) and (2.8) we have:

$$\Phi_q^{r,r-1} = \sum_{i=0}^{r-1} \chi \mathscr{P}^{q-i} : \bigoplus_{i=0}^{r-1} H\mathbf{Z}_p^{2n+2im}(X) \to H\mathbf{Z}_p^{2n+2qm}(X) .$$

This is defined for any $\mathbf{Z}_p$-vector and has no indeterminacy.

4.6.ii THE SECOND ORDER. Next we explicitly calculate the second to bottom row of our pyramid, comprised of secondary operations of the form $\Phi_q^{r,r-2}$ for $2 \leqq r \leqq N$. Let us pick an r in this range and consider the resulting operation of degree q, type $r-2$ and order 2.

This secondary operation will not be universally defined but will have as domain the subset of $\bigoplus_{i=0}^{r-2} HZ_p^{2n+2im}(X)$ consisting of vectors $\{x_i\}$, which lift to $\mathbf{Q}_p$-vectors $\{u_i\}$, which form $(q,r,r-2\}$-pairs with some splitting isomorphisms. This is evidently equivalent to the condition that $\{x_i\}$ be in the kernel of the primary operation $\Phi_q^{r-1,r-2}$. Let us choose such a $\mathbf{Z}_p$-vector $\{x_i\}$. Modulo the indeterminacy we define the coset value of the resulting secondary operation by:

$$\Phi_q^{r,r-2}\{x_i\} := \varrho'_*\left[\left(\sum_{i=0}^{r-2} \theta\mathfrak{J}^{-i}u_i\right)\Big/p\right] \in HZ_p^{2n+2qm}(X) .$$

The indeterminacy is computed as follows. There are two sources, the choice of $\mathbf{Q}_p$-lifting and the choice of splitting isomorphism. That is to say, our secondary operation is only well defined up to choice of $(q,r,r-2)$-pair. To calculate $Q(\Phi_q^{r,r-2})$, let us suppose that we have two $(q,r,r-2)$-pairs $[J,\{u_i\}]$ and $[L,\{v_i\}]$ both representing the same $\mathbf{Z}_p$-vector $\{x_i\}$. Let us firstly consider the effects of choosing two different $\mathbf{Q}_p$-liftings for $\{x_i\}$. Clearly for all $i \in [0,r-2]$, there is a $w_i \in H^{2n+2im}(X)$ such that

$$(4.7) \qquad u_i - v_i = pw_i .$$

Applying our homomorphism to (4.7) gives

$$(4.8) \qquad \sum_{i=0}^{r-2} \theta\mathfrak{J}^{-i}u_i = \sum_{i=0}^{r-2} \theta\mathfrak{J}^{-i}v_i + p\sum_{i=0}^{r-2} \theta\mathfrak{J}^{-i}w_i .$$

The other source of indeterminacy is the choice of splitting isomorphism. We can calculate the effect of this with the help of (2.14). Choosing $X=Y$ and $f^*=f_0=$ the identity map, we apply our deviation from naturality formula to (4.8). Modulo p^2 this gives:

$$(4.9) \qquad \sum_{i=0}^{r-2} [\theta\mathfrak{J}^{-i}u_i - \theta\mathfrak{L}^{-i}v_i] = p\sum_{i=0}^{r-2} [\theta\mathfrak{L}^{-i}w_i + \theta\mathfrak{L}^{-i-1}f_1 v_i] .$$

Since $[J,\{u_i\}]$ and $[L,\{v_i\}]$ may range over all possible $(q,r,r-2)$-pairs, we may conclude from (4.9) that

$$(4.10) \qquad Q(\Phi_q^{r,r-2}) = \operatorname{Im}\Phi_q^{r,r-1} .$$

Consequently, we have (since the primary operations are universally defined):

$$(4.11) \qquad \Phi_q^{r,r-2} : \operatorname{Ker} \Phi_q^{r-1,r-2} \subseteq \bigoplus_{i=0}^{r-2} HZ_p^{2n+2im}(X) \to \operatorname{Cok} \Phi_q^{r,r-1}$$

4.6.iii INTERMEZZO. The fact that a primary operation is fully determinate is reflected in the relative simplicity of the definitions of first and second order operations. For orders three and higher, the situation complicates somewhat. In order to deal properly with this increased complexity, we must have a rather tighter grasp on the indeterminacy. The following series of results is dedicated to this aim.

By way of preparation, we fix some notation: as always $X \in \mathscr{F}_p$; x_i and y_i will always denote elements in $HZ_p^{2n+2im}(X) \oplus \tilde{W}_i$; u_i, v_i, w_i and z_i will always lie in $H^{2n+2im}(X) \oplus W_i$; $\tilde{x}_i$ and $\tilde{y}_i$ shall denote the cosets of x_i and y_i, respectively, in $(HZ_p^{2n+2im}(X) \oplus \tilde{W}_i)/\tilde{W}_i$; η_i and γ_i will always represent element of W_i; J, K, L and M will denote splittings. The following seven results will take place in the context of $H^*(X) \oplus W_*$.

4.12. LEMMA. *Suppose we are given a non-zero class* mod p, $u_0 \in H^{2n}(X)$ *together with a splitting* J. *Suppose moreover that we are given a collection of r vectors, each of length s,* $\{v_i^j\}$, *together with r splittings* K^j ($v_i^j \in H^{2n+2im}(X)$ *for* $1 \leqq i \leqq s$ *and* $1 \leqq j \leqq r$). *Then, there exists a splitting* L *such that*

$$\theta_L^q u_0 = \theta_J^q u_0 + \sum_{j=1}^{r} \sum_{i=1}^{s} p^i \theta_{K^j}^{q-i} v_i^j .$$

PROOF. By (4.5) it is sufficient to determine the maps $\{f_i\}$, which will in turn yield a splitting L such that $L = J\sum_{i \geqq 0} f_i$. We define $f_i(u_0)$ to be $J^{-1}(\sum_{j=1}^r K^j v_i^j)$ for $1 \leqq i \leqq s$ and $f_i(u_0) = 0$ for $i \geqq s+1$. Now set $f_i = 0$ on all elements which are linearly independent of u_0 for all $i \geqq 1$. This defines on L with the desired properties.

4.13 LEMMA. *Suppose we are given a 2n-dimensional class* u_0 *together with two splittings* L *and* J. *Then there exists a class* v_1 *together with a splitting* K *such that* $Ju_0 = Ku_0$ *and such that*

$$\theta_L^q u_0 = \theta_J^q u_0 + p\theta_K^{q-1} v_1 .$$

PROOF. This follows from (2.14) and (4.5). Writing $J^{-1}L$ as $\sum f_i$ and $K^{-1}J$ as $\sum g_i$, we set $v_1 = f_1 u_0 + \xi_{t,1}$ for some t such that J does not cross $\xi_{t,i}$. Now define $g_i(v_i) := -K^{-1}Jf_{i+1}u_0$ for all $i \geqq 1$. Linearity now defines g_i on multiples of v_i. Set $g_i = 0$ for $i \geqq 1$ elsewhere. This defines a K with the required properties.

4.14. Lemma. *Let* $\{\eta_i\}$, $-1 \leqq i \leqq s$, *be given together with a splitting* L. *Then there exists a vector* $\{u_i\}$, $0 \leqq i \leqq s+1$, *and a splitting* K *such that*

$$\sum_{i=-1}^{s} \theta_L^{q-i}\eta_i = p\sum_{i=0}^{s+1} \theta_K^{q-i}u_i \,. \tag{4.15}$$

Proof. Choose a J such that it does not cross η_{i-1} for all $i \in [0, s+1]$. Then by (4.13), se have the existence of K and u_i such that

$$\theta_L^{q-i+1}\eta_{i-1} = \theta_J^{q-i+1}\eta_{i-1} + p\theta_K^{q-i}u_i \,,$$

for each appropriate i. This gives the result.

Now, we establish the converse to (4.14):

4.16. Lemma. *Let* $\{u_i\}$, $0 \leqq i \leqq s+1$ *be given together with some splitting* K. *Then there exists a vector* $\{\eta_i\}$, $-1 \leqq i \leqq s$, *and an* L *such that* (4.15) *holds.*

Proof. Choose a J and a vector $\{\eta_i\}$, $-1 \leqq i \leqq s$ such that none of the η_i are crossed by J nor are any $\eta_i \equiv 0 \pmod p$. Now we may apply (4.12) to get the result.

4.17. Lemma. *Let* $\{x_i\}$ *and* $\{y_i\}$, $0 \leqq i \leqq s$ *be given such that* $x_i - y_i \in \tilde{W}_i$ *for each appropriate* i. *Let* $\{u_i\}$ *and* $\{v_i\}$ *be* $\mathbf{Q}_p$*-representatives of* $\{x_i\}$ *and* $\{y_i\}$, *respectively. Suppose, moreover that we are given splittings* J *and* K. *Then there exists a vector* $\{\gamma_i\}$, $-1 \leqq i \leqq s$ *and a splitting* L *such that*

$$\sum_{i=-1}^{s} \theta_L^{q-i}\gamma_i = \sum_{i=0}^{s} [\theta_J^{q-i}u_i - \theta_K^{q-i}v_i] \,. \tag{4.18}$$

Proof. By (4.13) we have $\theta_J^{q-i}u_i - \theta_K^{q-i}u_i = p\theta_{L_i}^{q-i-1}z_{i+1}$, for each $i \in [0, s]$. Moreover by hypothesis we know that $u_i - v_i = \eta_i + pw_i$ for appropriate η_i and w_i. Consequently we have for each $i \in [0, s]$:

$$\theta_J^{q-i}u_i - \theta_K^{q-i}v_i = \theta_K^{q-i}\eta_i + p\theta_K^{q-i}w_i + p\theta_{L_i}^{q-i-1}z_{i+1} \,.$$

Now let us choose a vector $\{\xi_i\}$, $-1 \leqq i \leqq s$ such that K does not cross it. This gives:

$$\sum_{i=0}^{s} [\theta_J^{q-i}u_i - \theta_K^{q-i}v_i] = \theta_K^{q+1}\xi_{-1} + p\theta_K^{q}w_0$$

$$+\sum_{i=0}^{s-1}[\theta_K^{q-i}(\eta_i+\xi_i)+p\theta_K^{q-i-1}w_{i+1}+p\theta_{L_i}^{q-i-1}z_{i+1}]$$

$$+\theta_K^{q-s}(\eta_r+\xi_r)+p\theta_{L_r}^{q-s-1}z_{r+1}$$

$$=\sum_{i=-1}^{s}\theta_K^{q-i}\gamma_i+\sum_{i=-1}^{s}\sum_{j=1}^{r_i}\sum_{k=1}^{t_i}p^k\theta_{K^j}^{q-k-i}v_k^j\,,$$

where we have defined $\{\gamma_i\}$, $-1\leqq i\leqq s$ by $\gamma_{-1}:=\xi_{-1}$ and $\gamma_i:=\eta_i+\xi_i$, $0\leqq i\leqq s$. This gives the result as a consequnce of (4.12).

Conversely, we have:

4.19. Lemma. *Let $\{x_i\}$ and a $\mathbf{Q}_p$-lifting $\{u_i\}$, $0\leqq i\leqq s$, be given together with a splitting J. Suppose moreover that we are given a vector $\{\gamma_i\}$, $-1\leqq i\leqq s$, and a splitting L. Then there exists a $\{y_i\}$, $0\leqq i\leqq s$ with $\mathbf{Q}_p$-representative $\{v_i\}$ and a splitting K such that $x_i-y_i\in\tilde{W}_i$ for each appropriate i, and such that* (4.18) *holds.*

Proof. Choosing a vector $\{\xi_i\}$, $0\leqq i\leqq s$ such that J does not cross it, we may write by virute of (4.14):

$$\sum_{i=-1}^{s}\theta_L^{q-i}\gamma_i-\sum_{i=0}^{s}\theta_J^{q-i}u_i=p\sum_{i=0}^{s+1}\theta_T^{q-i}z_i-\sum_{i=0}^{s}\theta_J^{q-i}(u_i-\xi_i)\,. \tag{4.20}$$

Writing z for the $2n$-dimensional class $(J^{-1}Tpz_0)_{2n}$ we see thst the right-hand side of (4.20) equals:

$$\theta_J^q(\xi_0-u_0+z)+p\theta_T^{q-1}z_1+\sum_{i=1}^{s}[\theta_J^{q-i}(\xi_1-u_i)+p\theta_T^{q-i-1}z_{i+1}]\,.$$

Writing $v_i:=u_i-\xi_i$, $i\leqq i\leqq s$ and $v_0:=u_0-\xi_0-z$ and applying (4.12) gives the result.

4.21. Theorem. *Let $\{\tilde{x}_i\}$, $0\leqq i\leqq s$ be a coset in*

$$\bigoplus_{i=0}^{s}[H\mathbf{Z}_p^{2n+2im}(X)\oplus\tilde{W}_i]/\bigoplus_{i=0}^{s}\tilde{W}_i\cong\bigoplus_{i=0}^{s}H\mathbf{Z}_p^{2n+2im}(X)\,.$$

Let $\{u_i\}$ be any representative for $\{\tilde{x}_i\}$ in $\oplus_{i=0}^{s}[H^{2n+2im}(X)\oplus W_i]$. Suppose moreover that we are given some splitting isomorphism J. Then we have:

(i) *For any representative $\{v_i\}$ of $\{\tilde{x}_i\}$ and any K, there exists a splitting T and a vector $\{z_i\}$, $0\leqq i\leqq s+1$, such that the following holds:*

$$\sum_{i=0}^{s}[\theta_J^{q-i}u_i-\theta_K^{q-i}v_i]=p\sum_{i=0}^{s+1}\theta_T^{q-i}z_i\,. \tag{4.22}$$

(ii) *For any splitting* T *and vector* $\{z_i\}$, $0 \leqq i \leqq s+1$, *there exists a representative* $\{v_i\}$ *of* $\{\tilde{x}_i\}$ *together with a* K *such that* (4.22) *holds.*

PROOF. This follows from the combination of lemmas (4.14), (4.16), (4.17), and (4.19).

4.6.iv THE HIGHER ORDERS. Theorem (4.21) places us in a position to keep track of the indeterminacy in a rigorous fashion and allows us to proceed with the construction of the pyramid of operations $\{\Phi_q^{r,s}\}$. We shall define a linear homomorphism $\Phi_q^{r,s}$ on the kernel of $\Phi_q^{r-1,s}$ a subgroup of

$$\bigoplus_{i=0}^{s} HZ_p^{2n+2im}(X) \cong \bigoplus_{i=0}^{s} [HZ_p^{2n+2im}(X) \oplus \tilde{W}_i] / \bigoplus_{i=0}^{s} \tilde{W}_i .$$

This last identification will be tacitly assumed in what follows and we shall not explicitly distinguish between a vector $\{x_i\} \in \bigoplus_{i=0}^{s} HZ_p^{2n+2im}(X)$ and its associated coset $\{\tilde{x}_i\}$. When considering ranges, we shall identify $HZ_p^*(X)$ with the obvious subgroup of $HZ_p^*(X) \oplus \tilde{W}_*$. With these identifications in mind, we shall define an operation of degree q and order $(r-s)$:

$$\text{(4.23)} \qquad \Phi_q^{r,s} : \operatorname{Ker} \Phi_q^{r-1,s} \subseteqq \bigoplus_{i=0}^{s} HZ_p^{2n+2im}(X) \to \operatorname{Cok} \Phi_q^{r,s+1} .$$

Orders one and two have already been defined and possess, together with (4.23), the following three properties:

$$\text{(4.24)} \qquad \operatorname{Ker} \Phi_q^{r,s} \subseteqq \operatorname{Ker} \Phi_q^{r-1,s} ,$$

$$\text{(4.25)} \qquad \operatorname{Im} \Phi_q^{r,s} \supseteqq \operatorname{Im} \Phi_q^{r,s+1} ,$$

in the following strong sense: given a cohomology class $z \in \operatorname{Im} \Phi_q^{r,s+1}$, there exists a (q,r,s)-pair such that $\Phi_q^{r,s}$ defined using that pair yields precisely the same class z.

(4.26) Let $\{\tilde{x}_i\}$ be any vector upon which $\Phi_q^{r,s}$ is defined. Then there exists a vector $\{x_i\}$ representing $\{\tilde{x}_i\}$ which in turn has a $\mathbf{Q}_p$-representative $\{u_i\}$ together with a splitting J such that $[J, \{u_i\}]$ forms a (q,r,s)-pair. Modulo its indeterminacy, the value of this $(r-s)$th order operation is given by:

$$\text{(4.27)} \qquad \Phi_q^{r,s}\{x_i\} := \varrho'_* \left[\sum_{i=0}^{s} \theta_J^{q-i} u_i / p^{r-s-1} \right] .$$

4.28. REMARKS. (i) With the help of (4.26) and (4.27), we restate and prove (4.25). Given a cohomology class $z \in \operatorname{Im} \Phi_q^{r,s+1}$ resulting from a particular $(q,r,s+1)$-pair $[J, \{u_i\}]$, there exists a (q,r,s)-pair, $[L, \{v_i\}]$, say, such that:

$$\varrho'_*\left[\sum_{i=0}^{s} \theta_L^{q-i} v_i/p^{r-s-1}\right] = z\,,$$

with no indeterminacy. That this holds for any pair (r, s) with $r > s \geqq 0$, we see as follows:

Choosing an η_s such that J does not cross it, we may write

$$p\sum_{i=0}^{s+1} \theta_J^{q-i}(u_i) = \sum_{i=0}^{s-1} \theta_J^{q-i}(pu_i) + \theta_J^{q-s}(pu_s + \eta_s) + p\theta_J^{q-s-1}(u_{s+1})\,.$$

By (4.12), we may rewrite this as

$$p\sum_{i=0}^{s+1} \theta_J^{q-i}(u_i) = \sum_{i=0}^{s-1} \theta_J^{q-i}(pu_i) + \theta_K^{q-s}(pu_s + \eta_s) = \sum_{i=0}^{s} \theta_L^{q-i}(v_i)\,,$$

where $v_i = pu_i$ for $0 \leqq i \leqq s-1$, $v_s = pu + \eta_s$, $L = J$ in dimensions $2n$ though $2n + 2(s-1)m$ and $L = K$, the splitting determined by (4.12), in dimension $2n + 2sm$. Consequently we may write:

$$z = \varrho'_*\left[\sum_{i=0}^{s+1} \theta_J^{q-i}(u_i)/p^{r-s-2}\right] = \varrho'_*\left[p\sum_{i=0}^{s+1} \theta_J^{q-i}(u_i)/p^{r-s-1}\right]$$

$$= \varrho'_*\left[\sum_{i=0}^{s} \theta_L^{q-i}(v_i)/p^{r-s-1}\right].$$

Choosing $r = s + 2$, demonstrates that our already-defined second order operations exhibit property (4.25).

(ii) That our secondary operations obey (4.2.6), follows directly from theorem (4,21).

(iii) That the above-defined primary operations satisfy the four properties (4.23-4.26) is trivial.

(iv) Lastly we check that our primary and secondary operations are linear homomorphisms. Let $[J, \{u_i\}]$ and $[L, \{v_i\}]$ be two (q, r, s)-pairs. We may assume that u_i and v_i are linearly independent for all appropriate i, for were that not the case, we could replace u_i by $u_i + \eta_i$ for some η_i which is left uncrossed by J. Consequently we may define a new splitting K which agrees with J (L respectively) on the subspace spanned by u_i (v_i respectively). Thus, we may write:

$$\sum_{i=0}^{s} \theta_J^{q-i} u_i + \sum_{i=0}^{s} \theta_L^{q-i} v_i = \sum_{i=0}^{s} \theta_K^{q-i}(u_i + v_i)\,.$$

Now that we have seen that our operations defined in (4.6-i) and (4.6-ii) are linear cohomology homomorphisms which obey the four properties (4.23-4.26) we may proceed with the inductive step.

We assume, as inductive hypothesis, that we have constructed linear higher order operations satisfying properties (4.23-4.26) up to and including the bth order ($b \geqq 2$). We shall now construct a typical operation of order $b+1$ satisfying all the desired properties. Let $\{x_i\}$ be an element of the kernel of $\Phi_q^{r+b,r}$. By (4.26), there exists a J and a $\{u_i\}$ representing $\{x_i\}$ such that $\varrho'_*[\sum_{i=0}^r \theta_J^{q-i} u_i/p^{b-1}]$ is in the coset of zero. Thus, there exists a $(q, r+b, r+1)$-pair, $[T, \{z_i\}]$ such that

$$\varrho'_*\left[p \sum_{i=0}^{r+1} \theta_T^{q-i} z_i/p^{b-1}\right] = \varrho'_*\left[\sum_{i=0}^{r} \theta_J^{q-i} u_i/p^{b-1}\right].$$

Now by (4.21) we have the existence of a $(q, r+b+1, r)$-pair $[L, \{v_i\}]$, where $\{v_i\}$ represents $\{x_i\}$. We define the corresponding $(b+1)$st order operation by:

$$\Phi_q^{r+b+1,r}\{x_i\} = \varrho'_*\left[\sum_{i=0}^{r} \theta_L^{q-i} v_i/p^{b}\right] / Q(\Phi_q^{r+b+1,r}).$$

4.29. REMARKS. (i) That $Q(\Phi_q^{r+b+1,r}) = \operatorname{Im} \Phi_q^{r+b+1,r+1}$ follows from (4.22). Consequently, we see that (4.23) is satisfied.

The remaining properties (4.24-4.26) are easily seen to hold for the $(b+1)$st order operations by virtue of (4.21) and (4.28-i). Linearity follows as above. This completes the inductive construction of our pyramid of operations $\{\Phi_q^{r,s}\}$, $N \geqq r > s \geqq 0$.

(ii) Picorially, we may represent this pyramid (here, we have chosen $N=5$) as:

(4.30)

$$\begin{array}{ccccccccc}
 & \swarrow Q=\cup\mathrm{Im} & & & \Phi_q^{5,0} & & & \mathrm{Domain}=\cap\mathrm{Ker}\searrow & \\
 & & & \Phi_q^{5,1} & & \Phi_q^{4,0} & & & \\
 & & \Phi_q^{5,2} & & \Phi_q^{4,1} & & \Phi_q^{3,0} & & \\
 & \Phi_q^{5,3} & & \Phi_q^{4,2} & & \Phi_q^{3,1} & & \Phi_q^{2,0} & \\
\Phi_q^{5,4} & & \Phi_q^{4,3} & & \Phi_q^{3,2} & & \Phi_q^{2,1} & & \Phi_q^{1,0}
\end{array}$$

(iii) In addition to those mentioned above our operations also enjoy the properties of being natural with respect to maps of spaces and that of being stable under double suspension. This follows from (4.23) and (2.14).

We end this section by identifying our operations of higher order with those of C.R.F.Maunder ([11] and [12]). Let us consider the following chain complex in the sense of Maunder. Let q and N be integers such that $q \geqq N \geqq 1$. We define a series of left $\mathscr{A}_p$-modules C_n, $0 \leqq n \leqq N$. Each C_n shall be generated by elements, $\{c_n, c_{n,0}, c_{n,1}, \ldots, c_{n,n}\}$, where the dimension of c_n is $2qm+n-1$ and that of $c_{n,i}$ is $n+2(n-i)m$ for $0 \leqq i \leqq n$. We define the differentials of our chain complex $d_n: C_n \to C_{n-1}$ by:

$$\text{(i)} \quad d_n(c_n) = -\beta(c_{n-1}) + \sum_{i=0}^{n-1} \chi\mathscr{P}^{q-i}(c_{n-1,n-1-i}) \quad \text{and}$$

$$\text{(ii)} \quad d_n(c_{n,i}) = \beta(c_{n-1,i-1}) + \tau(c_{n-1,i}).$$

Here β denotes the Bockstein homomorphism and τ is $\mathscr{P}^1\beta - \beta\mathscr{P}^1$. Let $\{\Psi_q^{r,s}\}$, $N \geqq r > s \geqq 0$, denote the corresponding pyramid of operations associated with the above chain complex. We have:

4.30. Theorem. *Restricted to the category* $\mathscr{F}_p$, $\{\Psi_q^{r,s}\} = \{\Phi_q^{r,s}\}$ *modulo the indeterminacy.*

Proof. We only consider the case where $p=2$ as the odd prime case is proved analogously. Notice that in this case the above chain complex becomes the admissible complex of [12]. We proceed with the proof by induction.

Clearly in $\mathscr{F}_p$ one has $\Psi_q^{1,0} = \Phi_q^{1,0} = x\,\mathrm{Sq}^{2q}$. Consequently we see that the primary operations agree. Let us take as our inductive hypothesis that in $\mathscr{F}_p$, $\Psi_q^{r,0} = \Phi_q^{r,0}$ for all r, $2 \leqq r \leqq R$ some $R < N$. By axioms 1 and 2 of [11] and by (4.6) above we have agreement of the respective domains and ranges of $\Psi_q^{R+1,0}$ and $\Phi_q^{R+1,0}$. That these operations actually agree in $\mathscr{F}_p$ follows from Theorem 2 of [12] and (4.27). Thus, we have proved that in $\mathscr{F}_p$, $\Psi_q^{r,0} = \Phi_q^{r,0}$ for $N \geqq r \geqq 1$. The theorem now follows from axiom 0 of [11] and the definition of $\Psi_q^{r+a,a}$.

5. Pyramids of operations of the second kind.

As with θ_J^q the pseudo primary operations of the second kind also generate a system of higher order operations in $HZ_p^*(\mathscr{F}_p)$. This system, however, behaves rather differently from the pyramids of section 4 above. At the heart of this difference lie the two formulae (2.14) and (3.6). This seemingly minor contrast gives rise to some fundamental distinctions in behaviour. Let us begin by establishing some notation (we remain in the category $\mathscr{F}_p$):

5.1. Definition. (i) Let $\bigoplus_{i=0}^{s} \bar{\theta}_J^{q-i}$ be a direct sum of pseudo primary operations of the second kind. We shall say that such a sum is of type s and degree q ($q \geqq s$). (It will be defined on any element $u \in H^{2n}(X)$ and will take as value a vector of element in $\bigoplus_{i=0}^{s} H^{2n+2(q-i)m}(X)$.)

(ii) Let $n, q, r, s \in \mathbf{Z}^+$ be given such that $q \geqq r > s \geqq 0$. Let u be any element of $H^{2n}(X)$ such that, for some J:

$$\bigoplus_{i=0}^{s} \bar{\theta}_J^{q-i} u \equiv 0 \quad \bmod p^{r-s-1} \text{ and if } r-s-1 \geqq 1\,,$$

$$\bigoplus_{i=0}^{s} \bar{\theta}_J^{q-i-j} u \equiv 0 \quad \bmod p^{r-s-j}, \quad 1 \leqq j \leqq r-s-1\,.$$

Under such conditions we shall say that $[J, u]$ forms a (q, r, s)-pair of the second kind.

As above, we shall present our construction inductively. We shall explicitly define primary and secondary operations of the second kind and show that they satisfy certain properties. Then we will go on to define a higher order operation of the second kind of arbitrarily high order and we shall show that these operations satisfy the above-mentioned properties as well. This will complete our construction of a pyramid of the second kind. We shall be using the following notation throughout. We suppose we are given a space $X \in \mathscr{F}_p$ together with numbers $N, n, q \in \mathbf{N}$, where $q \geqq N \geqq 2$.

5.2.i Construction, The First Order. Our pyramid will be erected upon a base of primary operations which we shall denote by $\bar{\Phi}_q^{r,r-1}$, $(N \geqq r > 0)$. We define such an operation by:

$$\bar{\Phi}_q^{r,r-1} := \bigoplus_{i=0}^{r-1} \mathscr{P}^{q-i} : HZ_p^{2n}(X) \to \bigoplus_{i=0}^{r-1} HZ_p^{2n+2(q-i)m}(X)\,.$$

5.2.ii The Second Order. The second row of our pyramid of the second kind will consist of second order operations denoted by $\bar{\Phi}_q^{r,r-2}$, $(N \geqq r > 1)$. Such an operation shall be defined upon the kernel of the primary operation $\bar{\Phi}_q^{r,r-1}$ in the following way. Suppose we are given an

$$x \in HZ_p^{2n}(X) \cap \operatorname{Ker} \bar{\Phi}_q^{r,r-1}\,.$$

Then by (5.1.ii) there exists a $(q, r, r-2)$-pair of the second kind $[J, u]$ such that $\varrho'_* u = x$. We define our second order operation evaluated on x by:

$$\bar{\Phi}_q^{r,r-2}(x) := \varrho'_* \left[\bigoplus_{i=0}^{r-2} \bar{\theta}_J^{q-i} u/p \right] \Big/ Q(\bar{\Phi}_q^{r,r-2})\,.$$

The indeterminacy of this second order operation $Q(\bar{\Phi}_q^{r,r-2})$ will be calculated as follows. Given an element x in the kernel of a primary operation $\bar{\Phi}_q^{r,r-1}$ one must choose a suitable J and u such that $[J, u]$ is a $(q, r, r-2)$-pair of the second kind and such that $\varrho'_* u = x$. Suppose $[L, v]$ was another such pair. The indeterminacy $Q(\bar{\Phi}_q^{r,r-2})$ is clearly generated by:

$$\varrho' \bigoplus_{i=0}^{r-2} [(\bar{\theta}_J^{q-i} u - \bar{\theta}_L^{q-i} v)/p]\,. \tag{5.3}$$

It will turn out that unlike our higher order operations of the first kind the indeterminacy here will be solely generated by the choice of $\mathbf{Q}_p$-lifting of x. The choice of splitting offers no contribution to the value of the indeterminacy. We see this as follows. Let J and L be two splittings such that $[J, u]$ and $[L, u]$ are both $(q, r, r-2)$-pairs of the second kind for $\mathbf{Q}_p$-lift u of x. We apply (3.6) in the special case where $X = Y$ and where f is the identity morphism. Mod p^2 we get:

$$\bigoplus_{i=0}^{r-2} \bar{\theta}_J^{q-i} u = \bigoplus_{i=0}^{r-2} \bar{\theta}_L^{q-i} u + p \bigoplus_{i=0}^{r-2} f_1 \bar{\theta}_L^{q-i-1} u \,. \tag{5.4}$$

Because $[L, u]$ was assume to be a $(q, r, r-s)$-pair of the second kind, this becomes:

$$\bigoplus_{i=0}^{r-2} \bar{\theta}_J^{q-i} u = \bigoplus_{i=0}^{r-2} \bar{\theta}_L^{q-i} u, \qquad \operatorname{mod} p^2 \,. \tag{5.5}$$

We return to our calculation of $Q(\bar{\Phi}_q^{r,r-2})$. Since u and v were both assumed to be $\mathbf{Q}_p$-lifts of x, it is clear that

$$u - v = pw \,, \tag{5.6}$$

for some $w \in H^{2n}(X)$. Now, in view of (5.5), we may write (5.3) as:

$$\varrho'_* \bigoplus_{i=0}^{r-2} [(\bar{\theta}_J^{q-i} u - \bar{\theta}_L^{q-i} v)/p] = \varrho'_* \bigoplus_{i=0}^{r-2} \bar{\theta}_J^{q-i} w \,. \tag{5.7}$$

Moreover, given any two of u, v and w, clearly the third is uniquely determined. Consequently, one may conclude that $Q(\bar{\Phi}_q^{r,r-2})$ is precisely equal to the image of our primary operation $\operatorname{Im} \bar{\Phi}_q^{r-1,r-2}$. This value of $Q(\bar{\Phi}_q^{r,r-2})$ is, moreover, minimal.

5.2.iii The General Case. Here we shall define an operation of order $(r-s)$ and degree q which we shall denote by $\bar{\Phi}_q^{r,s}$. The coset value of the operation will be given by:

$$\bar{\Phi}_q^{r,s}(x) := \varrho'_* \left[\bigoplus_{i=0}^{s} \bar{\theta}_J^{q-i} u / p^{r-s-1} \right] / Q(\bar{\Phi}_q^{r,s}) \,. \tag{5.8}$$

Moreover, the following properties will be satisfied:

(5.9) $\bar{\Phi}_q^{r,s} : \operatorname{Ker} \bar{\Phi}_q^{r,s+1} \subseteqq H\mathbf{Z}_p^{2n}(X) \to \operatorname{Cok} \bar{\Phi}_q^{r-1,s}$,

(5.10) $\operatorname{Im} \bar{\Phi}_q^{r,s} \supseteqq \operatorname{Im} \bar{\Phi}_q^{r-1,s}$, in the strong sense of (4.25),

(5.11) $\operatorname{Ker} \bar{\Phi}_q^{r,s} \subseteqq \operatorname{Ker} \bar{\Phi}_q^{r,s+1}$,

(5.12) $\bar{\Phi}_q^{r,s}$ is a linear homomorphism and

(5.13) for any $x \in (H\mathbf{Z}_p^{2n}(X) \cap \operatorname{Ker} \bar{\Phi}_q^{r,s+1})$, there exists a (q, r, s)-pair of the second kind $[J, u]$ such that $\varrho'_* u = x$.

5.14. REMARKS. (i) The primary operations defined in (5.2.i) satisfy the properties (5.8–5.13), trivially.

(ii) The secondary operations defined in (5.2.ii) clearly satisfy properties (5.8–5.11). That (5.12) also holds for secondary operations follows from (5.8) and the fact that the choice of splitting isomorphism plays no rôle in the generation of indeterminacy. Finally property (5.13) is satisfied by virtue of the definition of a $(q, r, r-2)$-pair of the second kind.

(iii) Notice that the "full realisability" of the indeterminacy in the case of operations of the second kind was achieved directly (compare (5.7) with (4.22)) without an elaborate procedure corresponding to (4.6.iii). As such we may proceed straight away with the inductive step in our construction.

Let us assume as inductive hypothesis that we have constructed a truncated pyramid of operation satisfying the above properties up to and including the bth order ($b \geqq 2$). We shall now construct a typical element of $(b+1)$st-row of our pyramid $\bar{\Phi}_q^{r+b+1,r}$. To this end, we choose an element $x \in (HZ_p^{2n}(X) \cap \operatorname{Ker} \bar{\Phi}_q^{r+b+1,r+1})$. By inductive hypothesis, we are free to invoke (5.13). This provides us with a $(q, r+b+1, r)$-pair of the second kind $[J, u]$ where $\varrho'_* u = x$. We now define the coset value of our $(b+1)$st order operation by

$$\bar{\Phi}_q^{r+b+1,r}(x) := \varrho'_* \left[\bigoplus_{i=0}^{r} \bar{\theta}_J^{q-i} u / p^b \right] / Q(\bar{\Phi}_q^{r+b+1,r}) . \tag{5.15}$$

We compute the value of $Q(\bar{\Phi}_q^{r+b+1,r})$ as follows. This is clearly generated by the differences in the value of the operation given in (5.15) depending upon the various choices of $(q, r+b+1, r)$-pair. Let $[J, u]$ and $[L, v]$ be two such choices. As before, these assumptions imply that there exists an element $w \in H^{2n}(X)$ such that $u - v = pw$. Using this fact together with (3.6) (in the above-mentioned special case) we get modulo p^{b+1}:

$$\bigoplus_{i=0}^{r} [\bar{\theta}_J^{q-i} u - \bar{\theta}_L^{q-i} v] = \sum_{j=0}^{b-1} p^{j+1} \bigoplus_{i=0}^{r} f_j \bar{\theta}_L^{q-i-j} w + \sum_{j=1}^{b} p^j \bigoplus_{i=0}^{r} f_j \bar{\theta}_L^{q-i-j} v . \tag{5.16}$$

Now, because $[L, v]$ was assumed to be a $(q, r+b+1, r)$-pair of the second kind, we may rewrite (5.16) as:

$$\bigoplus_{i=0}^{r} [\bar{\theta}_J^{q-i} u \bar{\theta}_L^{q-i} v] = p \bigoplus_{i=0}^{r} \bar{\theta}_L{}^{q-i} w + \sum_{j=1}^{b-1} p^{j+1} \bigoplus_{i=0}^{r} f_j \bar{\theta}_L^{q-i-j} w , \tag{5.17}$$

modulo p^{b+1}. Here as in (5.2.ii), we see that the choice of splitting isomorphism contributes nothing to the value of Q. We may further reduce (5.17) to:

$$\bigoplus_{i=0}^{r} [\bar{\theta}_J^{q-i} u - \bar{\theta}_L^{q-i} v] = p \bigoplus_{i=0}^{r} \bar{\theta}_L^{q-i} w, \quad \operatorname{mod} p^{b+1} , \tag{5.18}$$

by virtue of the following:

5.19. LEMMA. *Under the above hypotheses, we have*

$$\bigoplus_{i=0}^{r} \bar{\theta}_L^{q-i-j} w \equiv 0\ (p^{b-j}), \qquad \text{for all } j \in [1, b-1]\,.$$

PROOF. It is clear that we lose no generality by assuming that $r=0$. We proceed by induction on $(b-j)$.

For $(b-j)=1$ we have:

$$\bar{\theta}_J^{q-(b-1)} u \equiv \bar{\theta}_L^{q-(b-1)} v + p\bar{\theta}_L^{q-(b-1)} w \equiv 0(p^2)\,. \tag{5.20}$$

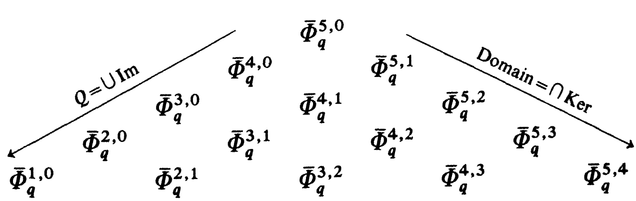
Since $[J, u]$ and $[L, v]$ were both assumed to be $(q, r+b+1, r)$-pairs of the second kind (5.20) implies that $\bar{\theta}_L^{q-(b-1)} w \equiv 0(p)$. This serves to start the induction. We take now as inductive hypothesis the validity of $\bar{\theta}_L^{q-j} w \equiv 0(p^{b-j})$, for all $(b-j) \in [1, b-t-i]$, for some $t \in [1, b-2]$. We must show that $\bar{\theta}_L^{q-t} w \equiv 0(p^{b-t})$. This follows, however, directly from the inductive hypothesis together with:

$$\bar{\theta}_J^{q-t} u \equiv \bar{\theta}_L^{q-t} v + p\bar{\theta}_L^{q-t} w + \sum_{k=1}^{b-t-1} p^{k+1} f_k \bar{\theta}_L^{q-t-k} \equiv 0(p^{b-t+1})\,.$$

5.21. REMARKS. (i) The fact that in (5.18) any two of u, v and w fully determine the third yields the "total realisability" of the indeterminacy of our higher order operations of the second kind. This together with (5.15) and the fact that the choice of splitting offers no contribution to the value of Q makes the properties (5.8–5.13) all direct consequences of the definitions. This completes the inductive construction of our pyramid of operations of the second kind, $\{\bar{\Phi}_q^{r,s}\}$, $q \geqq N \geqq r > s \geqq 0$.

(ii) Pictorially we may represent such a pyramid (here, we have chosen $N=5$) as:

(5.22)



(iii) In addition to the properties (5.8–5.13), these higher order operations of the second kind are also natural with respect to maps of spaces and are stable under the action of the double suspension. This is a direct consequence of (5.1.ii).

To end this section, we shall construct a chain complex dual in the sense of Theorem 4.3.1. of [11] to the complex constructed at the end of section 4. We shall see that the resulting pyramid will correspond to our pyramid of the second kind when we restrict to $\mathscr{F}_p$. Let q and N be integers such that $q \geqq N \geqq 1$ and consider the $\mathscr{A}_p$-modules C_n^*, $(0 \leqq n \leqq N)$ with generators, $\{e_n, e_{n,0}, e_{n,1}, \ldots, e_{n,N-n}\}$. The dimension of e_n is $2qm+N-n-1$ and that of $e_{n,i}$ is $N-n+2m(N-n-i)$, for $0 \leqq i \leqq N-n$. We define the differentials of our dual chain complex $d_n^*: C_n^* \to C_{n+1}^*$ by:

(i) $d_n^*(e_n) = -\beta(e_{n+1}) + \sum_{i=n}^{N-1} p^{q-i+n}(e_{n+1, N-1-i})$ and

(ii) $d_n^*(e_{n,i}) = \beta(e_{n+1,i-1}) + \tau(e_{n+1,i})$,

with β and τ as before. Let $\{\bar{\Psi}_q^{r,s}\}$, $N \geqq r > s \geqq 0$, denote the associated pyramid of operations. We have:

5.23. THEOREM. *Restricted to the category $\mathscr{F}_p$, $\{\bar{\Psi}_q^{r,s}\} = \{\bar{\Phi}_q^{r,s}\}$, modulo the indeterminacy.*

PROOF. By (4.30) we see that it is sufficient to show that $\bar{\Phi}_q^{r,s}$ is dual to $\Phi_q^{r,s}$. We do so by induction on the order of the operation. For primary operations, this follows directly from (3.5) above. Let us take as inductive hypothesis that $\bar{\Phi}_q^{r,s}$ is dual to $\Phi_q^{r,s}$, for all r and s, $N \geqq r > s \geqq 0$ such that $(r-s) \leqq t$ for some $t < N$. For r and s such that $(r-s) = t+1$, the result follows from the inductive hypothesis together with (4.23), (4.27), (5.8), and (5.9).

6. Two applications.

We shall now make good our promise of section 1 by presenting two applications of the theory developed above.

Let E_8 denote the exceptional Lie group and let X be a finite H-space such that, as an algebra, one has:

(6.1) $HZ_3^*(X) \cong HZ_3^*(E_8) \cong E(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes T_3(x_8, x_{20})$,

where E denotes an exterior algebra and T_3 denotes a polynomial algebra, truncated at height 3. Given a Hopf algebra A we shall write PA and QA for the primitives and indecomposables of A, respectively. We claim that the algebra structure alone is sufficient to prove that E_8 is not "irreducible mod 3". For E_8 with its entire algebra and coalgebra structure, this irreducibility mod 3 was shown in [10]. In [9] Kane proves (1.1) via a rather complicated BP argument. We deduce this result as an immediate corollary of the following:

6.2. THEOREM. *With X as above $\bar{\Phi}_3^{2,0}$ is defined on a generator a_{46} of $PHZ_3^{46}(\Omega X)$ and takes the coset of a_{58} a generator of $PHZ_3^{58}(\Omega X)$ as value.*

Before we proceed with the proof of (6.2), it behooves us to recall a few facts concerning the algebraic structure of (6.1). Firstly, we point out that the generators in dimensions 3, 7, 8, 15, 19, and 20 are all linked to one another by elements of the Steenrod algebra, as are the generators in dimensions 27, 35, 39, and 47 (see [9]). Consequently, if X were to be reducible mod 3, then x_{47} would have to be primitive for dimensional reasons. Secondly, passing to the $\mathbf{Z}_3$-homology of the loop space ΩX via the Eilenberg-Moore spectral sequence and then in turn computing the $\mathbf{Q}_3$-cohomology of ΩX, yields the following divided polynomial algebra:

$$(6.3) \qquad HQ_3^*(\Omega X) \cong T(b_2, b_{14}, b_{22}, b_{26}, b_{34}, b_{38}, b_{46}, b_{58}),$$

where the b_i were all chosen to be primitive classes. To allow ourselves to use our operations in the primitive submodule of (6.3), we shall require the following lemma which is due to J. R. Hubbuck:

6.4. LEMMA. *For $i \geqq 22$ one can choose primitive classes $u_i \in K_{2i}(\Omega X; \mathbf{Q}_3)$ such that u_i goes to b_i under the action of the quotient map.*

PROOF. Because $PK^0(\Omega X; \mathbf{Q}_3) \cong \mathrm{Hom}\,(QK_0(\Omega X; \mathbf{Q}_3), \mathbf{Q}_3)$, we see that it is sufficient to show that there exist classes a_i in the $2i$th K-homology skeletal filtration which do not become divisible by 3 when we pass to the indecomposable quotient. This, however, is clear since the α_i for $i \geqq 22$ generate a polynomial subalgebra.

PROOF OF 6.2. Let a_{22} be $\varrho'_* b_{22}$ in $PH\mathbf{Z}_3^{22}(\Omega X)$. In [9] it is shown that one has the following relation:

$$(6.5) \qquad P^9(a_{22}) = a_{58}.$$

Let k be any integer prime to 3 and q any natural number. It follows from (2.14) of [7] that

$$(6.6) \qquad (1-k^{qm})\bar{\theta}_J^q - \sum_{i=1}^{q-1} k^{im} T^{q-i}(k) \bar{\theta}_J^i \equiv 0 \ (3^q),$$

where $T^q(k)$ denotes the pseudo operation given in (3.7).

Applying (6.6) to b_{22}, taking $q=9$ and restricting our attention to the primitive submodule (which we may do by virute of (6.4)) gives:

$$(6.7) \qquad (1-k^{18})\bar{\theta}_J^9 b_{22} - k^{12} T^3(k) \bar{\theta}_J^6 b_{22} \equiv 0 \ (3^4).$$

This follows since $PH^{50}(\Omega X) = PH^{54}(\Omega X) = 0$. By (6.7) and (6.5) we see that

$$(6.8) \qquad (1-k^6)\bar{\theta}_J^3 b_{46} \equiv (1-k^{18}) b_{58}, \ \mathrm{mod}\ (3^4).$$

Now since $v_3(1-k^{qm}) = 1 + v_3(qm)$ (here v_p denote the p-adic valuation) (6.8) becomes

$$(6.9) \qquad \bar{\theta}_J^3 b_{46} \equiv 3b_{58}, \bmod 9 .$$

Since $PH_{54}(\Omega X)=0$, we see that $[J, b_{46}]$ forms a $(3,2,0)$-pair of the second kind. This gives the result.

PROOF OF 1.1. Let us assume that X is reducible mod 3. Thus, x_{47} is primitive. Restricting to the primitive submodule of $HZ_3^*(X)$ and using (5.23), we see that $\bar{\Psi}_3^{2,0}$ is defined on x_{47} and takes the value zero, since $PHZ_3^{59}(X)=0$. This contradicts the previous theorem as our operations commute with the classical map σ^*: $QHZ_p^0(X) \to PHZ_p^{l-1}(\Omega X)$. Hence, X is irreducible mod 3.

Our second application concerns calculations of our higher order operations of the first kind in the $\mathbf{Z}_2$-cohomology of $\mathbf{CP}^\infty$. We shall require firstly the following:

6.10. LEMMA. *If J is such that $J(u \cdot v) = J(u) \cdot J(v)$, then*:

$$\theta_J^q(u \cdot v) = \sum_{i+j=q} \theta_J^i(u) \cdot \theta_J^j(v) .$$

PROOF. This follows directly from (2.20) of [7] and (2.17) above.

Now, let z denote the canonical two-dimensional generator of $HZ^*(\mathbf{CP}^\infty)$. We shall write $\tilde{z}$ for the mod 2 reduction of z. Furthermore, we shall use the notation $\alpha(q)$ to denote the value of the function which assigns to any natural number the number of ones in its dyadic expansion. We shall use ν_2 to denote the 2-adic valuation. With this notation, we have the following:

6.11. THEOREM. *$\Phi_q^{\alpha(q+1),0}(\tilde{z})$ is defined and equal to $\tilde{z}^{q+1}$, with no indeterminacy.*

PROOF. Let J be the splitting isomorphism for $\mathbf{CP}^\infty$ which assigns γ, the Hopf bundle over $\mathbf{CP}^\infty$ to the generator z. The value of the Chern character of γ in dimension $2(q+1)$ is then clearly $z^{q+1}/(q+1)!$. Using the fact that $\nu_2(r!) = r - \alpha(r)$ and the definition of θ_J^q we get:

$$(6.12) \qquad \theta_J^q z = 2^{\alpha(q+1)-1} z^{q+1} .$$

Consequently, we may conclude that $[J, z]$ is a $(q, \alpha(q+1), 0)$-pair of the first kind. Thus, clearly,

$$(6.13) \qquad \Phi_q^{\alpha(q+1),0}\, \tilde{z} = \tilde{z}^{q+1} \bmod Q, \qquad \text{for all } q \geqq 1 .$$

Now we must show that the indeterminacy Q is zero. By (4.25), we see that it is sufficient to show that

(6.14) $$\varrho'_*[(\theta_J^q z + \theta_J^{q-1} z^2)/2^{\alpha(q+1)-2}] = 0 .$$

Now J can clearly be taken to be a ring homomorphism so that we may apply (6.10) to calculate $\theta_J^{q-1} z^2$. Using (6.12) we see that (6.14) is equivalent to the following statement:

(6.15) $$k = 2^{\alpha(q+1)-1} + \sum_{i=0}^{q-1} 2^{\alpha(q-i)+\alpha(i+1)-2} \text{ is divisible by } 2^{\alpha(q+1)-1} .$$

We consider the two cases, q being even or odd, separately. Firstly, we let q be an even natural number. Then k becomes:

(6.16) $$2^{\alpha(q+1)-1} + \sum_{i=0}^{(\frac{1}{2}q-1)} 2^{\alpha(q-i)+\alpha(i+1)-1} .$$

Since $\alpha(a)+\alpha(b) \geqq \alpha(a+b)$, we have shown (6.15) to hold.

Now let q be equal to $2s+1$ for some s. In this case k becomes:

(6.17) $$2^{\alpha(q+1)-1} + \sum_{i=0}^{s-1} 2^{\alpha(q-i)+\alpha(i+1)-1} + 2^{2(\alpha(s+1)-1)} .$$

Now, (6.15) follows in the case that q is odd, as well, since $2(\alpha(s+1)-1) \geqq \alpha(s+1)-1$. This completes the proof.

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