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NOTES ON CARTAN SUBALGEBRAS IN TYPE II_1 FACTORS

SORIN POPA

1. Introduction.

Let M be a von Neumann algebra. A Cartan subalgebra of M is a maximal abelian $*$-subalgebra of M whose normalizer generates M and which is the range of a normal conditional expectation ([5], [7], [14]).

For instance, if M arises by the classical group measure space construction from a free action of a discrete countable group G on a measure space (X,μ), then $L^\infty(X,\mu)$ is naturally imbedded in M as a Cartan subalgebra ([11]). In fact the von Neumann algebra constructed in this way depends only on the orbit equivalence relation R_G given by the group G ([6]). A general construction of a von Neumann algebra from a countable measured equivalence relation R over (X,μ) was given by Feldman and Moore in [7]. Actually, their construction depends also on a two cocycle τ over R and they show that every separable von Neumann algebra M with a Cartan subalgebra A arises like this, i.e. there exists a countable measured equivalence relation R on a space (X,μ) and a 2-cocycle τ over R such that M is isomorphic to the von Neumann algebra $M(R,\tau)$ constructed from R and τ in such a way that A is carried onto the natural imbedding of $L^\infty(X,\mu)$ in $M(R,\tau)$.

In [3], A. Connes, J. Feldman and B. Weiss proved the striking result that any countable amenable measured equivalence relation is generated by a single transformation of the space. When translated in the von Neumann algebra context (via the Feldman-Moore construction) this theorem, together with Krieger's theorem ([10]), show that if M is an injective von Neumann algebra then any two Cartan subalgebras are conjugated by an automorphism of M.

This paper originated in the author's attempt to give a proof of the Connes-Feldman-Weiss theorem using only operator algebras techniques. We are indeed able to do this, but the proof we present here is not a new one, it uses the main ideas of the original proof and may be regarded rather

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as a translation into operator algebra terms of [3]. We treat only the factorial type II_1 case where many simplifications occur, but we believe that the proof of this case gives also an image of what happens in the general case. The proof is elementary and uses only basic results in operator theory.

When we prepare the setting for the proof of the Connes-Feldman-Weiss theorem we also obtain some results of independent interest. For instance we show that if M is a type II_1 factor with a subfactor N that contains a Cartan subalgebra A of M, then there exist some unitaries $\{u_i\}_{i\in I}$ in the normalizer of A in M such that $\{Nu_i\}_{i\in I}$ are mutually orthogonal subspaces with respect to the trace of M and such that Nu_i fill up M. This shows in particular that in this case if $[M:N]$ denotes the index of N in M as was recently introduced by V. Jones in [8], then $[M:N]$ is an integer, if finite. Also, as when M is a group measure algebra over A, it follows that M may be decomposed in a direct sum $M = \Sigma_n Au_n$, where u_n are unitaries in the normalizer of A in M such that each $u_m^* u_n$ acts properly outer on A. We also prove that if M has a Cartan subalgebra then it is single generated.

Let us mention here that recently Connes and Jones gave an example of a separable type II_1 factor with two nonconjugate Cartan subalgebras, but there are no known examples of separable type II_1 factors without Cartan subalgebras.

It is our hope that this expositary account will be helpful to those which are more familiar with operator theory than with ergodic theory to the understanding of this remarkable theorem of Connes, Feldman and Weiss and its beautiful proof.

2. Decomposition relative to a Cartan subalgebra.

From now on M will be a type II_1 factor, with trace $\tau, \tau(1) = 1$, and $\|x\|_2 = \tau(x^*x)^{1/2}$ will be the Hilbert norm given by τ on M. The completion of M in the norm $\|\cdot\|_2$ is identified with $L^2(M,\tau)$, the Hilbert space of square integrable operators affiliated with M. If $B \subset M$ is a von Neumann subalgebra of M, then E_B denotes the unique normal trace preserving conditional expectation of M on B. The closure $\bar{B}^\tau$ of B in $L^2(M,\tau)$ is $L^2(B,\tau_{|B})$ and if $x \in M$, then $E_B(x)$ is the orthogonal projection of x on $L^2(B,\tau_{|B})$. More generally if $v \in M$ is a partial isometry with left support in B, then an easy computation shows that $x - E_B(xv^*)v$ is orthogonal to Bv, so that the orthogonal projection of x on the (not necessary closed) vector subspace $Bv \subset M \subset L^2(M,\tau)$ exists and is equal to $E_B(xv^*)v$. It will be denoted in the sequel by $E_{Bv}(x)$.

Let $A \subset M$ be a maximal abelian $*$-subalgebra of M and $B \subset M$ a von

Neumann algebra that contains A. Then $\mathcal{N}_B(A)$ denotes the normalizer of A in B, that is

$$\mathcal{N}_B(A) = \{u \in B \mid u \text{ unitary}, uAu^* = A\},$$

and $\mathcal{G}\mathcal{N}_B(A)$ denotes the normalizing groupoid of A in B,

$$\mathcal{G}\mathcal{N}_B(A) = \{v \in B \mid v \text{ partial isometry}, vv^*, v^*v \in A,\ vAv^* = Av^*v\}.$$

Since B is finite it follows that $v \in \mathcal{G}\mathcal{N}_B(A)$ if and only if there exist a unitary $u \in \mathcal{N}_B(A)$ and a projection $e \in A$ such that $v = ue$ (see for instance [9, 2.1]). In other words: any partial isometry that normalize A extends to a unitary in $\mathcal{N}_B(A)$. Note that if $v_1,\ v_2 \in \mathcal{G}\mathcal{N}_M(A)$, then $v_1 v_2 \in \mathcal{G}\mathcal{N}_M(A)$ and if in addition $v_2^* v_1 = v_1 v_2^* = 0$. then $v_1 + v_2 \in \mathcal{G}\mathcal{N}_M(A)$. In particular, if $v \in \mathcal{G}\mathcal{N}_M(A)$ and e_1, e_2 are projections in A, then $e_1 v e_2 \in \mathcal{G}\mathcal{N}_M(A)$ and

$$v - e_1 v e_2 = v - (e_1 v e_2 v^*)v = (1 - e_1 v e_2 v^*)v \in \mathcal{G}\mathcal{N}_M(A).$$

Suppose that $A \subset B \subset M$ are von Neumann subalgebras, A is maximal abelian in M and $v \in \mathcal{G}\mathcal{N}_M(A)$. By a result of H. Dye (cf. [6]; see also [9, 2.2]), there exists a unique projection $e \in A$ such that $E_B(v) = ev$ and $e \leqq vv^*$. In particular it follows that $E_B(\mathcal{G}\mathcal{N}_M(A)) = \mathcal{G}\mathcal{N}_B(A)$. This result, with the preceding remarks, easily yields the following:

2.1 LEMMA. a) *Let $v_0, v \in \mathcal{G}\mathcal{N}_M(A)$. There exists a unique projection $e \in A$ such that $E_{Bv_0}(v) = ev$, $e \leqq vv^*$. There exists a unique partial isometry $b \in \mathcal{G}\mathcal{N}_B(A)$ such that $E_{Bv_0}(v) = bv_0, b^*b \leqq v_0 v_0^*$.*

b) *Let $v_1, v_2,\ v \in \mathcal{G}\mathcal{N}_M(A)$ and suppose that Bv_1, Bv_2 are mutually orthogonal subspaces. Then the left (respective right) supports of $E_{Bv_1}(v)$, E_{Bv_2} are mutually orthogonal.*

PROOF. a) We have $E_{Bv_0}(v) = E_B(vv_0^*)v_0 = bv_0$, where $b = evv_0^*$ for some projection e with $e \leqq vv_0^* v_0 v^*$. If $b_1 \in \mathcal{G}\mathcal{N}_B(A)$, $b_1^* b_1 \leqq v_0 v_0^*$ and $e_1 \in A$, $e_1 \leqq vv^*$, are such that $b_1 v_0 = bv_0$, $ev = e_1 v$, then clearly $e = e_1, b = b_1$.

b). By a) there exist projections $e_1, e_2 \in A$ such that $E_{Bv_1}(v) = e_1 v$, $E_{Bv_2}(v) = e_2 v$, $e_1, e_2 \leqq vv^*$. Since $Bv_1 \perp Bv_2$ we have

$$0 = \tau(v^* e_1 e_2 v) = \tau(e_1 e_2 vv^*) = \tau(e_1 e_2)$$

so that $0 = e_1 e_2 = e_1 vv^* e_2$.

Let $A \subset B \subset M$ be as before and consider a family of partial isometries $\{v_i\}_{i \in J}$ in $\mathcal{G}\mathcal{N}_M(A)$ such that the spaces $\{Bv_i\}_{i \in J}$ are mutually orthogonal. Then we denote by

$$\sum_{i \in J} Bv_i = \{x \in M \mid x = \sum_{i \in J} b_i v_i \text{ for some } b_i \in B \text{ with } \sum_i \| b_i v_i \|_2^2 < \infty\},$$

(the sums are considered in $L^2(M,\tau)$ and make sense because b_jv_j are mutually orthogonal vectors). Note that $\Sigma_{j\in J} Bv_j$ is closed in M in the norm $\|\cdot\|_2$. Indeed, because if $x\in M$ then the projection of x on the closure of $\Sigma_{j\in J}Bv_j$ in $L^2(M,\tau)$ is just $\Sigma_{j\in J} E_B(xv_j^*)v_j$ so that it is of the form $\Sigma_{j\in J}b_jv_j$ for some $b_j\in B$. Thus, if x is in $\overline{\Sigma_{j\in J} Bv_j}^{\tau} \cap M$, then

$$x = \sum_{j\in J} b_jv_j \in \sum_{j\in J} Bv_j.$$

By the preceding lemma, if $v\in\mathscr{G}\mathscr{N}_M(A)$, then $E_{Bv_i}(v) = e_iv$ for some projection $e_i\in A$, $e_i \leqq vv^*$. Since $\{Bv_i\}_{i\in J}$ are mutually orthogonal, $\{e_i\}_{i\in J}$ are mutually orthogonal, so that $e = \Sigma_{i\in J}e_i$ is a projection in A,

$$ev = \sum_{i\in J} e_iv \in \sum_{i\in J} Bv_i$$

and $v - ev$ is orthogonal to $\Sigma_{i\in J}Bv_i$. It follows that the projection of v on $\Sigma_{i\in J}Bv_i$ exists and it is of the form ev for some $e\in A$, uniquely determined if we require $e \leqq vv^*$.

Now let us recall that a von Neumann subalgebra $A \subset M$ is a Cartan subalgebra of M, if it is maximal abelian in M and if $\mathscr{N}_M(A)$ generates M as a von Neumann algebra. Since $\mathscr{N}_M(A)$ is a group we always have

$$\mathscr{N}_M(A)'' = \overline{\text{span}}^w\ \mathscr{N}_M(A) = \overline{\text{span}}^s\ \mathscr{N}_M(A)$$

so that if A is a Cartan subalgebra then span $\mathscr{N}_M(A)$ is dense in M in the norm $\|\cdot\|_2$ and of course span $\mathscr{G}\mathscr{N}_M(A)$ is also dense in M. Since

$$E_B(\mathscr{G}\mathscr{N}_M(A)) \subset \mathscr{G}\mathscr{N}_B(A) \ \textit{for}\ \ A \subset B \subset M.$$

it follows that if A is a Cartan subalgebra in M, then it is a Cartan subalgebra in B ([6]).

Let us also mention that if A is a Cartan subalgebra of the II_1 factor M, then given any two projections e_1, e_2 in A, with the same dimension in M, there exists a partial isometry $v\in\mathscr{G}\mathscr{N}_M(A)$ such that $v^*v = e_1$, $vv^* = e_2$. This fact follows easily by a maximality argument (see for instance [12, 3.4]).

2.2 Proposition. *If $A \subset M$ is a Cartan subalgebra of M and $B \subset M$ is a von Neumann subalgebra that contains A, then there exists a family $\{v_j\}_{j\in J}$ of nonzero partial isometries in $\mathscr{G}\mathscr{N}_M(A)$ such that $\{Bv_j\}_{j\in J}$ are mutually orthogonal subspaces and $\Sigma_{j\in J} Bv_j = M$.*

Proof. Let $\{v_j\}_{j\in J}$ be a maximal family of nonzero partial isometries such that $\{Bv_j\}_{j\in J}$ are mutually orthogonal. If $\Sigma_{j\in J} Bv_j \neq M$, then there exists a partial isometry $v\in\mathscr{G}\mathscr{N}_M(A)$ such that $v\notin\Sigma_{j\in J} Bv_j$. By the preceding remarks there exists a projection $e \in A$ such that $ev \in \Sigma_{j\in J} Bv_j$ and $v - ev$ is orthogonal to $\Sigma_{j\in J} Bv_j$. Since $v \notin \Sigma_{j\in J} Bv_j$ we have $v - ev \neq 0$, so that

$$\{v_j\}_{j\in J} \cup \{v - ev\}$$

contradicts the maximality of $\{v_j\}_{j\in J}$.

We now prove the main result of this section:

2.3 THEOREM. *Let M be a type II_1 factor with a Cartan subalgebra A. If $N \subset M$ is a subfactor that contains A then there exists a family of unitaries $\{u_j\}_{j\in J}$ in $\mathcal{N}_M(A)$ such that $\{Nu_j\}_{j\in J}$ are mutually orthogonal subspaces and $\Sigma Nu_j = M$.*

PROOF. Let $\{v_j\}_{j\in J}$ be a maximal family of unitaries in $\mathcal{N}_M(A)$ such that $\{Nv_j\}_{j\in J}$ are mutually orthogonal. If $\Sigma_{j\in J} Nv_j = M$, then we are done. If not, then, as in the proof of the preceding proposition, there exists a nonzero partial isometry $v \in \mathcal{G}\mathcal{N}_M(A)$ orthogonal to $\Sigma_{j\in J} Nv_j$. Let

$$W = \{w \in \mathcal{G}\mathcal{N}_M(A) \mid w \text{ orthogonal to } \sum_j Nv_j, wv^*v = v\}.$$

Given $w_1, w_2 \in W$, we write $w_1 \leqq w_2$ if $w_2 w_1^* w_1 = w_1, w_2 \neq w_1$ (that is, w_1 is a restriction of w_2). $(W, \leqq)$ is clearly inductively ordered, so we can take $w \in W$ a maximal element. Then w is not a unitary element, because of the maximality of the family $\{v_j\}_{j\in J}$. Let $e \leqq 1 - w^*w$ be a nonzero projection in A. We claim that one can find a set of unitaries $\{u_j^0\}_{j\in J}$ in M_e satisfying the statement for the triple $A_e \subset N_e \subset M_e$. Indeed, since A is a Cartan subalgebra in N, there exist partial isometries $\{w_j\}_{j\in J}$ in $\mathcal{G}\mathcal{N}_N(A)$ such that $w_j v_j e v_j^* w_j^* = w_j w_j^* = e, j \in J$. Then $u_j^0 = w_j v_j$ are all in $\mathcal{G}\mathcal{N}_M(A)$ and they are unitary elements when regarded in M_e. It is easy to see that $\{N_e u_j^0\}_{j\in J}$ are mutually orthogonal in M_e. If $\Sigma_{j\in J} N_e u_j^0 \neq M_e$, then let $w_0 \in \mathcal{G}\mathcal{N}_{M_e}(A_e)$ be a nonzero partial isometry orthogonal to all $N_e u_j^0$. Take v_0 a partial isometry in $\mathcal{G}\mathcal{N}_N(A)$ such that $v_0^* v_0 = w_0 w_0^*$ and $v_0 v_0^* \leqq 1 - ww^*$. This is possible because

$$\tau(w_0 w_0^*) \leqq \tau(e) \leqq \tau(1 - w^*w) = \tau(1 - ww^*)$$

and also because A is a Cartan subalgebra in N. So $w + v_0 w_0$ is in W and $w \leqq w + v_0 w_0$, contradicting the maximality of w.

We have thus obtained a family $\{u_j^0\}_{j\in J}$ in M_e satisfying the conditions for $A_e \subset N_e \subset M_e$. In particular we can take e so that $\tau(e)^{-1}$ is an integer. Let $e = e_0, e_1, \ldots, e_n$ be projections in A such that $\tau(e_i) = \tau(e)$, $0 \leqq i \leqq n$, $\Sigma_{i=0}^n e_i = 1$. Since A is Cartan in N, there exist partial isometries $\{e_{0i}\}_{1 \leqq i \leqq n}$ in $\mathcal{G}\mathcal{N}_N(A)$ such that $e_{0i} e_{0i}^* = e_0, e_{0i}^* e_{0i} = e_i$. Finally, take

$$u_j = \sum_{i=0}^{n} e_{0i}^* u_i^0 e_{0i},\ j \in J.$$

An easy computation shows that $\{u_j\}_{j\in J}$ satisfy the conditions for $A \subset N \subset M$.

2.4. COROLLARY. *Let M be a type II_1 factor and $N \subset M$ a subfactor that contains a Cartan subalgebra of M. Then the index $[M:N]$ of N in M is an integer or ∞ (for the definition of the index of a subfactor see* [8]).

PROOF. By the preceding theorem, there exist unitaries $\{u_j\}_{j\in J}$ such that $\{Nu_j\}_{j\in J}$ are mutually orthogonal and $\sum_{j\in J} Nu_j = M$. If P_{Nu_j} denotes the extension by continuity of E_{Nu_j} to $L^2(M,\tau)$, then $P_{Nu_j} \in N'$ and

$$P_{Nu_j} = (Ju_j^* J)(P_{N1_M})(Ju_j J),$$

where J is the canonical conjugation. Since $Ju_j J \in M' \subset N'$ it follows that all the projections P_{Nu_j} are equivalent to P_{N1_M} in N'. Since $\Sigma_{j\in J} P_{Nu_j} = 1$ it follows by the definition of the index of a subfactor that $[M:N] = \operatorname{card} J$.

2.5 COROLLARY. *Let M be a separable type II_1 factor with a Cartan subalgebra $A \subset M$. There exists a sequence of unitaries $\{u_n\}_{n\geqq 0}$ in the normalizer of A in M such that $\{Au_n\}_{n\geqq 0}$ are mutually orthogonal and $\Sigma_{n\geqq 0} Au_n = M$.*

PROOF. Let $\{M_n\}_{n\geqq 1}$ be an increasing sequence of type I_{2^n} subfactors of M, each of them with a set of matrix units $\{e_{ij}^n\}_{1\geqq i,j\leqq 2^n}$ satisfying:

1) $e_{ij}^n \in \mathscr{G}\mathscr{N}_M(A),\ n \geqq 1,\ 2^n \geqq i,j \geqq 1;$
2) $e_{ij}^n = e_{2i-1,2j-1}^{n+1} + e_{2i,2j}^{n+1},\ n \geqq 1,\ 2^n \geqq i,j \geqq 1;$
3) A is generated by the projections $\{e_{ii}^n\}_{2^n\geqq i\geqq 1; n\geqq 1}$

(see for instance [12, 3.4]).

Denote by $R = \overline{\bigcup_n M_n}^w$. Then R is the hyperfinite factor and $A \subset R \subset M$. Moreover if $v = \Sigma_{n\geqq 1} e_{2,2^n-1}^n$, then it is easy to verify that v is a unitary in the normalizer of A and that v and A generate R. Thus v acts ergodically on A so that $\{Av^k\}_{k\in \mathbf{Z}}$ are mutually orthogonal and $\Sigma_{k\in\mathbf{Z}} Av^k = R$. Since M is separable, by Theorem 2.3 we get a sequence of unitaries $\{v_n\}_{n\geqq 1}$ in $\mathscr{N}_M(A)$ such that $\{Rv_n\}_{n\geqq 1}$ are mutually orthogonal and $\Sigma_n Rv_n = M$. Thus $\{Av^k v_n\}_{k\in\mathbf{Z}, n\in\mathbf{N}}$, are mutually orthogonal subspaces and fill up M.

2.6. COROLLARY. *If M is a separable type II_1 factor with a Cartan subalgebra A, then there exists an orthogonal basis of $L^2(M,\tau)$ with all the elements unitaries from $\mathcal{N}_M(A)$.*

PROOF. Since A is diffuse and separable, there exists a unitary $u \in A$ such that $\tau(u^k) = 0$, $k \in \mathbb{Z} \setminus \{0\}$, and $\overline{\operatorname{span}}^w\{u^k | k \in \mathbb{Z}\} = A$. Thus, if $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_M(A)$ are such that $\{Au_n\}_{n\in\mathbb{N}}$ decompose M as in the preceding corollary, then $\{u^k u_n\}_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}}}$, satisfy the conditions.

3. The algebra generated by A and JAJ.

Let A be a maximal abelian $*$-subalgebra of the type II_1 factor M and denote by $\mathcal{A}$ the von Neumann algebra generated in $\mathcal{B}(L^2(M,\tau))$ by A and JAJ, where J is the canonical conjugation in the standard representation of M.

The algebra $\mathcal{A}$ associated in this way with a Cartan subalgebra A in M plays a fundamental role in the proof of the Connes-Feldman-Weiss theorem. Using the decompositions obtained in Section 2, we give now a more detailed description of $\mathcal{A}$.

If $\mathcal{B} \subset \mathcal{B}(L^2(M,\tau))$ is a von Neumann algebra and $\xi \in L^2(M,\tau)$, then we denote by $P_{\mathcal{B}\xi}$ the orthogonal projection on $\overline{\mathcal{B}\xi}$. Thus $P_{\mathcal{B}\xi} \in \mathcal{B}'$.

3.1 LEMMA. *If $v \in \mathcal{G}\mathcal{N}_M(A)$, then $P_{Av} = P_{\mathcal{A}v} = P_{\mathcal{A}'v} \in \mathcal{A}$, $AP_{Av} = \mathcal{A}P_{Av} = A'P_{Av}$.*

PROOF. The equality $P_{Av} = P_{\mathcal{A}v}$ follows easily, since $\overline{Av}^\tau = \overline{vA}^\tau = \overline{AvA}^\tau$. We first prove the rest of the statement for $v = 1 \in L^2(M,\tau)$. So, let $\{e^n_k\}_{2^n \geqq k \geqq 1, n \geqq 0} \subset A$ be a set of projections such that:

1) $e^0_1 = 1$, $e^n_i = e^{n+1}_{2i-1} + e^{n+1}_{2i}$, $2^n \geqq i \geqq 1$, $n \geqq 0$;
2) $\{e^n_k\}_{n,k}$ generate A.

As shown in [12], since A is maximal abelian in M,

$$\left\| \sum_{k=1}^{2^n} e^n_k x e^n_k - E_A(x) \right\|_2 \to 0 \text{ for all } x \in M.$$

Thus, if $P_n = \sum_{k=1}^{2^n} e^n_k J e^n_k J$, then

$$\| P_n(x) - P_{A1_M}(x) \|_2 \to 0 \text{ for all } x \in M.$$

But $P_n \in \mathcal{A}$ and $P_n^2 = P_n = P_n^*$ are projections so that P_n converge to P_{A1} in the strong operator topology (since M is dense in $L^2(M,\tau)$) and thus $P_{A1} \in \mathcal{A}$. It follows that $P_{A1} = P_{\mathcal{A}1} = P_{\mathcal{A}'1}$ because $P_{\mathcal{A}'1} \geqq P_{\mathcal{A}1} = P_{A1} \in \mathcal{A}$

and $P_{\mathscr{A}'1}$ is the smallest projection p in $\mathscr{A}$ such that $p(1)=1$. Moreover, since the closure of A in $L^2(M,\tau)$ is $P_{A1}(L^2(M,\tau))=L^2(A,\tau_{|A})$, the restriction of A to $P_{A1}(L^2(M,\tau))$ is the standard form of A given by $\tau_{|A}$. Thus AP_{A1} is maximal abelian in $P_{A1}(L^2(M,\tau))$ and since $\mathscr{A}P_{A1}$ is abelian we get $\mathscr{A}P_{A1}$ maximal abelian and $AP_{A1}=\mathscr{A}P_{A1}=\mathscr{A}'P_{A1}$.

Now, for arbitrary $v\in\mathscr{GN}_M(A)$ let $e=v^*v$, $f=vv^*$. Then $x\to vxv^*$ is an isomorphism from $\mathscr{A}e$ onto $\mathscr{A}f$ (because v commutes with JAJ) so that $vP_{A1}v^*\in\mathscr{A}$ and $v(\mathscr{A}P_{A1})v^*$ is maximal abelian on $vP_{A1}v^*$. Since $vP_{A1}v^*$ is the projection on the left support of vP_{A1}, that is on $\overline{vA}^{\tau}=\overline{Av}^{\tau}$, the rest of the statement follows by spatial isomorphism.

From the preceding lemma it follows the result by Feldman and Moore ([7, 2.9]) that if A is a Cartan subalgebra of M then $\mathscr{A}$ is maximal abelian. More precisely:

3.2 COROLLARY. *If B is the von Neumann algebra generated by $\mathscr{N}_M(A)$, then $P_{B1}\in\mathscr{A}$ and $\mathscr{A}P_{B1}=\mathscr{A}'P_{B1}$. In particular if A is a Cartan subalgebra of M, then $\mathscr{A}$ is maximal abelian in $\mathscr{B}(L^2(M,\tau))$.*

PROOF. By the preceding lemma if $u\in\mathscr{N}_M(A)$, then $P_{Au}\in\mathscr{A}\subset\mathscr{A}'$ and $\mathscr{A}P_{Au}=\mathscr{A}'P_{Au}$. But

$$\overline{B1}^{\tau}=\bigvee_{u\in\mathscr{N}_M(A)}\overline{Au}^{\tau}$$

so that

$$P_{B1}=\bigvee_{u\in\mathscr{N}_M(A)}P_{Au}\in\mathscr{A}\quad\text{and}\quad\mathscr{A}P_{B1}=\mathscr{A}'P_{B1}.$$

Suppose now that A is a Cartan subalgebra of M. If $\{v_n\}_{n\geqq 0}$ is a sequence of elements in $\mathscr{GN}_M(A)$, then $\Sigma_{n\geqq 0}P_{Av_n}=1$ if and only if $\{Av_n\}_{n\geqq 0}$ are mutually orthogonal subspaces and $\Sigma_{n\geqq 0}Av_n=M$. By Lemma 3.1 in this case we can identify $\mathscr{A}$ with the algebra of elements $\Sigma_{n\geqq 0}a_nP_{Av_n}$, where $\{a_n\}_{n\geqq 0}$ are norm bounded sequences in A. The expression of $x\in\mathscr{A}$ as $x=\Sigma_{n\geqq 0}a_nP_{Av_n}$ is unique if we require a_n to be supported on $v_nv_n^*$. This is of course the case, if v_n are all unitaries.

Let φ be a normal state on A. Then one can define in a natural way a normal semifinite weight $\tilde\varphi$ on $\mathscr{A}$ such that if $a\in A_+$, $v\in\mathscr{GN}_M(A)$, $a\leq vv^*$, then $\tilde\varphi(aP_{Av})=\varphi(v^*av)$: if $\{u_n\}_{n\geq 0}$ are unitaries in $\mathscr{N}_M(A)$ and $\Sigma_{n\geq 0}P_{Au_n}=1$, then $\tilde\varphi$ acts on $\mathscr{A}$ as

$$\tilde\varphi\Big(\sum_{n\geqq 0}a_nP_{Au_n}\Big)\overset{\text{def}}{=}\sum_n\varphi(u_n^*a_nu_n),$$

where $\Sigma_{n\geqq 0}a_nP_{Au_n}\in\mathscr{A}_+$. This is of course well defined so we need only to show that $\tilde{\varphi}(aP_{Av})=\varphi(v^*av)$. Let $v=\Sigma_{k\geqq 0}s_ku_k$, where s_k are partial isometries in A and $\Sigma_{k\geqq 0}|s_k|=vv^*$ (cf. section 2). Since $\mathscr{A}$ is maximal abelian and $\{|s_k|u_k\}_{k\geqq 0}$ are mutually orthogonal we get

$$P_{Av}=P_{\mathscr{A}v}=P_{\mathscr{A}(\Sigma s_ku_k)}=\sum_{k\geqq 0}P_{\mathscr{A}s_ku_k}=\sum_{k\geqq 0}|s_k|P_{\mathscr{A}u_k}=\sum_{k\geqq 0}|s_k|P_{Au_k}.$$

Thus $aP_{Av}=\Sigma_{k\geqq 0}a|s_k|P_{Au_k}$ and since $s_kv=|s_k|u_k$ we obtain

$$\tilde{\varphi}(aP_{Av})=\tilde{\varphi}\left(\sum_k a|s_k|P_{Au_k}\right)=\sum_{k\geqq 0}\varphi(u_k^*a|s_k|u_k)=\sum_{k\geqq 0}\varphi(v^*a|s_k|v)$$
$$=\varphi(v^*a(\Sigma|s_k|)v)=\varphi(v^*av).$$

In particular we can associate to $\tau_{|A}$ the weight $\tilde{\tau}$ on $\mathscr{A}$. In this case

$$\tilde{\tau}\left(\sum_n a_nP_{Au_n}\right)=\sum_n\tau(u_n^*a_nu_n)=\sum_n\tau(a_n).$$

Let us show that $\tilde{\tau}$ is invariant to the automorphisms implemented on $\mathscr{A}$ by the unitaries in $\mathscr{N}_M(A)$. So let $v\in\mathscr{G}\mathscr{N}_M(A)$ and for each $k\geqq 0$ decompose vu_k as $vu_k=\Sigma_ns_{kn}u_n$, where s_{kn} are partial isometries in A and $\Sigma_{k\geqq 0}|s_k|=vv^*$ (cf. section 2). Since $\mathscr{A}$ is maximal abelian and $\{|s_k|u_k\}_{k\geqq 0}$ are mutually orthogonal we get

$$P_{Av}=P_{\mathscr{A}v}=P_{\mathscr{A}(\Sigma s_ku_k)}=\sum_{k\geqq 0}P_{\mathscr{A}s_ku_k}\underset{a}{P}_{Av}=\sum_{k\geqq 0}a|s_k|P_{Au_k}$$ and since $s_kv=|s_k|u_k$

we obtain

$$\tilde{\varphi}(aP_{Av})=\tilde{\varphi}\sum_k a|s_k|P_{Au_k})=\sum_{k\geqq 0}\varphi(u_k^*a|s_k|u_k)$$
$$=\sum_{k\geqq 0}\varphi(v^*a|s_k|v)=\varphi(v^*a(\Sigma|s_k|)v)=\varphi(v^*av).$$

In particular we can associate to $\tau_{|A}$ the weight $\tilde{\tau}$ on $\mathscr{A}$. In this case

$$\tilde{\tau}(\sum_n a_nP_{Au_n})=\sum_n\tau(u_n^*a_nu_n)=\sum_n\tau(a_n).$$

Let us show that $\tilde{\tau}$ is invariant to the automorphisms implemented on $\mathscr{A}$ by the unitaries in $\mathscr{N}_M(A)$. So let $v\in\mathscr{G}\mathscr{N}_M(A)$ and for each $k\geqq 0$ decompose vu_k as $vu_k=\Sigma_ns_{kn}u_n$, where s_{kn} are partial isometrics in A and $\Sigma_n|s_{kn}|=vv^*$ for all $k\geqq 0$ (in fact it is easy to see that also $\Sigma_k|s_{kn}|=vv^*$ for all $n\geqq 0$). If $\tilde{a}=\Sigma_na_nP_{Au_n}\in\mathscr{A}$, then

$$v\tilde{a}v^* = v(\sum_{k\geqq 0} a_k P_{Au_k})v^* = \sum_{k\geqq 0} va_kv^*vP_{Au_k}c^*$$
$$= \sum_{k\geqq 0} va_kv^*P_{Avu_k} = \sum_{n\geqq 0}\left(\sum_{k\geqq 0} va_kv^*|s_{kn}|\right)P_{Au_n},$$

so that if $\tilde{a} \geqq 0$, then

$$\tilde{\tau}(v\tilde{a}v^*) = \tilde{\tau}(v\left(\sum_n a_nP_{Au_n}\right)v^*) = \sum_{k\geqq 0}\tau\left(\sum_{k\geqq 0} va_kv^*|s_{kn}|\right)$$
$$= \sum_k \tau(a_kv^*\sum_n |s_{kn}|v) = \sum_k \tau(a_kv^*v) = \tilde{\tau}(\tilde{a}v^*v).$$

These computations will be used in section 4.

3.4 REMARK. Since any automorphism of M is spatial in $\mathcal{B}(L^2(M,\tau))$, it follows that the type of the algebra $\mathcal{A}'$ is an invariant for the maximal abelian $*$-subalgebra A of M. This invariant was first considered by W. Ambrose and I. Singer (unpublished) and by L. Pukanszky who showed that in the hyperfinite II_1 factor there are singular maximal abelian $*$-subalgebras A_n, $n \geqq 1$, such that the corresponding algebras $\mathcal{A}_n$ (where $\mathcal{A}_n$ is the von Neumann algebra generated by A_n and JA_nJ), are homogeneous of type I_n on $1 - P_{A_n 1}$, $n \geqq 1$ (on $P_{A_n 1}$ they are always commutative by 3.1). Our preceding corollary shows that in fact, if $\mathcal{A}_n'$ is I_n homogeneous on $1 - P_{A_n 1}$ and $n \geqq 2$, then A_n has trivial normalizer so that it is automatically singular.

We end this section with the following:

3.4 THEOREM. *If the separable type II_1 factor M has a maximal abelian $*$-subalgebra A such that the von Neumann algebra $\mathcal{A}$ generated by A and JAJ is maximal abelian in $\mathcal{B}(L^2(M,\tau))$, then M is single generated. In particular if M has a Cartan subalgebra then M is single generated.*

The idea of the proof is as follows: since $\mathcal{A}$ is maximal abelian over $L^2(M,\tau)$, it has a bicyclic vector, i.e. there exists a square integrable operator ξ affiliated with M such that $\overline{\mathrm{span}}^{\tau} A\xi A = L^2(M,\tau)$. If ξ would be selfadjoint, then take $h_0 \in M_+$ such that ξ is affiliated with the von Neumann algebra A_0 generated by h_0. Take also $h \in A_+$, $\{h\}'' = A$. Then the von Neumann algebra M_0 generated by $h + ih_0$ contains A and A_0 so that

$$\overline{M_0}^{\tau} \supset \overline{\mathrm{span}}^{\tau} AA_0A \supset \overline{\mathrm{span}}^{\tau} A\xi A = \overline{\mathrm{span}}^{\tau} AJAJ\xi = \overline{\mathcal{A}\xi}^{\tau} = L^2(M,\tau)$$

and $M_0 = M$. If ξ is not selfadjoint, then we use the following:

3.5 LEMMA. *Let $B \in \mathcal{B}(\mathcal{H})$ be a commutative von Neumann algebra and for $\zeta \in \mathcal{H}$ denote by P_ζ the orthogonal projection $P_{B'\zeta}$ onto $\overline{B'\zeta}$. If $\xi, \eta \in \mathcal{H}$ then the set*

$$L = \{\lambda \in \mathbf{C} \setminus \{0\} \mid P_{\xi+\lambda\eta} \neq P_\xi \vee P_\eta\}$$

is countable.

PROOF. For $\xi \in \mathscr{H}$ denote by P'_ξ the orthogonal projection $P_{B\xi}$ onto $B\xi$. Thus $P_\xi \in B$, $P'_\xi \in B'$, and since B is commutative, $B \subset B'$ so that $P'_\xi \leqq P_\xi$. Now, if $P_\xi P_\eta = 0$, then $P'_\xi P'_\eta = 0$ so that by ([13, § 7]) $P_{\xi+\lambda\eta} = P_\xi + P_\eta$ for any $\lambda \neq 0$. So in this case L is empty. If $P = P_\xi \ P_\eta \neq 0$, then P is a projection (since P_ξ, P_η are in B which is abelian) and we denote by $\xi_1 = (P_\xi - P)(\xi)$, $\xi_2 = P(\xi)$, $\eta_1 = (P_\eta - P)(\eta)$, $\eta_2 = P(\eta)$. It follows that $P_{\xi_1} = P_\xi - P$, $P_{\eta_1} = P_\eta - P$, $P_{\xi_2} = P_{\eta_2} = P$, and $P_{\xi_2+\lambda\eta_2} = PP_{\xi+\lambda\eta} \leqq P$, for all $\lambda \in \mathbf{C}$. Since P_{ξ_1}, P_{η_1} and $P_{\xi_2+\lambda\eta_2}$ are mutually orthogonal projections, P'_{ξ_1}, P'_{η_1}, and $P'_{\xi_2+\lambda\eta_2}$ are also mutually orthogonal (because $B' \supset B$), so that for any $\lambda \neq 0$ we have

$$P_{\xi+\lambda\eta} = P_{\xi_1+(\xi_2+\lambda\eta_2)+\lambda\eta_1} = P_{\xi_1} + P_{\xi_2+\lambda\eta_2} + P_{\eta_1} = P_\xi \vee P_\eta - (P - P_{\xi_2+\lambda\eta_2}).$$

Thus the set L may be characterized as

$$L = \{\lambda \in \mathbf{C} \setminus \{0\} \mid P - PP_{\xi+\lambda\eta} \neq 0\}.$$

We infer that if $\lambda_1, \lambda_2 \in \mathbf{C} \setminus \{0\}$, $\lambda_1 \neq \lambda_2$, are such that

$$P - PP_{\xi+\lambda_1\eta} \neq 0 \text{ and } P - PP_{\xi+\lambda_2\eta} \neq 0,$$

then $P - PP_{\xi+\lambda_1\eta}$ and $P - PP_{\xi+\lambda_2\eta}$ are mutually orthogonal. Indeed, because then the projection $q = (P - PP_{\xi+\lambda_1\eta})(P - PP_{\xi+\lambda_2\eta})$ satisfies $q(\xi + \lambda_1\eta) = q(\xi + \lambda_2\eta) = 0$, so that $\lambda_1 q(\eta) = -q(\xi) = \lambda_2 q(\eta)$ and thus $q(\xi) = q(\eta) = 0$; since $q \leqq P_\xi$, $q \leqq P_\eta$, this is impossible unless $q = 0$. We have thus proved that the projections $\{P - PP_{\xi+\lambda\eta}\}_{\lambda \in L}$ are mutually orthogonal. Since they are all dominated by P, which is countably decomposable in B, L is countable.

To end the proof of the theorem, let $\zeta = \zeta_1 + i\zeta_2$, where ζ_1, ζ_2 are selfadjoint square integrable operators affiliated with M. By the preceding lemma there exists $t \in \mathbf{R} \setminus \{0\}$ such that

$$\overline{\mathscr{A}(\zeta_1 + t\zeta_2)} = \overline{\mathscr{A}\zeta_1} \vee \overline{\mathscr{A}\zeta_2} = L^2(M, \tau).$$

Since $\zeta_1 + t\zeta_2$ is selfadjoint, we are done.

4. The Connes–Feldmann–Weiss theorem.

4.1. THEOREM (Connes-Feldman-Weiss [3]). *If A_1, A_2 are Cartan subalgebras of the hyperfinite type* II_1 *factor R, then there exists an automorphism $\theta \in \mathrm{Aut}(R)$ such that $\theta(A_1) = A_2$.*

From the hyperfiniteness of R the proof will use only the existence of a hypertrace Φ_0 on R, i.e. of a (nonormal) state Φ_0 on $\mathscr{B}(L^2(M,\tau))$ having R in its centralizer. Actually one needs only the existence of a state on $\mathscr{A}$ invariant to all the automorphisms of $\mathscr{A}$ implemented by unitaries in the normalizer of A in R. The restriction to $\mathscr{A}$ of the hypertrace Φ_0 satisfies this property, since if $v \in \mathscr{G}\mathscr{N}_M(A)$, then v is in the centralizer of Φ_0 hence

$$\Phi_0(v\tilde{a}v^*) = \Phi_0\,(\tilde{a}v^*v) \text{ for all } \tilde{a} \in \mathscr{A}.$$

So in fact it will be proved that if M is a separable type II_1 factor with hypertrace and A is a Cartan subalgebra of M, then there exists an increasing sequence of $2^n \times 2^n$ matrix subalgebras $\{M_n\}_{n \geqq 1}$ in M each of them with a set of matrix units $\{e_{ij}^n\}_{2^n \geqq ij \geqq 1}$ satisfying

$$e_{2i-1,2j-1}^{n+1} + e_{2i,2j}^{n+1} = e_{ij}^n,$$

such that $\{e_{ii}^n\}_{i,n}$ are all in A and generate it as a von Neumann algebra and such that $(\bigcup_{n \geqq 1} M_n)^- = M$. This clearly implies the statement and besides it shows that an injective type II_1 factor having a Cartan subalgebra is isomorphic to the hyperfinite factor R, although the proof will not use [2].

The whole proof of the theorem relies on the following.

4.2. LEMMA ([3, 9]). *Let A be a Cartan subalgebra of R, $\{v_1, v_2, \ldots, v_n\}$ a selfadjoint set of partial isometries in $\mathscr{G}\mathscr{N}_R(A)$ and $\varepsilon > 0$. There exists a type I_{2^m} subfactor M_0 of R with a set of matrix units in the normalizer $\mathscr{G}\mathscr{N}_R(A)$ such that $\|E_{M_0}(v_i) - v_i\|_2 < \varepsilon$, $n \geqq i \geqq 1$.*

The idea is to prove first that ([3, Lemma 8]):

(*) *There exixts a matrix subalgebra $N_0 \subset R$ with the unit e_0 in A and with a set of matrix units in $\mathscr{G}\mathscr{N}_R(A)$ such that*

$$\| E_{M_0}(e_0 v_i e_0) - (v_i - (1-e_0)v_i(1-e_0)) \|_2 < \varepsilon \| e_0 \|_2, \; n \geqq i \geqq 1.$$

Then by a maximality argument the lemma will follow easily.

To prove (*) one uses Day's trick. So, let

$$\mathscr{L} = \{\Psi(v_i \cdot v_i^*) - \Psi(\cdot\, v_i^* v_i))_{n \geqq i \geqq 1} \,|\, \Psi \text{ a normal state on } \mathscr{A}\}.$$

Then $\mathscr{L}$ is a convex subset in $(\mathscr{A}_*)^n$ and its closure in $(\mathscr{A}^*)^n$ in the duality topology $\tau((\mathscr{A}^*)^n, \mathscr{A}^n)$ contains all elements of the form $(\Phi(v_i \cdot v_i^*) - \Phi(\cdot\, v_i^* v_i))_{n \geqq i \geqq 1}$ with Φ a state on $\mathscr{A}$. In particular it contains

$$(0, \ldots, 0) = (\Phi_0(\cdot\, v_i^* v_i))_{n \geqq i \geqq 1}.$$

Thus, the $\sigma((\mathscr{A}_*)^n, \mathscr{A}^n)$ closure of $\mathscr{L}$ in $(\mathscr{A}_*)^n$ contains 0. Since $\mathscr{L}$ is convex, this coincides with the norm closure of $\mathscr{L}$ in $(\mathscr{A}_*)^n$ so that there exists a normal state $\Psi \in \mathscr{A}_*$ such that

$$\|\Psi(v_i \cdot v_i^*) - \Psi(\cdot\, v_i^* v_i)\| < \varepsilon^2/2n,\ n \geqq i \geqq 1.$$

Since the normal semifinite weight $\tilde{\tau}$ on $\mathscr{A}$ constructed from the trace $\tau_{|A}$, as in section 3, is faithfull on $\mathscr{A}$, Ψ is a Radon-Nykodim derivative of $\tilde{\tau}$ and so it may be approximated in norm with a state of the form $\tilde{\tau}(\cdot\,\tilde{a})$, where

$$\tilde{a} = \sum_{k=0}^{m} a_k P_{Au_k} \in \mathscr{A}_+,\ \tilde{\tau}(\tilde{a}) = \Sigma\tau(a_k) = 1,$$

to have:

(1) $\quad |\tilde{\tau}(v_i\tilde{x}v_i^*\tilde{a}) - \tilde{\tau}(\tilde{x}v_i^*v_i\tilde{a})| < \varepsilon^2/2n\|\tilde{x}\|,\ \tilde{x}\in\mathscr{A},\ \tilde{x}\neq 0,\ n\geqq i\geqq 1.$

As we pointed out that $\tilde{\tau}$ is invariant to conjugation by elements in $\mathscr{G}\mathscr{N}_R(A)$, we get:

(2) $\quad |\tilde{\tau}(\tilde{x}(v_i^*\tilde{a}v_i - \tilde{a}v_i^*v_i))| < \varepsilon^2/2n\|\tilde{x}\|,\ n\geqq i\geqq 1.$

In particular, for each $n\geqq i\geqq 1$ take $\tilde{x}$ to be the adjoint of the partial isometry in the polar decomposition of $v_i^*\tilde{a}v_i - \tilde{a}v_i^*v_i\in\mathscr{A}$. Since $\tilde{\tau}(\tilde{a}) = 1$ it follows that:

(3) $$\sum_i \tilde{\tau}(|v_i^*\tilde{a}v_i - \tilde{a}v_i^*v_i|) < (\varepsilon^2/2)\tilde{\tau}(\tilde{a}).$$

By the Namioka trick (see [2, 2.1.4] or [1, 1.2.2]) this last inequality yields a similar one satisfied by a nonzero spectral projection of $\tilde{a}$. So we may assume that in (3), $a = \Sigma_{k=0}^m a_k P_{Au_k}$ is a nonzero projection in $\mathscr{A}$ or equivalently that a_k are projections and at least one is nonzero. Let $v_i^* u_k = \Sigma_{j\geqq 0} s_{kj}^i u_j$ be the decomposition of $v_i^* u_k\in\mathscr{G}\mathscr{N}_R(A)$, where s_{kj}^i are partial isometries in A and

$$\sum_{j\geqq 0}|s_{kj}^i| = v_i^*v_i,\ k\geqq 0,\ n\geqq i\geqq 1.$$

As was shown in section 3 we have

$$v_i^*\tilde{a}v_i = \sum_{j\geqq 0}\left(\sum_{k=0}^{m} v_i^* a_k v_i |s_{kj}^i|\right)P_{Au_j},$$

and by the definition of $\tilde{\tau}$ we get:

(4)
$$\sum_{i=1}^{n}\sum_{j=0}^{m}\tau\left(u_j^*\left|\sum_{k=0}^{m} a_j v_i^* a_k v_i |s_{kj}^i| - a_j v_i^* v_i\right| u_j\right)$$
$$\leqq \sum_{i=1}^{n}\sum_{j\geqq 0}\tau\left(u_j^*\left|\sum_{k=0}^{m} v_i^* a_k v_i |s_{kj}^i| - a_j v_i^* v_i\right| u_j\right) < (\varepsilon/2)\sum_{j=0}^{m}\tau(u_j^* a_j u_j)$$

Since all the terms in the sums implicated in the above inequality are in A

which is commutative, it follows that there exists a nonzero projection e in A such that:

(5) $$\sum_{i=1}^{n}\sum_{j=0}^{m} eu_j^*\left|\sum_{k=0}^{m} a_j v_i^* a_k v_i |s_{kj}^i| - a_j v_i^* v_i\right| u_j < (\varepsilon^2/2)\sum_{j=0}^{m} eu_j^* a_j u_j.$$

Now we take into account that $|s_{kj}^i| v_i^* u_k = s_{kj}^i u_j$ so that $|s_{kj}^i| v_i^* = s_{kj}^i u_j u_k^*$ and we get:

(6) $$\sum_{i=1}^{n}\sum_{j=0}^{m} eu_j^*\left|\sum_{k=0}^{m} s_{kj}^i a_j u_j u_k^* a_k v_i - a_j v_i^* v_i\right| u_j < (\varepsilon^2/2)\sum_{j=0}^{m} eu_j^* a_j u_j.$$

Any nonzero projection in A smaller than e still satisfy (6) so one can choose it such that the partial isometries $eu_0^*, eu_1^*, \ldots, eu_m^*$ have the right supports mutually orthogonal and compatible with $a_0, a_1, \ldots, a_m$ respectively, that is, $eu_j^* a_j = eu_j^*$ or $eu_j^* a_j = 0$ for each $m \geqq j \geqq 0$. This is possible because for all $n \neq m$, $u_n^* u_m$ acts properly outer on A. Let $I = \{0 \leqq j \leqq m | eu_j^* = eu_j^* a_j\}$, and for $j, k \in I$ let $e_{jk} = u_j eu_k^*$. Then clearly $\{e_{jk}\}_{j,k\in I}$ are matrix units, $e_{jk} \in \mathcal{G}\mathcal{N}_R(A)$ and we get from (6):

(7) $$\begin{aligned}&\sum_{i=1}^{n}\sum_{j\in I}\tau\left(e_{jj}\left|\sum_{k=0}^{m} s_{kj}^i a_j u_j u_k^* a_k v_i - a_j v_i v_i^*\right|\right)\\ &\quad = \sum_{i=1}^{n}\sum_{j=0}^{m}\tau\left(eu_j^*\left|a_j\sum_{k=0}^{m} s_{kj}^i a_j u_j u_k^* a_k v_i - a_j v_i v_i^*\right| u_j\right)\\ &\quad < (\varepsilon^2/2)\sum_{j=0}^{m}\tau(eu_j^* a_j u_j) = (\varepsilon^2/2)\sum_{j\in I}\tau(e_{jj}).\end{aligned}$$

(Note that the strict inequality in (6) implies that $I \neq \varnothing$). But $e_{jj} a_j u_j u_k^* a_k = e_{jj} u_j eu_k^* a_k$ is equal to zero if $k \notin I$ and to e_{jk} if $k \in I$ so that (7) becomes:

(8) $$\sum_i \sum_{j\in I}\tau\left(\left|\sum_{k\in I} s_{kj}^i e_{jk} v_i - e_{jj} v_i^* v_i\right|\right) < (\varepsilon^2/2)\tau(e_0).$$

where $e_0 = \sum_{j\in I} e_{jj}$. Finally we obtain:

(9) $$\begin{aligned}&\sum_{i=1}^{n}\left\|\sum_{j\in I}\sum_{k\in I} s_{kj}^i e_{jk} v_i v_i^* - e_0 v_i^*\right\|_2^2\\ &\quad = \sum_{i=1}^{n}\sum_{j\in I}\sum_{k\in I}\left\|s_{kj}^i e_{jk} v_i v_i^* - e_{jj} v_i^*\right\|_2^2\\ &\quad = \sum_{i=1}^{n}\sum_{j\in I}\left\|\sum_{k\in I} s_{kj}^i e_{jk} v_i - e_{jj} v_i^* v_i\right\|_2^2\\ &\quad = \sum_{i=1}^{n}\sum_{j\in I}\tau\left(\left|\sum_{k\in I} s_{kj}^i e_{jk} v_i - e_{jj} v_i^* v_i\right|\right) < (\varepsilon^2/2)\tau(e_0) = (\varepsilon^2/2)\|e_0\|_2^2\end{aligned}$$

We can now take a set of matrix units $\{e_{rs}^0\}$ in $\mathcal{GN}_R(A)$, refining $\{e_{ij}\}$, with the same support e_0, such that if N_0 denotes the algebra generated by $\{e_{rs}^0\}$, then $\| E_{N_0}(s_{kj}^i e_0) - s_{kj}^i e_0 \|_2$ and $\| E_{N_0}(v_i v_i^* e_0) - v_i v_i^* e_0 \|_2$ are small enough to insure that still we have:

$$\sum_{i=1}^{n} \left\| E_{N_0}\left(\sum_{k,j} s_{kj}^i e_{jk} v_i v_i^* \right) - e_0 v_i^* \right\|_2^2 < (\varepsilon^2/2) \| e_0 \|_2^2 \tag{10}$$

(this is possible because all $v_i v_i^* e_0$ and $s_{kj}^i e_0$ belong to Ae_0).

Since $E_{N_0}(e_0 v_i e_0)$ is the closest point to $v_i e_0$ in N_0 we get:

$$\sum_{i=1}^{n} \left\| E_{N_0}(e_0 v_i e_0) - v_i e_0 \right\|_2^2 < (\varepsilon^2/2) \| e_0 \|_2^2. \tag{11}$$

In particular, since $\{v_i\}_{n \geqq i \geqq 1} = \{v_i^*\}_{n \geqq i \geqq 1}$, we have

$$\sum_{i=1}^{m} \left\| e_0 v_i e_0 - v_i e_0 \right\|_2^2 = \sum_{i=1}^{n} \left\| e_0 v_i (1 - e_0) \right\|_2^2 = \sum_{i=1}^{n} \left\| (1 - e_0) v_i e_0 \right\|_2^2$$
$$< (\varepsilon^2/2) \left\| e_0 \right\|_2^2,$$

so that:

$$\sum_{i=1}^{n} \left\| E_{N_0}(e_0 v_i e_0) - (v_i - (1 - e_0) v_i (1 - e_0)) \right\|_2^2 < \varepsilon^2 \| e_0 \|_2^2 \tag{12}$$

and $(*)$ is proved.

Consider now the set of all families of matrix subalgebras $\{N_j\}$ having mutually orthogonal supports $\{e_j\}$ in A, such that each N_j is generated by some matrix units in $\mathcal{GN}_R(A)$ and such that if $e = \Sigma e_j$, then

$$\sum_i \| E_{\bigoplus_j N_j}(e v_i e) - (v_i - (1 - e) v_i (1 - e)) \|_2^2 < \varepsilon^2 \| e \|_2^2.$$

This set is clearly inductively ordered with respect to inclusion, so we can take a maximal family $\{N_j^0\}_{j \in J}$; suppose that the supports e_j^0 of N_j^0 do not fill up the unity and take $e = \Sigma e_j^0$, $f = 1 - e \neq 0$. By the first part of the proof applied to the partial isometries $f v_i f$ in R_f, there exists a nonzero matrix algebra $N_0 \subset R_f$, supported on some $e_0 \in A$, generated by matrix units from $\mathcal{GN}_R(A)$ and satisfying:

$$\sum_i \left\| E_{N_0}(e_0 v_i e_0) - (f v_i f - (f - e_0) v_i (f - e_0)) \right\|_2^2 < \varepsilon^2 \| e_0 \|_2^2.$$

It follows that:

$$\sum_i \| E_{\underset{j}{\oplus} N_j^0 \oplus N_0}((e+e_0)v_i(e+e_0)) - (v_i - (1-(e+e_0))v_i(1-(e+e_0))) \|_2^2$$

$$= \sum_i \| E_{\underset{j}{\oplus} N_j^0}(ev_ie) - (v_i - (1-e)v_i(1-e) \|_2^2$$

$$+ \sum_i \| E_{N_0}(e_0v_ie_0) - (fv_if - (f-e_0)v_i(f-e_0)) \|_2^2$$

$$< \varepsilon^2 \| e \|_2^2 + \varepsilon^2 \| e_0 \|_2^2 = \varepsilon^2 \| e + e_0 \|_2^2,$$

which is in contradiction with the maximality of the family $\{N_j^0\}_j$.

So at this moment we have a family of matrix algebras $\{N_j^0\}_{j \in J}$ with the supports $\{e_j^0\}$ in A satisfying $\Sigma e_j^0 = 1$ such that each N_j^0 is generated by matrix units from $\mathcal{G}\mathcal{N}_R(A)$ and

$$\sum_{i=1}^n \| v_i - E_{\underset{j}{\oplus} N_j^0}(v_i) \|_2^2 < \varepsilon^2,$$

since R is separable the family is countable so we may suppose $J = \mathbf{N}$. Thus, for m sufficiently large:

$$\sum_{i=1}^n \| v_i - E_{\overset{m}{\underset{j=1}{\oplus}} N_j^0 \oplus \mathbf{C}}(v_i) \|_2^2 < \varepsilon^2.$$

A standard argument shows that we can slightly modify $N_j^0, m \geqq j \geqq 1$, such that their minimal projections have dimension of the form $k2^{-r}$, $k,r \in \mathbf{N}$. Then we can find a $2^r \times 2^r$ matrix algebra M_0 with $1_{M_0} = 1_R$, with a set of matrix units in $\mathcal{G}\mathcal{N}_R(A)$, such that $N_j^0 \subset M_0$, $m \geqq j \geqq q$, and the proof of 4.2 is completed.

The proof of 4.1 is now quite simple: Suppose $A \subset R$ is a Cartan subalgebra. Let $\{v_i\}_{i \in \mathbf{N}} \subset \mathcal{G}\mathcal{N}_R(A)$ be a dense subset (in the norm $\| \cdot \|_2$). We construct by induction an increasing sequence of matrix subalgebras $\{M_n\}_{n \geqq 1}$ of R, each of them with a set of matrix units $\{e_{ij}^n\}_{2^{k_n} \geqq i,j \geqq 1}$, such that

1) $e_{ii}^n \in A$, $\Sigma_i e_{ii}^n = 1$;
2) e_{ij}^n are in $\mathcal{G}\mathcal{N}_R(A)$ for all i,j,n;
3) every e_{rs}^p, for $p \leqq n$, is the sum of some e_{ij}^n;
4) $\| E_{M_n}(v_i) - v_i \|_2 < 2^{-n}$, $1 \leqq i \leqq n$.

Suppose we constructed these algebras for $n \leqq m$. Consider the set

$$V = \{e_{1r}v_ie_{s1} \,|\, m+1 \geqq i \geqq 1.\ 2^{k_m} \geqq r,s \geqq 1\}.$$

Then V is a finite set of partial isometries in the normalizer of Ae_{11}^m in $R_{e_{11}^m}$.

So by 4.2 there exists a set of matrix units $\{e'_{ij}\}_{2' \geqq i,j \geqq 1}$ in $\mathscr{G}\mathscr{N}_{R_{e^m_{11}}}(A_{e^m_{11}})$ such that if N_0 denotes the algebra generated by $\{e'_{ij}\}$, then $1_{N_0} = e^m_{11}$ and

$$\| E_{N_0}(e_{1r}v_i e_{s1}) - e_{1r}v_i e_{s1} \|_2 < 2^{-(m+1)}2^{-k_m}.$$

It follows that if $\{e^{m+1}_{ij}\} = \{e'_{kl}e^m_{rs}\}$ and M_{m+1} is the algebra generated by $\{e^{m+1}_{ij}\}$ then the conditions 1)–4) are fulfilled. Indeed, because

$$\begin{aligned}\| E_{M_{m+1}}(v_i) - v_i \|_2^2 &= \sum_{r,s} \left\| E_{M_{m+1}}(e^m_{rr}v_i e^m_{ss}) - e^m_{rr}v_i e_{ss} \right\|_2^2 \\ &= \sum_{r,s} \left\| E_{N_0}(e^m_{1r}v_i e^m_{s1} \right\|_2^2 < 2^{k_m} \cdot 2^{k_m}2^{-2(m+1)} \cdot 2^{-2k_m} \\ &= 2^{-2(m+1)}, \quad m+1 \geqq i \geqq 1.\end{aligned}$$

Since $\{v_i\}_{i \in \mathbb{N}}$ are dense in $\mathscr{G}\mathscr{N}_R(A)$ from 4), we get $\mathscr{G}\mathscr{N}_R(A) \subset (\bigcup_n M_n)^-$ so that $(\bigcup_n M^w_n)^- = R$. Moreover $A_0 = \{e^n_{ii}\}''_{i,n} \subset A$ and since A_0 is maximal abelian in $(\bigcup_n M^w_n)^-$, $A_0 = A$. This ends the proof of 4.1.

FINAL REMARK. The proof of the general case of the Connes-Feldman-Weiss theorem is not much more complicated, than the type II_1 case. In fact, if M has a Cartan subalgebras A and φ is a normal faithful state on A, then $\varphi \circ E_A$ is a normal faithful state on M and we may suppose $M \subset \mathscr{B}(\mathscr{H})$ is represented so that $\omega_{\xi_0} = \varphi \circ E_A$ for some bicyclic vector $\xi_0 \in \mathscr{H}$. Then a similar result as Proposition 2.2 holds. Moreover, if J is the canonical conjugation associated with ξ_0, then by [12, 2.1], it follows that the proof of 3.1 works as well to show that $\mathscr{A} = (A \cup JAJ)''$ is maximal abelian in $\mathscr{B}(\mathscr{H})$ and that using an analogue of 2.2 one can decompose $\mathscr{A}$ as in Section 3.

Then to prove Lemma 4.2 one defines the weight $\tilde{\varphi}$ on $\mathscr{A}$ associated with $\varphi \circ E_A$ (as in Section 3) and instead of using the invariance of $\tilde{\varphi}$ to conjugation by elements u in $\mathscr{N}_M(A)$ one uses the fact that $\tilde{\varphi} \circ \mathrm{Ad}u$ is a Radon-Nykodim derivative of $\tilde{\varphi}$.

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SORIN POPA
DEPARTMENT OF MATHEMATICS
INCREST
BD. PACIL 220
79622 BUCHAREST
ROMANIA