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PRODUCT STRUCTURES IN PYRAMIDS OF HIGHER ORDER COHOMOLOGY OPERATIONS

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1. Introduction.

Let p be a prime number and let X be a topological space, the integral cohomology of which is free of p-torsion. In [8], [10], and [11] certain pseudo primary operations, related to the Chern character, were defined that act upon the p-localised cohomology of X. In addition, pyramids, in the sense of Maunder ([14] and [15]), of higher order operations were defined on $HZ_p^*(X)$ using the p-divisibility of these pseudo operations. Several properties and applications of the resulting pyramids were presented in [10] and [11]. In this paper, we turn our attention to the interplay between our higher order operations and the ring structure of $HZ_p^*(X)$. Our results include several higher order Cartan-like formulae which both improve and generalise results in the literature, improve in the sense of decreasing indeterminacy and generalise in that of admitting arbitrarily high orders.

We organise our presentation in the following manner. We begin by recalling in sections 2 and 3 the basic definitions and results with which we shall be working. Next, in section 4, we consider the multiplicative behaviour of our pseudo primary and higher order operations and we present several applications of the machinery developed.

During the writing of [8], from which this paper is excerpted, the author has indebted himself to several institutions and individuals. It is with great pleasure that I exploit this forum to express my gratitude. Thanks are due, in the first instance, to Linacre College, Oxford, to the Catholic University of Nijmegen and to St. Josephs College of Brooklyn for their generous financial support. Secondly, professors I. M. James, H. O. Singh Varma, S. Gitler and E. G. Rees are all owed thanks for a great deal of help and guidance that they have extended to the author. Lastly, and perhaps most importantly of all, the author's very deep gratitude is due to Professor J. R. Hubbuck for the generous giving of his time, knowledge and perspective during the writing of [8] and thereafter.

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2. Definitions and basic results.

Let us begin by establishing the context within which we shall be working. Unless we state otherwise, we shall choose all of our spaces from the category, the objects of which are topological spaces with the homotopy type of a CW complex of finite type and for which the integral cohomology is free of p-torsion. The morphisms of our category are the homotopy classes of continuous maps of such spaces. We denote this category by $\mathscr{F}_p$.

Let $\mathbf{Q}_p$ indicate the subring of the rational numbers where the denominators are all relatively prime to p. We will write $H^*(X)$ and $K(X)$ in place of $H\mathbf{Q}_p^*(X)$ and $K\mathbf{Q}_p^0(X)$, the cohomology and zero-graded unitary K-theory, respectively, of a space X with coefficients in $\mathbf{Q}_p$. Let $\mathbf{Z}_p$ denote the integers modulo p $\mathbf{Z}/p\mathbf{Z}$. The obvious homomorphisms: $\varrho\colon \mathbf{Z} \to \mathbf{Z}_p$, $\varrho'\colon \mathbf{Q}_p \to \mathbf{Z}_p$, $k\colon \mathbf{Z} \to \mathbf{Q}$, $k'\colon \mathbf{Z} \to \mathbf{Q}_p$, and $l\colon \mathbf{Q}_p \to \mathbf{Q}$ induce the coefficient homomorphism in cohomology: ϱ_*, ϱ'_*, k_*, k'_*, and l_*, respectively.

Given a space X in $\mathscr{F}_p$, we write the standard sketal filtration as follows:

$$K(X) = K_0(X) \supseteqq K_1(X) \supseteqq \ldots \supseteqq K_n(X) \supseteqq \ldots \supseteqq * . \tag{2.1}$$

Because we will be working frequently with the residue classes mod $(p-1)$, we shall fix the notation $m=p-1$. We will write ch_n for the component of the Chern character in dimension $2n$.

Before we can define our pseudo primary operations, we must make several observations about K-theory in our category $\mathscr{F}_p$. We know from [2] and [12] that p-localised unitary K-theory splits up into a direct sum of K-theories, one for each of the mod m residue classes. Thus, we have:

$$K(X) = \bigoplus_{i=0}^{m-1} K(X)^{(i)} . \tag{2.2}$$

Such a decomposition is respected by the action of the Adams operations ψ^k and it induces a mod m splitting on the skeletal filtration (2.1). The following theorem which is due to Adams and Hubbuck ([2], [3], and [12]) makes this more precise.

2.3. THEOREM. *There is a canonical direct sum splitting given by* (2.2) *such that*:

(i) *each $K(X)^{(i)}$ is closed under the action of Ψ^k for each k in $\mathbf{Z}$*;

(ii) *the associated graded group is defined by*

$$K_{2n}(X)^{(i)}/K_{2n+1}(X)^{(i)} := G_{2n}K(X)^{(i)}$$

and it equals the usual associated graded group $G_{2n}K(X)$ if and only if $n \equiv i$ mod m. *If $n \not\equiv i$* mod m, $G_{2n}K(X)^{(i)} = 0$.

Now given that the p-local K-theory breaks up into m summands, we may consider the associated split, local cohomology. This is related to the split K-theory as follows:

2.4. PROPOSITION. *There exists an isomorphism $J: H^{ev}(X) \to K(X)$, for every $X \in \mathscr{F}_p$, such that:*

(i) $J(H^{2n}(X)) \subseteq K_{2n}(X)$;

(ii) *the composition of J with the quotient map $I_{2n}: K_{2n}(X) \to K_{2n}(X)/K_{2n+1}(X) \cong H^{2n}(X)$, is the identity map on $H^{2n}(X)$;*

(iii) *we may decompose J into a direct sum*

$$\bigoplus_{i=0}^{m-1} J^{(i)}$$

such that $J^{(i)}: H^{2n}(X)^{(i)} \to K_{2n}(X)^{(i)}$, where $H^{2n}(X)^{(i)}$ is defined to be $K_{2n}(X)^{(i)}/K_{2n+1}(X)^{(i)}$.

PROOF. Let $(x_1, \ldots, x_t)$ be a basis for $H^{2n}(X)^{(i)}$, for some fixed i, $0 \leqq i \leqq m-1$. For each x_j $(1 \leqq j \leqq t)$ one can choose elements $u_j \in K_{2n}(X)^{(i)}$ of exact filtration $2n$ such that $I_{2n}^{(i)}(u_j) = x_j$. Let us define $J^{(i)}$ by $x_j \mapsto u_j$, for $1 \leqq j \leqq t$. Now define J to be the direct sum of the $J^{(i)}$, $0 \leqq i \leqq m-1$. This gives us, in view of (2.3), the desired results.

From this point onward, we shall only consider "splitting isomorphisms" of this form, namely those J's which satisfy (2.4).

2.5. DEFINITION. A cohomology class $x \in HQ^n(X)$ is said to be integral mod p if it lies in the image of l_*: $H^n(X) \to HQ^n(X)$.

We have the following theorem of Adams [1]:

2.6. THEOREM. *Let η be a complex vector bundle over a* CW *complex X such that η is trivial when restricted to the $(2q-1)$-skeleton of X. Then $p^r \mathrm{ch}_{q+rm}\eta$ is integral* mod p.

We are now in a position to define our pseudo primary cohomology operations. These will be homomorphisms on cohomology groups defined on and evaluated in $H^{ev}(\mathscr{F}_p)$, the subring of H^* with even grading and arguments taken from our category $\mathscr{F}_p$.

2.7. DEFINITION. Let J be a splitting (satisfying (2.4)) and let u be any element of $H^{2n}(X)$, for $X \in \mathscr{F}_p$ and $n \in \mathbf{Z}^+$, the non-negative integers. Then for each

$q \geqq 0$ we define a pseudo primary cohomology operation of the first kind and of degree q by:

$$l_*^{-1} p^q \mathrm{ch}_{n+qm} J(u) \,.$$

We shall denote this "operation" by θ^j: $H^{2n}(X) \to H^{2n+2qm}(X)$. We set the convention that $\theta_q^j(u)=0$ for $q<0$.

Invoking Theorem 2 of [1] yields:

2.8. Proposition. *Let $X \in \mathscr{F}_p$. Then the following diagram commutes:*

$$\begin{array}{ccc} H^{2n}(X) & \xrightarrow{\theta_J^q} & H^{2n+2qm}(X) \\ {\scriptstyle \varrho_*}\downarrow & & \downarrow{\scriptstyle \varrho_m'} \\ HZ_p^{2n}(X) & \xrightarrow{\chi\mathscr{P}^q} & HZ_p^{2n+2qm}(X) \end{array}$$

Here and throughout this paper we shall use χ to denote the canonical anti-automorphism of the Steenrod algebra. Moreover, we shall always take $\mathscr{P}^q$ to mean Sq^{2q} in the case $p=2$.

These pseudo primary operations are not proper cohomology operations in the usual sense because they fail to be natural. This failure, however, is not total and the extent to which the θ_J^q deviate from naturality can be explicitly calculated.

Let X and Y be spaces in $\mathscr{F}_p$. Let f: $Y \to X$ be a morphism in this category. We may choose splittings:

$$J\colon H^{\mathrm{ev}}(X) \to K(X) \quad \text{and} \quad L\colon H^{\mathrm{ev}}(Y) \to K(Y) \,.$$

We define a homomorphism f_{JL} by requiring the commutativity of the following diagram:

$$\text{(2.9)} \qquad \begin{array}{ccc} H^{\mathrm{ev}}(X) & \xrightarrow{J} & K(X) \\ {\scriptstyle f_{JL}}\downarrow & & \downarrow{\scriptstyle f^!} \\ H^{\mathrm{ev}}(Y) & \xrightarrow{L} & K(Y) \,. \end{array}$$

By virtue of (2.3) we see that f_{JL} can be written as a sum of linear maps:

$$\text{(2.10)} \qquad f_{JL} = \sum_{i \geqq 0} f_i \,,$$

where each f_i raises degree by $2im$ and where $f_0=f^*$. We have the following important formula which measures the dediation from naturality of our pseudo primary operations:

2.11. THEOREM. *Under the above hypothese*:

$$f^*\theta_J^q = \sum_{i=0}^{q} p^{q-i}\theta_L^i f_{q-i} : H^{2n}(X) \to H^{2n+2qm}(Y) .$$

PROOF. See [11].

One also has a second kind of pseudo primary cohomology operation, dual in the sense of [16] to the θ_J^q.

2.12. DEFINITION. Let J be a splitting and u be an element of $H^{2n}(X)$, $X \in \mathscr{F}_p$. Then for each $q \in \mathbf{Z}^+$ we may define a pseudo primary operation of the second kind, $\bar{\theta}_J^q$, by the formula:

$$\sum_{i=0}^{q} \bar{\theta}_J^{q-i}\theta_J^i(u) = 0 \in H^{2n+2qm}(X) ,$$

where we define $\bar{\theta}_J^0(u) = u$ and $\bar{\theta}_J^q(u) = 0$ for $q < 0$. The result is a homomorphism of $\mathbf{Q}_p$-cohomology groups of spaces in $\mathscr{F}_p$:

$$\bar{\theta}_J^q : H^{2n}(X) \to H^{2n+2qm}(X) ,$$

the formal inverse of θ_J^q.

Dual to the deviation from naturality formula for θ_J^q one has:

2.13. THEOREM. *With the notation of* (2.11):

$$\bar{\theta}_L^q f^* = \sum_{i=0}^{q} p^{q-i} f_{q-i}\bar{\theta}_J^i : H^{2n}(X) \to H^{2n+2qm}(Y) .$$

3. Pyramids of higher order operations.

In this section we shall sketch the inductive construction of pyramids (in the sense of Maunder) of higher order cohomology operations based upon the mod p integral "p-divisibility" of our pseudo operations. Both the homomorphisms θ_J^q and $\bar{\theta}_J^q$ will yield such systems, the main characteristics of which will be presented below. Let us begin by establishing some notation.

3.1. DEFINITIONS. (i) Let $\{u_i\}$ be a vector in the $\mathbf{Q}_p$-cohomology of some space, $X \in \mathscr{F}_p$, where $u_i \in H^{2n+2im}(X)$, for i between 0 and some given s; $s, n \in \mathbf{Z}^+$. For a given $q > s$, we shall denote a sum of pseudo primary cohomology operations of the first kind of degree q and type s by the expression

$$\sum_{i=0}^{s} \theta_J^{q-i}$$

defined upon a vector $\{u_i\}$ and taking values in $H^{2n+2qm}(X)$.

(ii) Let $\{u_i\}$ be as above. Suppose that there exists a J such that

$$\sum_{i=0}^{s} \theta_J^{q-i} u_i = v$$

is divisible in $H^{2n+2qm}(X)$ by p^{r-s-1} for some $r>s$. Under these conditions we shall say that $[J,\{u_i\}]$ is a (q,r,s)-pair of the first kind. (Pairs of the second kind shall be defined below. Until then, and where confusion is unlikely, we shall suppress explicit mention of kind.)

We construct pyramids of higher order operations from these sums of pseudo operations in the following way. Suppose, for a given n, s, r and $q \in \mathbf{Z}^+$ (with $q \geqq r > s$) and an $X \in \mathscr{F}_p$, we have a $\mathbf{Z}_p$-cohomology vector $\{x_i\}$ with $x_i \in H\mathbf{Z}_p^{2n+2im}(X)$, for $0 \leqq i \leqq s$, such that the following property is satisfied: there exists a splitting J such that $[J,\{u_i\}]$ forms a (q,r,s)-pair for some $\mathbf{Q}_p$-lifting u_i of the given $\mathbf{Z}_p$-vector. If this is the case, we define an operation (of the first kind) of degree q, type s and order $(r-s)$ by:

$$\Phi_q^{r,s}\{x_i\} = \varrho'_*\left[\sum_{i=0}^{s} \theta_J^{q-i} u_i \Big/ p^{r-s-1}\right]\Big/ Q(\Phi_q^{r,s}), \tag{3.2}$$

where $Q(\Psi)$ denotes the indeterminacy of the operation Ψ and where J and $\{u_i\}$ run over all possible (q,r,s)-pairs associated with the vector $\{x_i\}$ in the sense described above. It will turn out that the domain of definition of the operation given in (3.2), the set of $\mathbf{Z}_p$-vectors which admit at least one (q,r,s)-pair, will coincide precisely with the kernel of the operation $\Phi_q^{r-1,s}$. Moreover, the indeterminacy of the operation which is generated by the various choices made in finding a splitting and a $\mathbf{Q}_p$-lifting will be exactly the image of the operation $\Phi_q^{r,s+1}$.

For a fixed $N, q, n \in \mathbf{Z}^+$ and some fixed $X \in \mathscr{F}_p$, we construct a pyramid of cohomology operations $\{\Phi_q^{r,s}\}$ where $q \geqq N \geqq r > s \geqq 0$. The construction will be inductive on the order of rhe operations. For any r and s such that $N \geqq r > s \geqq 0$ and $r-s=1$, we get, by virtue of (3.2) and (2.8):

$$\Phi_q^{r,s} = \sum_{i=0}^{s} \chi\mathscr{P}^{q-i} : \bigoplus_{i=0}^{s} H\mathbf{Z}_p^{2n+2im}(X) \to H\mathbf{Z}_p^{2n+2qm}(X)\,. \tag{3.3}$$

This is a primary operation and, consequently, is universally defined and has no indeterminacy. It serves to start the induction. Higher order operations can be defined in a way consistent with (3.2) and such that:

$$\Phi_q^{r,s} : \operatorname{Ker} \Phi_q^{r-1,s} \subseteqq \bigoplus_{i=0}^{s} HZ_p^{2n+2im}(X) \to \operatorname{Cok} \Phi_q^{r,s+1} . \tag{3.4}$$

Furthermore, the operations will obey:

$$\operatorname{Ker} \Phi_q^{r,s} \subseteqq \operatorname{Ker} \Phi_q^{r-1,s} \quad \text{and} \tag{3.5}$$

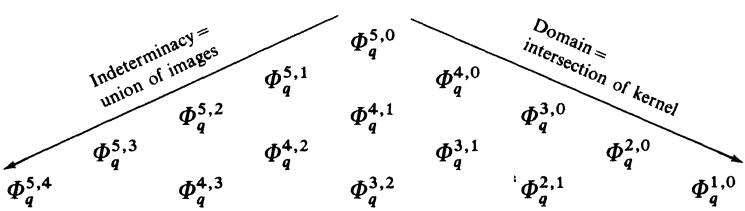
$$\operatorname{Im} \Phi_q^{r,s} \supseteqq \operatorname{Im} \Phi_q^{r,s+1} \tag{3.6}$$

where (3.6) is to be interpreted in the following strong sense. Given a cohomology class z in the image of $\Phi_q^{r,s+1}$ coming from a particular $(q,r,s+1)$-pair, $[J,\{u_i\}]$, there exists a (q,r,s)-pair $[L,\{v_i\}]$, say, such that $\Phi_q^{r,s}$ defined using L and $\{v_i\}$ gives precisely the same cohomology class z with no indeterminacy.

One can prove the following (see [11]):

3.7. THEOREM. *With the above notation, the pyramid* $\{\Phi_q^{r,s}\}$, $N \geqq r > s \geqq 0$, *forms a well-defined system of cohomology operations satisfying* (3.2), (3.4), (3.5) *and* (3.6).

Pictorially, we may represent this pyramid (where $N=5$, for example) by Figure 3.8.



3.8. FIGURE. A typical pyramid of the first kind.

3.9. REMARKS. (1) The above diagram represents a pyramid of cohomology operations generated by the pseudo primary operation θ_J^q. Here we have taken $N=5$ and have allowed r and s to run over all values in the range $5 \geqq r > s \geqq 0$. By (3.4) and (3.5), we see that every operation is defined on the kernel of the operation one step down and to the right of it. By (3.4) and (3.6), we calculate the indeterminacy of a given operation by computing the image if the operation one step below and to the left. Because (3.6) holds in the strong sense given above, we find that the union of the images on a diagonal coincides with the image of the uppermost operation.

(ii) The operations in (3.7) are natural and stable under double suspension.

3.10. THEOREM. *There exists a chain complex in the sense of Maunder* [14] *such that the associated pyramid of operations coincides, modulo indeterminacy, with* $\{\Phi_q^{r,s}\}$ *when the former is restricted to the category* $\mathscr{F}_p$.

The pseudo operations of the second kind also admit a similar construction and provide us with a system of higher order operations, dual to those of (3.7). We shall require some notation:

3.11. DEFINITIONS. (i) Let $\oplus_{i=0}^{s}\bar{\theta}_J^{q-i}$ be a direct sum of pseudo primary operations of the second kind. We shall say that such a sum is of type s and of degree q ($q \geqq s$). It will be defined on any element $u \in H^{2n}(X)$ and will take as value a vector of elements in $\oplus_{i=0}^{s} H^{2n+2(q-i)m}(X)$.

(ii) Let $n, q, r, s \in \mathbf{Z}^+$ be given such that $q \geqq r > s \geqq 0$. Let u be an element of $H^{2n}(X)$ such that:

$$1)\quad \bigoplus_{i=0}^{s} \bar{\theta}_J^{q-i}u \equiv 0 \bmod p^{r-s-1} \qquad \text{and}$$

$$2)\quad \bigoplus_{i=0}^{s} \bar{\theta}_J^{q-i-j}u \equiv 0 \bmod p^{r-s-j}, \qquad 1 \leqq j \leqq r-s-1\,.$$

Under such conditions we shall say that $[J, u]$ forms a (q, r, s)-pair of the second kind.

The construction of our pyramids of the second kind will mimic our above presentation to some extent. We shall proceed as follows. Suppose for a given $n, q, r, s \in \mathbf{Z}^+$ as above, and an $X \in \mathscr{F}_p$, we have a cohomology class $x \in H\mathbf{Z}_p^{2n}(X)$ such that there exists a splitting J which forms a (q, r, s)-pair of the second kind for some $\mathbf{Q}_p$-lifting u of our given class x. Then it follows that

$$\bigoplus_{i=0}^{s} \bar{\theta}_J^{q-i}u = \{y_i\} \equiv 0 \bmod p^{r-s-1}$$

and thus we are free to divide $\{y_i\}$ by p^{r-s-1} in the context of $\mathbf{Q}_p$-cohomology. Doing so and reducing mod p determines the coset value of our $(r-s)$th order operation of the second kind acting on x. We denote this by:

$$\bar{\Phi}_q^{r,s}x = \varrho'_*\left[\bigoplus_{i=0}^{s} \bar{\theta}_J^{q-i}u \Big/ p^{r-s-1}\right]\Big/ Q(\bar{\Phi}_q^{r,s})\,, \tag{3.12}$$

where J and u are allowed to run over all suitable values. As before, it is the choices involved in producing a (q, r, s)-pair associated with x which generate the indeterminacy $Q(\bar{\Phi}_q^{r,s})$. In contrast to the higher order operations of the first kind, however, the choice of splitting offers no contribution to the indeterminacy. This is a result of the rather more stringent conditions that pairs of the second kind must satisfy in contrast to pairs of the first kind. It will turn out that the value of the indeterminacy of $\bar{\Phi}_q^{r,s}$ shall be precisely the image of the operation $\bar{\Phi}_q^{r-1,s}$ with no indeterminacy. The domain of definition will be the kernel of the operation $\bar{\Phi}_q^{r,s+1}$. That is to say, our operations of the second kind will be defined such that:

$$\bar{\Phi}_q^{r,s}: \operatorname{Ker} \bar{\Phi}_q^{r,s+1} \subseteqq HZ_p^{2n}(X) \to \operatorname{Cok} \bar{\Phi}_q^{r-1,s}. \tag{3.13}$$

Moreover, the following two properties will be exhibited:

(3.14) $\quad \operatorname{Im} \bar{\Phi}_q^{r,s} \supseteqq \operatorname{Im} \bar{\Phi}_q^{r-1,s}$ (in the strong sense of (3.6)) and

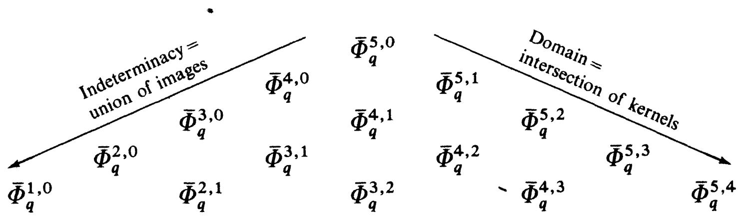
(3.15) $\quad \operatorname{Ker} \bar{\Phi}_q^{r,s} \subseteqq \operatorname{Ker} \bar{\Phi}_q^{r,s+1}$.

Now, for any fixed N between q and 1 we have the following result (see [11]):

3.16. THEOREM. (i) *With the above notation, the pyramid* $(\bar{\Phi}_q^{r,s})$, $N \geqq r > s \geqq 0$, *forms a well-defined system of higher order operations satisfying* (3.12-3.15).

(ii) *The operations in such a pyramid are Spanier dual to those in the corresponding pyramid given in* (3.7), *when both sets of operations are defined.*

We may depict such a pyramid as in Figure 3.17 (again we have chosen $N=5$).



3.17. FIGURE. A typical pyramid of the second kind.

3.18. REMARKS. (i) In comparison with (3.8), the pyramid of operations of the second kind is reflected about a vertical line through its centre. After this reflection, the domains and indeterminacies are computed as in (3.8).

(ii) Again, as in (3.9), we point out that the indeterminacy can be expressed in terms of the image of a single operation with no indeterminacy. This is a result of the strong inclusion given in (3.14).

(iii) The operations of the second kind are natural and stable under double suspension.

3.19. THEOREM. *There exists a chain complex in the sense of Maunder* [14] *such that the associated pyramid of operations coincides, modulo indeterminacy, with the operations of* (3.16), *when the former is restricted to the category* $\mathscr{F}_p$.

3.20. REMARKS. The operations $\Phi_q^{r,s}$ and $\bar{\Phi}_q^{r,s}$ turn out to coincide with several specific examples of higher order operations that appear repeatedly in the literature. For example in $\mathscr{F}_p$:

(i) $\bar{\Phi}_q^{2,0}$ is the operation Φ_q of [13] for p an odd prime. That is to say, $\bar{\Phi}_q^{2,0}$ is the stable secondary cohomology operation associated with the following Adem relation:

$$(\mathscr{P}^1\beta)\mathscr{P}^{q-1} - (q-1)\beta\mathscr{P}^q - \mathscr{P}^q\beta = 0 \ (\text{odd } p).$$

(ii) For $p=2$, $\bar{\Phi}_{2j}^{2,0}$ is Φ_{4j} of [6] and [7]. Thus, $\bar{\Phi}_{2j}^{2,0}$ turns out to be the secondary operation associated with:

$$\mathrm{Sq}^1\mathrm{Sq}^{4j} + (\mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{4j-2} + \mathrm{Sq}^{4j}\mathrm{Sq}^1 = 0.$$

(iii) When $p=2$ and q is arbitrary, $\bar{\Phi}_q^{2,0}$ is the secondary operation studied in [4], [5], and [6] (to name a few) based upon the relation:

$$\mathrm{Sq}^1\mathrm{Sq}^{2q} + (\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{2q-2} + \mathrm{Sq}^{2q}\mathrm{Sq}^1 = 0.$$

4. Products.

Let us make a brief examination of the ring structures of the $\mathbf{Q}_p$-modules with which we have beeen working, $H^{\mathrm{ev}}(X)$ and $K(X)$, $X \in \mathscr{F}_p$. Before we can turn our attention to the multiplicative behaviour of our higher order operations, we must consider the effect of our splitting isomorphisms on the product structure of these two modules.

The first point to be made is that it will not be possible, in general to find a splitting isomorphism $J\colon H^{\mathrm{ev}}(X) \to K(X)$ that will be a ring isomorphism. However, for any two given elements of positive grading x and y, say, in $H^{\mathrm{ev}}(X)$, it will indeed always be possible to find a J such that $J(x)\cdot J(y) = J(xy)$. In general, however, multiplication of two elements $J(x)$ and $J(y)$ produces "error terms" in filtration higher than that of $J(x)\cdot J(y)$. However, we do have:

4.1. PROPOSITION. (i) *Let $x \in H^{2n}(X)$ and $y \in H^{2t}(X)$ for some $X \in \mathscr{F}_p$. Let J be any splitting isomorphism satisfying* (2.4). *Then,*

$$J(x)\cdot J(y) = J(xy) \bmod K_{2n+2t+1}(X).$$

(ii) *$H\mathbf{Q}^{\mathrm{ev}}(X) \cong K\mathbf{Q}(X)$ as rings.*

PROOF. This is obvious.

In order to deal with these error terms in a systematic way, we make the following definition.

4.2. DEFINITION. Let $M_i^J\colon H^{2n}(X)\times H^{2k}(X)\to H^{2n+2k+2im}(X)$ be the mapping defined by $(u,v)\mapsto J^{-1}(J(u)\cdot J(v))$, restricted to the dimension $2n+2k+2im$, for $i\geqq 0$.

It is evident from the definition that:

$$J^{-1}(J(u)\cdot J(v)) = \sum_{i\geqq 0} M_i^J(u,v) \tag{4.3}$$

and that $M_0^J(u,v)=u\cdot v$. Where confusion is unlikely, we shall suppress the superscript J. Using the above notation we have ([8]):

4.4. PROPOSITION.

(i)
$$\bar\theta_J^q(u\cdot v) = \sum_{r=0}^{q}\ \sum_{i+j=q-r} p^r M_r(\bar\theta_J^i(u),\bar\theta_J^j(v));$$

(ii)
$$\sum_{r=0}^{q} p^r\theta_J^{q-r}(M_r(u,v)) = \sum_{i+j=q} \theta_J^i(u)\cdot\theta_J^j(v)\,.$$

4.5. COROLLARY. (i) *If J is a ring homomorphism, we have*:

$$\bar\theta_J^q(u\cdot v) = \sum_{i+j=q} \bar\theta_J^i(u)\cdot\bar\theta_J^j(v);$$

(ii) *if J is such that $J(u\cdot v)=J(u)\cdot J(v)$, then*:

$$\theta_J^q(u\cdot v) = \sum_{i+j=q} \theta_J^i(u)\cdot\theta_J^j(v)\,.$$

What (4.4) tells us is that on the pseudo primary level, a product formula with a general splitting isomorphism contains a series of error terms, the images of the M_i's, for $i\geqq 1$. One of the main results of this section is that the presence of these terms is a primary effect only. That is to say a pyramid of higher order operations exhibits a product behaviour that does not include these error terms.

4.6. DEFINITION. Suppose we are given a relation of the form:

$$\overset{(-)}{\theta}{}_J^q(u\cdot v) = \sum_{i=1}^{r} \alpha_i\,\overset{(-)}{\theta}{}_J^{q-s_i}(u)\cdot\overset{(-)}{\theta}{}_J^{s_i}(v)\,,$$

where all of the coefficients α_i are prime to p and where all of the operations are either of the first kind or all of the second kind. Suppose, moreover, that there exist pairs $[J,u\cdot v]$, $[J,u]$ and $[J,v]$ such that:

(i) $[J, u\cdot v]$ is a $(q, N, 0)$-pair of the first (second) kind,

(ii) $[J, u]$ *simultaneously* forms $(q-s_1, A_1, 0)$, $(q-s_2, A_2, 0), \ldots$ and $(q-s_r, A_r, 0)$-pairs of the first (second) kind.

(iii) $[J, v]$ *simultaneously* forms $(s_1, B_1, 0), (s_2, B_2, 0), \ldots$ and $(s_r, B_r, 0)$-pairs of the the first (second) kind, where $A_i+B_i=N+1$ for all i between 1 and r. Suppose now that N is the greatest integer for which this can be done. Under these circumstances we shall say that $[J, u*v]$ is an admissible product pair of order N for the given relation.

Suppose now that u and v are elements in $H^{2n}(X)$ and $H^{2t}(X)$, respectively for some space X in $\mathscr{F}_p$. Let us denote by x and y the respective $\mathbf{Z}_p$-reductions of these elements. We shall now show that when provided with the appropriate admissible product pairs one may obtain a pair of "higher order Cartan formulae" that improve upon and generalise several such formulae in the literature.

4.7. THEOREM. (i) *With the above notation, we suppose that $[J, u*v]$ is an admissible product pair of order N for the relation*:

(a) $$\bar{\theta}^q_J(u\cdot v) = \sum_{i+j=q} \bar{\theta}^i_J(u)\cdot \bar{\theta}^j_J(v)\,.$$

Then, modulo the total indeterminacy, we have:

(b) $$\bar{\Phi}^{N,0}_q(x\cdot y) = \sum_{i+j=q} \bar{\Phi}^{A_i,0}_i(x)\cdot \bar{\Phi}^{B_j,0}_j(y)$$

where $A_i+B_j=N+1$ for each (i,j) pair such that $i+j=q$.

(ii) *With u, v, x and y as above, let us suppose that $[J, u*v]$ is an admissible product pair of order N for the relation*:

(c) $$\theta^q_J(u\cdot v) = \sum_{i+j=q} \theta^i_J(u)\cdot \theta^j_J(v) - \sum_{r=1}^{q} p^r \theta^{q-r}_J(M_r(u,v))\,.$$

Then modulo the total indeterminacy we have:

(d) $$\Phi^{N,0}_q(x\cdot y) = \sum_{i+j=q} \Phi^{A_i,0}_i(x)\cdot \Phi^{B_j,0}_j(y)$$

where $A_i+B_j=N+1$ for each (i,j) pair such that $i+j=q$.

PROOF. (i) We shall proceed by induction on the order N. The primary case is trivial as it is just the ordinary Cartan formula for the reduced Steenrod powers.

Consider the case of $N=2$. By (4.4-i) we have mod p^2:

(e) $$\bar{\theta}^q_J(u\cdot v) = \sum_{i+j=q} \bar{\theta}^i_J(u)\cdot \bar{\theta}^j_J(v) + p \sum_{i+j=q-1} M_1(\bar{\theta}^i_J(u), \bar{\theta}^j_J(v))\,.$$

By hypothesis we know that every term in the equation above is divisible by p. Which half of a product of terms is actually divisible by p is determined by the existence of pairs. Dividing (e) by p yields, modulo p:

$$\text{(f)} \qquad \bar{\Phi}_q^{2,0}(x \cdot y) = \sum \bar{\Phi}_i^{A_i,0}(x) \cdot \bar{\Phi}_j^{B_j,0}(y) + T.$$

Here, for each pair (i,j) such that $i+j=q$, it is clear that (A_i, B_j) is either $(1,2)$ or $(2,1)$, depending upon which term of the product admits a pair of order 2. The value of T is given by:

$$\text{(g)} \qquad T = \varrho'_*\left[\sum_{i+j=q-1} M_1(\bar{\theta}^i_J(u), \bar{\theta}^j_J(v))\right].$$

Let us consider an arbitrary term of T, say $M_1(\bar{\theta}^i_J(u), \bar{\theta}^j_J(v))$ for $i+j=q-1$. By definition, this is the first "error term" in the product $\bar{\theta}^i_J(u) \cdot \bar{\theta}^j_J(v)$. But by hypothesis $[J, u * v]$ is an admissible product pair of the second kind for (a). It follows that $\bar{\theta}^i_J(u) \cdot \bar{\theta}^j_J(v)$ is p-divisible. Now by definition of M_1 we see that this term must also be p-divisible. It follows that $T=0$. This proves the theorem for second order operations of the second kind.

As inductive hypothesis let us assume that we have the formula (b) for $N \leqq s$. We wish to show that (b) holds for operations of order $(s+1)$ as well. Consider:

$$\text{(h)} \qquad \bar{\Phi}_q^{s+1,0}(x \cdot y) = \sum_{i+j=q} \bar{\Phi}_i^{A_i,0}(x) \cdot \bar{\Phi}_j^{B_j,0}(y) + T'.$$

This is (4.4-i) evaluated $\bmod p^{s+1}$ divided by p^s and reduced $\bmod p$. As above T' will consist of the error terms of the operations of order up to and including s. By inductive hypothesis, however, these operations all have no error terms. It follows that $T'=0$. This completes the proof of (i).

(ii) The second half of (4.7) follows almost immediately from (4.4-ii). Evaluating this equation $\bmod p^N$, dividing by p^{N-1} and reducing $\bmod p$ gives:

$$\text{(k)} \qquad \Phi_q^{N,0}(x \cdot y) + \sum_{r=1}^{q} \Phi_{q-r}^{N-r,0}(M_r(x,y)) = \sum_{i+j=q} \Phi_i^{A_i,0}(x) \cdot \Phi_j^{B_j,0}(y).$$

Since $\operatorname{Im} \Phi_{q-r}^{N-r,0}$ is contained in $\operatorname{Im} \Phi_q^{N,r}$ and this is contained in the indeterminacy of the right-hand side of (k), we have the desired result. This completes the proof of (4.7).

4.8. REMARKS. (i) Notice how in both halves of this proof, the actual values of the $M_i(\cdot,\cdot)$ were not needed. This, however, should not be surprising as the value of $M_i(\cdot,\cdot)$ is dependent upon J and higher order operations allow J to vary.

(ii) Note that we have avoided the use of operations of type greater than zero in (4.7). This was done purely for the sake of notational simplicity. No generality was lost.

(iii) There are several second order "Cartan formulae" in the literature to which (4.7) might be compared. Let us consider one in particular, (3.4) of [13]. If we take p to be odd, $q=3$ and $N=2$ and apply (4.7-i), above, we get that if a J exists such that $[J, u*v]$ is an admissible pair for the relation:

$$\bar{\theta}_J^3(u\cdot v) = \sum_{i=0}^{3} \bar{\theta}_J^{3-i}(u)\cdot\bar{\theta}_J^i(v),$$

then

$$\bar{\Phi}_3^{2,0}(x\cdot y) = \sum_{i=0}^{3} \bar{\Phi}_{3-i}^{A_i,0}(x)\cdot\bar{\Phi}_i^{B_i,0}(y)$$

in $H\mathbf{Z}_p^{2n+2t+6m}(X)/\mathrm{Im}\,\mathscr{P}^3$, with $A_0=A_1=B_2=B_3=2$.

If we take these same hypotheses and apply (3.4) of [13] (where we have taken $k=3$ and $j=1$), we get the same result but with greater indeterminacy. The value of Q in Kobayashi's result is

$$\mathrm{Im}\,[\mathscr{P}^3 + f^*H\mathbf{Z}_p^{2n+2t+6m}(K(\mathbf{Z},2n)\times K(\mathbf{Z},2t))],$$

where f is the map, $f\colon X \to K(\mathbf{Z},2n)\times K(\mathbf{Z},2t)$ defined by $f(z)=(g(z),h(z))$, where $g\colon X \to K(\mathbf{Z},2n)$ and $h\colon X \to K(\mathbf{Z},2t)$ are such that $g^*(\gamma^{2n})=x$ and $h^*(\gamma^{2t})=y$, with γ^{2i} the mod p reduction of the fundamental class of $H\mathbf{Z}^{2i}(K(\mathbf{Z},2i))$.

(iv) We note that $\mathbf{Q}_r(\tilde{H}^{\mathrm{ev}}(\mathscr{F}_p))=\tilde{H}^{\mathrm{ev}}/(\tilde{H}^{\mathrm{ev}}\otimes\ldots\otimes\tilde{H}^{\mathrm{ev}})$ (where we have factored out $(r+1)$-fold products), the rth indecomposable quotient associated with $\mathbf{Q}_p$-cohomology theory applied to the spaces in $\mathscr{F}_p$, is the associated graded ring of $\mathbf{Q}_r(\tilde{K}^0(\mathscr{F}_p))$. Consequently, our higher order operations, Φ and $\bar{\Phi}$, restrict to indecomposable quotients in $\tilde{H}\mathbf{Z}_p^{\mathrm{ev}}(\mathscr{F}_p)$.

We conclude with a pair of particularly "nice" product formulae for certain higher order operations. Let x, y, u and v be as above. We have:

4.9. THEOREM. *Suppose that $\bar{\Phi}_{2q}^{2q,q}$ is defined on x and on y. Then $\bar{\Phi}_{2q}^{q,0}$ is defined on x, on y and on $x\cdot y$ and modulo the total indeterminacy one has:*

$$\bar{\Phi}_{2q}^{q,0}(x\cdot y) = \bar{\Phi}_{2q}^{q,0}(x)\cdot y + x\cdot\bar{\Phi}_{2q}^{q,0}(y).$$

PROOF. The hypotheses imply that the following congruences are satisfied for pairs $[J,u]$ and $[J,v]$:

(i) $\bar{\theta}_J^{2q-i}(u) \equiv \bar{\theta}_J^{2q-i}(v) \equiv 0 \mod p^{q-1}$, for $0 \leqq i \leqq q+1$, and

(ii) $\bar{\theta}_J^{2q-j}(u) \equiv \bar{\theta}_J^{2q-j}(v) \equiv 0 \mod p^{2q-j}$, for $q+2 \leqq j \leqq 2q-1$.

By (4.4-i), we have $\mod p^q$:

(iii) $\bar{\theta}_J^{2q}(u \cdot v) = \sum_{i=0}^{2q} \bar{\theta}_J^{2q-i}(u) \cdot \bar{\theta}_J^i(v) + \sum_{r=1}^{q-1} p^r \sum_{i=0}^{2q-r} M_r(\bar{\theta}_J^{2q-r-i}(u), \bar{\theta}_J^i(v))$.

But (i) and (ii) above imply that (iii) is equivalent to:

(iv) $\bar{\theta}_J^{2q}(u \cdot v) = \bar{\theta}_J^{2q}(u) \cdot v + u \cdot \bar{\theta}_J^{2q}(v)$, $\mod p^q$.

Moreover, the hypotheses imply that each term in (iv) is individually divisible by p^{q-1}. Stronger yet, they guarantee that each term in (iv) admits the definition of $\bar{\Phi}_{2q}^{q,0}$. The result now follows from the definitions.

4.10. REMARK. Taking $q=2$ in (4.9) yields the results of theorem (8.4) of [4]. As in (4.8-iii), however, the value of the indeterminacy has been reduced in our case. The difference in our value of Q and that of Adem is, once again, the image of $f^*H^{2n+2t+2qm}(K(\mathbf{Z}, 2n) \times K(\mathbf{Z}, 2t))$.

A similar result to (4.9), couched in rather different terms, is the following. We take x, y, u and v to be as above. Suppose that $[J, u]$ and $[J, v]$ are both, simultaneously, a $(2q, q, 0)$- and a $(q, q, 0)$-pair of the second kind, for some splitting isomorphism J. Then we have:

4.11. THEOREM. *With the hypotheses given above $[J, u*v]$ is an admissible pair for*:

(i) $\bar{\theta}_J^{2q}(u \cdot v) = \sum_{i+j=2q} \bar{\theta}_J^i(u) \cdot \bar{\theta}_J^j(v)$ *and*

(ii) $\bar{\theta}_J^q(u \cdot v) = \sum_{i+j=q} \bar{\theta}_J^i(u) \cdot \bar{\theta}_J^j(v)$.

Moreover, modulo the total indeterminacy, we have that

(iii) $\bar{\Phi}_{2q}^{q,0}(x \cdot y) = \bar{\Phi}_{2q}^{q,0}(x) \cdot y + x \cdot \bar{\Phi}_{2q}^{q,0}(y)$ and

(iv) $\bar{\Phi}_q^{q,0}(x \cdot y) = \bar{\Phi}_q^{q,0}(x) \cdot y + x \cdot \bar{\Phi}_q^{q,0}(y)$.

PROOF. The result follows from the following set of congruences which are, in turn, implied by the hypotheses:

(a) $\bar{\theta}_J^{2q-i}(u) \equiv \bar{\theta}_J^{2q-i}(v) \equiv 0 \mod p^{q-i}$, for $1 \leqq i \leqq q-1$,

(b) $\bar{\theta}_J^{q-i}(u) \equiv \bar{\theta}_J^{q-i}(v) \equiv 0 \mod p^{q-i}$, for $1 \leqq i \leqq q-1$ and

(c) $\bar{\theta}_J^{2q}(u) \equiv \bar{\theta}_J^{2q}(v) \equiv \bar{\theta}_J^q(u) \equiv \bar{\theta}_J^q(v) \equiv 0 \mod p^{q-1}$.

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