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INTEGRABLE-ERGODIC C*-DYNAMICAL SYSTEMS ON ABELIAN GROUPS

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Abstract.

In this paper we introduce a notion of combined integrability and ergodicity for actions of locally compact, abelian groups on C*-algebras. We prove that for a fixed, second countable group G, the set $[G]$ of all covariantly non-isomorphic, integrable-ergodic and faithful C*-dynamical systems $(\mathscr{A}, G, \beta)$, can be classified by means of $H_b^2(\hat{G}, \boldsymbol{T})$, the second Borel-cohomology group over $\hat{G}$. This is a direct generalization of a result of D. Olesen, G. K. Pedersen, and M. Takesaki on ergodic systems over compact, abelian groups.

Introduction.

In [10], D. Olesen, G. K. Pedersen, and M. Takesaki classify all ergodic and faithful W*-dynamical systems $(\mathscr{M}, G, \alpha)$ on a fixed, compact, abelian group G.

They show that the set $[G]$ of covariantly non-equivalent, ergodic and faithful W*-dynamical systems over G admits a multiplication, so that $[G], \times$ is isomorphic to $\chi^2(\hat{G}, \boldsymbol{T})$, the group of anti-symmetric bicharacters of $\hat{G}$. The classification in the C*-case then became trivial, since they could prove that under the conditions of ergodicity and faithfulness, we have a one-to-one correspondence between W*- and C*-systems.

These results, as far as the W*-case is concerned, have been generalized in a number of different ways. In [13], A. Wassermann was successful in giving the classification for systems on non-abelian groups. H. H. Zettl, in [15], shows how the ergodicity of the action can be weakened down to the condition that the fixed-point algebra $\mathscr{M}^\alpha$ is contained in the centre $Z(\mathscr{M})$ of $\mathscr{M}$. Finally, turning the attention towards locally compact, abelian groups it is proved in [4] that here the proper setting for the W*-classification theorem is that of integrable, ergodic and faithful systems.

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In this paper, we concentrate on the C*-case. We introduce a notion of integrable-ergodic C*-dynamical systems, and – by showing that to each such system corresponds a unique system of [4] – we classify them by means of $H_b^2(\hat{G},\boldsymbol{T})$, the second Borel-cohomology group of $\hat{G}$. The conclusion will be that every integrable-ergodic (in short I-E) and faithful C*-dynamical system $(\mathscr{A},G,\beta)$ on a second countable, locally compact, abelian group G, is of the form $(C^*_{r,\omega}(\hat{G}), \mathrm{ad}\, v)$, where $C^*_{r,\omega}(\hat{G})$ is the twisted, reduced group C*-algebra of $\hat{G}$, and $(v_s f)(p) = \langle s,p\rangle f(p)$, $f\in L^2(\hat{G})$, $s\in G$, $p\in\hat{G}$.

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1. Definition and basic facts.

Throughout these notes G will denote a second countable, abelian, locally compact Hausdorff group with Haarmeasure ds. $(\mathscr{M},G,\alpha)$, respectively $(\mathscr{A},G,\beta)$, will be a W*-, respectively C*-dynamical system over G with a continuous, faithful and ergodic action α, respectively β. The condition of ergodicity on β is that the fixedpoint algebra $\mathscr{A}^\beta$ must be either $\mathbf{C}.1$ or $\{0\}$, depending on whether or not $\mathscr{A}$ is unital.

By η_α we denote the set of all $x\in\mathscr{M}$ such that $\int\varphi\alpha_s(x^*x)ds<\infty$ for all $\varphi\in(\mathscr{M}_*)_+$ and μ_α will stand for $\eta_\alpha\eta_\alpha$. It is well-known that $(\mu_\alpha)_+$ consists precisely of those elements in $\mathscr{M}_+$ which are α-integrable. More specifically, in the ergodic setting the conditions

1) $x\in(\mu_\alpha)_+$, and
2) there exists a $r_x\in\mathbf{R}_+$ so that $\int\varphi\alpha_s(x)ds = r_x\,.\,\|\varphi\|$ for all $\varphi\in(\mathscr{M}_*)_+$,

are equivalent, where 2) expresses that $\int\alpha_s(x)ds = r_x\,.\,1$. By a definition of A. Connes and M. Takesaki in [2] the action α is called integrable when μ_α is σ-weakly dense in $\mathscr{M}$. For $(\mathscr{A},G,\beta)$ a similar definition is possible.

1.1. Definition. An $x\in\mathscr{A}_+$ is called β-integrable whenever there exists a $r_x\in\mathbf{R}_+$ so that

$$\int\varphi\beta_s(x)ds = r_x\|\varphi\|,\quad \varphi\in(\mathscr{A}^*)_+.$$

The system $(\mathscr{A},G,\beta)$ is integrable if the set

$$\mu_\beta = \operatorname{span}\{x\in\mathscr{A}_+\,|x \text{ is } \beta\text{-integrable}\}$$

is norm-dense in $\mathscr{A}$.

Let us give one example.

1.2 EXAMPLE. The easiest and best-known integrable, ergodic and faithful W*-dynamical systems is $(L^\infty(G), G, \alpha^1)$, where $\alpha_s^{-1}(f)(t) = f(t-s)$. It contains the C*-system $(C_0(G), G, \beta^1)$, $\beta^1 = \alpha^1|_{C_0(G)}$, which is still faithful and ergodic. Since $(C_0(G)^*)_+$ consists of all positive, bounded measures on G, we have for any $f \in C_c(G)$:

$$\int \varphi\beta_s^1(f)ds = \int\int f(t-s)dsd\mu_\varphi(t)$$
$$= \int f(s)ds \,.\, \mu_\varphi(G),$$

where $\varphi = \mu_\varphi$ in $(C_0(G)^*)_+$.

One of the main statements of [4] is that for an ergodic and faithful W*-dynamical system the integrability of α is equivalent to the condition that for each $p \in \hat{G}$ there exists a unitary operator $u_p \in \mathcal{M}$, so that

$$\alpha_s(u_p) = \langle s,p\rangle u_p, \quad s \in G.$$

Example 1.2 shows that this is no longer true in the C*-case. Since $C_0(G)$ is non-unital it does not contain unitaries. This observation also holds in general.

1.3. LEMMA. *Let G be non-compact and $(\mathcal{A}, G, \beta)$ an integrable-ergodic C*-dynamical system, then $\mathcal{A}$ is non-unital.*

PROOF. Suppose that $\mathcal{A}$ contains 1 and let $\varphi \in \mathcal{A}^*$ be a state. Then, for each $x \in \mathcal{A}$ with $\|x-1\| < \frac{1}{2}$ we have $|\varphi\beta_s(x) - 1| < \frac{1}{2}$ for all $s \in G$. But then $\int\varphi\beta_s(x)ds = \infty$, so that $x \notin \mu_\beta$ and $\bar{\mu}_\beta \neq \mathcal{A}$.

Opposite to what we have in the W*-setting, the map $\varepsilon: x \in \mu_\beta \to \int\beta_s(x)ds$ has its image in $\mathcal{A}^{**}$ and not in $\mathcal{A}$. In fact, using the ergodicity and Lemma 1.3, $\varepsilon(x) \in \mathcal{A}$ implies $x = 0$. On the other hand, putting $\mathcal{A}_1 = \mathcal{A} + \mathbf{C}.1$, our definition states that $\varepsilon(x)$ is a scalar multiple of 1, which is precisely what we have in the W*-case.

We conclude the section with a second, more sofisticated example, which was brougt to our attention by H. H. Zettl.

1.4. EXAMPLE. Let $p \in \hat{G}$ and denote $\tilde{p}$ the action of G on $C(\boldsymbol{T})$ defined by $(\tilde{p}_s f)(v) = f(\overline{\langle s,p\rangle} v)$. For $G = \mathbf{Z}$ and taking the crossed product $G \times_{\tilde{p}} C(\boldsymbol{T})$, we obtain all rational and irrational rotation C*-algebras, as they were introduced by M. Rieffel in [12]. More in general, for G discrete, $G \times_{\tilde{p}} C(\boldsymbol{T})$ are the generalized rotation algebras, studied by M. De Branbanter and H. H. Zettl in [3].

On $G \times_{\tilde{p}} C(\boldsymbol{T})$ we have the dual action $(\tilde{p})^\wedge$ of $\hat{G}$, implemented by the unitary representation $q \to v_q \otimes 1$ of $\hat{G}$ on $L^2(G) \otimes L^2(\boldsymbol{T})$, where $(v_q f)(s) = \langle s,q \rangle f(s)$. There is also a second action γ of $\boldsymbol{T}$ on $G \times_{\tilde{p}} C(\boldsymbol{T})$, given by $\gamma_\mu = \mathrm{ad}(1 \otimes T_\mu)$, with $(T_\mu \xi)(v) = \xi(\bar{\mu} v)$, $\xi \in L^2(\boldsymbol{T})$. This gives rise to a C*-dynamical system $(G \times_{\tilde{p}} C(\boldsymbol{T}), \hat{G} \times \boldsymbol{T}, \beta)$, where

$$\beta_{(q,\mu)} \doteq (\tilde{p})^{\wedge}_q \circ \gamma_\mu .$$

We show that it is faithful and integrable-ergodic.

To see this, take $\xi \in C(\boldsymbol{T})$ and $f \in L^1(G)$, then the operators $\pi(\xi)\lambda_f$, with

$$(\pi(\xi)(g \otimes \xi))(t,v) \doteq (\tilde{p}_{-t}(\xi))(v) . g(t) . \xi(v)$$

and

$$(\lambda_f(g \otimes \xi))(t,v) = \int f(s) g(t-s) ds . \xi(v),$$

are norm-dense in $G \times_{\tilde{p}} C(\boldsymbol{T})$. Denoting $\xi^\mu(v) = \xi(\bar{\mu} v)$, one verifies that

$$\beta_{(q,\mu)}(\pi(\xi)\lambda_f) = \pi(\xi^\mu)\lambda_{\langle \cdot, q \rangle f},$$

from which the faithfulness of β is obtained.

To prove the ergodicity, first observe that $s \to (v \to \langle s,p \rangle v)$ determines a continuous homomorphism of G in the group of all isometries on $\boldsymbol{T}$. So, by [6; Proposition 3.3], $(C(\boldsymbol{T}), G, p)$ is an almost periodic system in the sense of [6; Definition 3.2], and from [6; Theorem 4.8] it follows that $G \times_{\tilde{p}} C(\boldsymbol{T})$ is contained in

$$C^*\{m_f \lambda_g \mid f,g \in C_u^b(G)\} \otimes C(\boldsymbol{T}).$$

Thus,

$$G \times_{\tilde{p}} C(\boldsymbol{T}) \subset \mathscr{B}(L^2(G)) \otimes C(\boldsymbol{T}).$$

Next, let x be a fixedpoint for β in $G \times_{\tilde{p}} C(\boldsymbol{T})$. To any $\varphi \in C(\boldsymbol{T})^*$ we can associate a bounded operator

$$(1 \otimes \varphi) : \mathscr{B}(L^2(G)) \otimes C(\boldsymbol{T}) \to \mathscr{B}(L^2(G))$$

defined by

$$(1 \otimes \varphi)(x \otimes \xi) = x . \varphi(\xi), \quad x \in \mathscr{B}(L^2(G)), \quad \xi \in C(\boldsymbol{T}).$$

For $q \in \hat{G}$ we then have

$$\begin{aligned}(1 \otimes \varphi)(x) &= (1 \otimes \varphi)(\beta_{(q,1)}(x)) \\ &= (1 \otimes \varphi)((\mathrm{ad} v_q \otimes 1)(x)) \\ &= \mathrm{ad} v_q((1 \otimes \varphi)(x)),\end{aligned}$$

and since $\{v_q | q \in \hat{G}\}' = L^\infty(G)$, there exists a $f_\varphi \in L^\infty(G)$ so that $(1 \otimes \varphi)(x) = m_{f_\varphi}$.

By a similar argument we also obtain that for each $\psi \in \mathscr{B}(L^2(G))^*$ there is a $\lambda_\psi \in \mathbf{C}$ satisfying $(\psi \otimes 1)(x) = \lambda_\psi$. Combining the 2 relations, we get

$$\psi(m_{f_\varphi}) = (\psi \otimes \varphi)(x) = \lambda_\psi \,.\, \varphi(1),$$

so that the operator $\varphi(1)^{-1} m_{f_\varphi}$ is obviously independent of φ. In addition we conclude from the above that

$$(\psi \otimes \varphi)(\varphi(1)^{-1} \,.\, m_{f_\varphi} \otimes 1) = (\psi \otimes \varphi)(x),$$

which, by weak*-density of $\mathscr{B}(L^2(G))^* \otimes C(\boldsymbol{T})^*$ in $(\mathscr{B}(L^2(G)) \otimes C(\boldsymbol{T}))^*$, means that $x = \varphi(1)^{-1} \,.\, m_{f_\varphi} \otimes 1$. Finally, using [7; Theorem 4.10] and $x \in G \times_{\tilde{p}} C(\boldsymbol{T})$, we get that f_φ must be translation-invariant. Thus, $f_\varphi \in \mathbf{C}.1$ and β is ergodic.

The major problem, however, is the integrability. Let $\varphi \in (G \times_{\tilde{p}} C(\boldsymbol{T}))$, then by [11; Proposition 7.6.8] there exists a norm-continuous, bounded function $\phi : G \to C(\boldsymbol{T})^*$ so that for $n \in \mathbf{N}_0, s_1, s_2, \ldots, s_n \in G$ and $\xi_1, \xi_2, \ldots, \xi_n \in C(\boldsymbol{T})$

$$(1) \qquad \sum_{i,j}^{n} \phi(s_i - s_j)(\tilde{p}_{-s_j}(\xi_j^* \xi_i)) \geqq 0,$$

and, for $\xi \in C(\boldsymbol{T})$ and $f \in L^1(G)$

$$\varphi(\pi(\xi)\lambda_f) = \int \phi(s)(\xi) \,.\, f(s)\, ds.$$

Furthermore, $\phi(s) \in C(\boldsymbol{T})^*$, so that for some bounded Radon-measure $m_{\varphi,s}$ on $\boldsymbol{T}$

$$\phi(s)(\xi) = \int \xi(v)\, dm_{\varphi,s}(v).$$

Now let $K^1(G)$ denote the set of all $L^1(G)$-functions such that $\hat{f}$ has compact support and take $f \in K^1(G)$. Then

$$\begin{aligned} &\int_{\hat{G}} \int_{\boldsymbol{T}} \varphi(\beta_{(q,\mu)}(\pi(\xi)\lambda_f))dq\,d\mu \\ &\quad = \int_{\hat{G}} \int_{\boldsymbol{T}} \int_G \int_{\boldsymbol{T}} \xi(\bar{\mu}v) \langle s,q \rangle f(s) dm_{\varphi,s}(v) ds\, d\mu\, dq \\ &\quad = \int_{\boldsymbol{T}} \xi(\mu) d\mu \int_{\hat{G}} \int_G m_{\varphi,s}(\boldsymbol{T}) \langle s,q \rangle f(s) ds\, dq. \end{aligned}$$

Using (1), with ξ_i equal to a constant function $v \to \lambda_i$, we get

$$\sum_{i,j}^{n} \int_{\boldsymbol{T}} \bar{\lambda}_j \lambda_i\, dm_{\varphi.s_i - s_j}(v) \geqq 0.$$

So, since $s \to m_{\varphi,s}(\boldsymbol{T}) = \phi(s)(1)$ is clearly continuous and bounded, it is also positive-definite. Bochner's theorem may be applied, so that there exists a positive, bounded Radon-measure m_ϕ on $\hat{G}$, with $\phi(\cdot)(1) = m_\phi^\wedge$. Then, by Fubini's theorem, [8; Theorem 31.27] and the inversion theorem, we get

$$\begin{aligned}\int_{\hat{G}} \int_G m_{\varphi,s}(\pi)\langle s,q\rangle f(s)\,ds\,dq &= \int_{\hat{G}} (\phi(\cdot)(1)\,.\,f)^\wedge(-q)\,dq \\ &= \int_{\hat{G}} \int_{\hat{G}} \hat{f}(q-r)\,dm_\phi(r)\,dq \\ &= \int_{\hat{G}} \hat{f}(q)\,dq\,.\,\phi(0)(1).\end{aligned}$$

Next, by positive-definiteness of $\phi(\cdot)(1)$ again, $\phi(0)(1) \geqq 0$, so that [11; p. 258] gives $\phi(0)(1) = \|\phi(0)\| = \|\varphi\|$. Therefore we may conclude that

$$\begin{aligned}\int_{\hat{G}}\int_T \varphi(\beta_{(q,\mu)}(\pi(\xi)\lambda_f))\,dq\,d\mu &= \int_T \xi(\mu)\,d\mu\,.\int_{\hat{G}} \hat{f}(q)\,dq\,.\,\|\varphi\|\end{aligned}$$

and since the operators $\pi(\xi)\lambda_f$, $f \in K^1(G)$, are norm-dense, β is integrable.

2. The classification theorem.

Let $(\mathscr{A}, G, \beta)$ be a faithful, integrable-ergodic C*-dynamical system, then $\tau : \mathscr{A}_+ \to [0,\infty]$, $x \to \int \varphi\beta_s(x)ds$, where φ is a state on $\mathscr{A}$, defines a faithful, lower semi-continuous weight on $\mathscr{A}$. The lower semi-continuity of τ follows from the existence of a net $\{\tau_i\}_{i\in I}$ of continuous functions $\tau_i : \mathscr{A}_+ \to [0, +\infty]$,

$$x \to \int_{K_i} \varphi\beta_s(x)ds,$$

where K_i is compact in G, such that $\tau = \sup_{i\in I} \tau_i$.

Denote $\eta_\tau = \{x \in \mathscr{A} \,|\, \tau(x^*x) < \infty\}$ and let $\mathscr{H}_\tau$ be the completion of the pre-Hilbert space η_τ for the norm arising from the inner product $\langle \xi_x, \xi_y \rangle = \tau(y^*.x)$, $x,y \in \eta_\tau$. We then get a faithful, non-degenerate *-representation of $\mathscr{A}$ on $\mathscr{H}_\tau$, given by $\pi_\tau(x)\xi_y = \xi_{xy}$.

Next, let $\mathscr{M}$ be the σ-weak completion of $\pi_\tau(\mathscr{A})$ in $\mathscr{B}(\mathscr{H})$. We can see that β is unitary implemented on $\mathscr{H}_\tau$ by U, where $U_s\xi_x = \xi_{\beta_s(x)}$, so that the action of G on $\mathscr{A}$ can be extended to $\mathscr{M}$. The extended action $\alpha = \mathrm{ad}\,U$ is continuous, since τ is lower semi-continuous. So, we obtain a W*-dynamical system $(\mathscr{M}, G, \alpha)$.

2.1. LEMMA. *$(\mathscr{M}, G, \alpha)$ is an integrable, ergodic and faithful W*-dynamical system.*

PROOF. The faithfulness of ad U is clear, while the integrability follows from the integrability of β. To prove the ergodicity, let x_0 be a fixed point in $\mathscr{M}$ and take a and b integrable in $\mathscr{M}$, denoting

$$a_0 = \int_G U_s a U_s^* ds, \quad b_0 = \int_G U_s b U_s^* ds.$$

Then,

$$\int_G U_t (\int_G U_s a U_s^* x_0 b ds) U_t^* dt = a_0 x_0 b_0.$$

Thus, if

$$\mathscr{R} = \{ \int_G U_s x U_s^* ds | x \text{ integrable in } \mathscr{M} \}'' = \mathrm{C}.1$$

and $y \in \mathscr{R}'$, $ya_0x_0b_0 = a_0x_0b_0y$, so that $a_0yx_0b_0 = ax_0yb_0$. But since $a_0, b_0 \in \mathrm{C}.1$, we have $yx_0 = x_0y$ or $x_0 \in \mathscr{R}'' = \mathrm{C}.1$.

In [10], for a compact group G, D. Olesen, G. K. Pedersen, and M. Takesaki prove that $(\mathscr{A}, \beta)$ can be reconstructed from $(\mathscr{M}, \alpha)$ by taking $(\mathscr{M}^c, \alpha|_{\mathscr{M}^c})$, where

$$\mathscr{M}^c = \{x \in \mathscr{M} | s \to \alpha_s(x) \text{ is norm-continuous}\}.$$

In the locally compact case, however, $\mathscr{M}^c$ can no longer be used for this purpose, as we can see from the abelian G-system $(L^\infty(G), G, \alpha^1)$. Since $(L^\infty(G))^c = C^b(G)$, this algebra is unital, contradicting Lemma 1.3.

What we will prove here is that $\mathscr{A} = \mathscr{M}^{ic}$, with $\mathscr{M}^{ic} = (\mathscr{M}^c \cap \mu_\alpha)^{-\|\cdot\|}$. To do this we want to use the classification theorem 2.7 of [4], for integrable, ergodic and faithful W*-dynamical systems. Unfortunately, this theorem was formulated for W*-systems with a separable predual, and although this is a very natural condition to impose on von Neumann algebra's, the equivalent condition on an underlying C*-subalgebra is completely unacceptable. This problem is solved in the following lemma.

2.2 LEMMA. *Let $(\mathscr{M}, G, \alpha)$ be a* W**-dynamical system, with α integrable, ergodic and faithful, and G second countable, then $\mathscr{M}_*$ is separable.*

PROOF. First observe that with the technique of [4; Lemma 1.13] we can construct a continuous cross-section $p \to u_p$ from a neighbourhood of any point in $\hat{G}$ into the group G_α, of unitary eigenoperators for α. Using the second countability of $\hat{G}$, these can be linked together, so that we obtain a Borel-measurable cross-section $p \to u_p$ of $\hat{G}$ onto G_α, which is continuous at 0. Also, for any $x \in \mu_\alpha$, the map

$$p \in \hat{G} \to \hat{x}(p) = \int \langle s,p \rangle^{-1} \alpha_s(x) ds$$

is continuous.

Next, let $\mathscr{M}$ be faithfully represented on a Hilbert space $\mathscr{H}$ and take $\xi \in \mathscr{H}$, $\|\xi\| = 1$. By the continuity-conditions on the maps u, and $\hat{x}(\cdot)$ and, again, the second countability of $\hat{G}$, it is not hard to see that $\{u_p \xi | p \in \hat{G}\}$ and $\{\hat{x}(p)\xi | p \in \hat{G}\}$ generate separable sub-Hilbert spaces $\mathscr{H}_\xi^u$ and $\mathscr{H}_\xi^x$ of $\mathscr{H}$.

Since $\alpha_s(\hat{x}(p)) = \langle s,p\rangle \hat{x}(p)$, α is ergodic and $\|\hat{x}(p)\| \leqq |\rho(x)|$ (see [4; Lemma 1.13]), we can define a bounded, scalar function f_x on $\hat{G}$ by $f_x(p) = u_p^* \hat{x}(p)$. With F we denote the map $x \in \mu_\alpha \to f_x$. Observe that by the separability of $\mathscr{H}_\xi^u$ and $\mathscr{H}_\xi^x$, there exists a countable, orthonormal set $\{\xi_i\}_{i\in \mathbf{N}_0}$ in $\mathscr{H}$, so that for all $p \in \hat{G}$

$$f_x(p) = \sum_{i=1}^{\infty} \langle \hat{x}(p)\xi,\xi_i\rangle \langle \xi_i,u_p\xi\rangle.$$

Therefore, $f_x \in L^\infty(\hat{G})$.

We now extend $\{\xi_i\}_{i\in \mathbf{N}_0}$ to a total orthonormal basis $\{\xi_j\}_{j\in J}$ of $\mathscr{H}$. By arguments similar to the ones of [4; Lemma 1.10] and using the monotone convergence theorem for nets of lower semi-continuous functions as it was formulated in [1; Proposition 5], we have

$$\begin{aligned}\|x\|_\rho^2 &= \int_G \langle \alpha_s(x^*x)\xi,\xi\rangle ds \\ &= \int_G \sum_{j\in J} |\langle \alpha_s(x)\xi,\xi_j\rangle|^2 ds \\ &= \sum_{j\in J} \int_{\hat{G}} |\langle \alpha_{\cdot}(x)\xi,\xi_j\rangle^\wedge(p)|^2 dp \\ &= \int_{\hat{G}} \sum_{j\in J} |\langle \hat{x}(p)\xi,\xi_j\rangle|^2 dp \\ &= \int_{\hat{G}} \langle x^*(p)x(p)\xi,\xi\rangle dp \\ &= \|f_x\|_2^2,\end{aligned}$$

which shows that $F:\mu_\alpha \to L^2(\hat{G}) \cap L^\infty(\hat{G})$, $x \to f_x$, is an isometry of a dense part of the Hilbert space $\mathscr{H}_\rho$ associated to the left Hilbert algebra $\eta_\alpha \cap \eta_\alpha^*$ into $L^2(\hat{G})$. Thus, $\dim \mathscr{H}_\rho \leqq \dim L^2(\hat{G}) = \chi_0$, since $\hat{G}$ is second countable.

So, by [4; Theorem 2.7], $(\mathscr{M}, G, \alpha)$ is covariantly isomorphic to some $(\hat{G} \times_\omega \mathbf{C}, G, \operatorname{ad} v)$, where $\omega \in Z_b^2(\hat{G},\mathbf{T})$, the group of the Borel-measurable 2-cocycles of $\hat{G}$ in $\mathbf{T}$, and $(v_s f)(p) = \langle s,p\rangle f \in L^2(\hat{G})$. If the 2-cocycle is trivial, we get the abelian case $(\mathscr{M}(\hat{G}),G,\operatorname{ad} v)$, where $\mathscr{M}(\hat{G})$ is the group von Neumann algebra of $\hat{G}$. Obviously, this system is covariantly isomorphic to $(L^\infty(G),G,\alpha^1)$ of Example 1.2 and under the same isomorphism, $(C_0(G),G,\beta^1)$ corresponds with $(C_r^*(\hat{G}),G,\operatorname{ad} v)$, where $C_r^*(\hat{G})$ is the reduced group C*-algebra of $\hat{G}$.

This can easily be generalized towards the non-abelian case. First, from [5], we recall some facts on the twisted group algebras $L_\omega^1(\hat{G})$.

Let $\omega \in Z_b^2(\hat{G},\mathbf{T})$ and $f,g \in L^1(\hat{G})$, then the function $f *_\omega g$ and $f^{*\omega}$, $\hat{G} \to \mathbf{C}$, defined by

$$(f *_\omega g)(p) = \int f(p-q)\omega(p-q,q)g(p)dq,$$

and

$$f^{*\omega}(p) = \omega(p,-p)^{-1}\overline{f(-p)},$$

are both in $L^1(\hat{G})$. One easily verifies that $L^1_\omega(\hat{G}) = (L^1(\hat{G}), *_\omega, *^\omega)$ is an involutive Banach algebra.

Next recall that $\boldsymbol{T} \times_\omega \hat{G}$ denotes the locally compact group of all (μ,p), $\mu \in \boldsymbol{T}$ and $p \in \hat{G}$, with multiplication defined by

$$(\mu,p)\,.\,(\nu,q) = (\mu\,.\,\nu\,.\,\omega(p,q), p+q),\ \mu,\nu \in \boldsymbol{T},\ \ p,q \in \hat{G}.$$

C. M. Edwards and J. T. Lewis then consider 2 maps, which we will denote Λ and Ω, linking $L^1_\omega(\hat{G})$ to $L^1(\boldsymbol{T} \times_\omega \hat{G})$. The map $\Lambda: L^1_\omega(\hat{G}) \to L^1(\boldsymbol{T} \times_\omega \hat{G})$ is defined by

$$(\Lambda f)(\mu,p) = \mu f(p),\ f \in L^1_\omega(\hat{G}),$$

while $\Omega: L^1(\boldsymbol{T} \times_\omega \hat{G}) \to L^1_\omega(\hat{G})$ is given by

$$(\Omega F)(p) = \int F(\mu,p)^{-1} d\mu,\ \ F \in L^1(\boldsymbol{T} \times_\omega \hat{G}).$$

The following statements summarize the results given in [5; Lemma 3.1 and Lemma 3.2].

2.3. LEMMA. *Λ is an isometric *-isomorphism from $L^1_\omega(\hat{G})$ onto a closed 2-sided ideal of $L^1(\boldsymbol{T} \times_\omega \hat{G})$. Ω is a norm non-increasing *-homomorphism from $L^1(\boldsymbol{T} \times_\omega \hat{G})$ onto $L^1_\omega(\hat{G})$.*

Next, let $m_{\omega(\cdot,p)}$ and λ_p denote the multiplication-operator by $\omega(\cdot,p)$ and the translationoperator by p on $L^2(\hat{G})$. Then, a faithful, non-degenerate representation λ^ω of $L^1_\omega(\hat{G})$ is defined by

$$\lambda^\omega(f) = \int f(p)\lambda_p m_{\omega(\cdot,p)} dp.$$

Taking the σ-weak completion of $\lambda^\omega(L^1_\omega(\hat{G}))$ in $\mathscr{B}(L^2(\hat{G}))$, we obtain the twisted group von Neumann algebra $\mathscr{M}_\omega(\hat{G})$ of $\hat{G}$. If instead, we take the norm completion of it, the twisted, reduced group C*-algebra $C^*_{r,\omega}(\hat{G})$ of $\hat{G}$ is obtained. Of course, these notions are no different from the twisted cross-products $\mathrm{C} \times_\omega \hat{G}$, as they were defined in [14].

What we will show is the following. Suppose that $p \to u_p$ is a Borel cross-section for $(\mathscr{M},\alpha)$. Then $\mathscr{M} \cong \mathscr{M}_\omega(\hat{G})$ under the natural isomorphism φ, $\varphi(u_p) = \lambda_p m_{\omega(\cdot,p)}$, of [4; Lemma 2.6]. We prove that $\varphi(\mathscr{A}) = \varphi(\mathscr{M}^{ic}) = C^*_{r,\omega}(\hat{G})$, so that in particular, for each integrable-ergodic, faithful C*-G-system, there exists $\omega \in Z^2_b(\hat{G},\boldsymbol{T})$, so that $(\mathscr{A},\beta) \cong (C^*_{r,\omega}(\hat{G}), \mathrm{ad}\, v)$.

First, one technical result.

2.4. LEMMA. *Let $x \in \mu_\alpha$, then there exists a function $\theta_x \in L^\infty(\text{supp}\,\hat{x}, T)$, so that for all $f \in L^1(\hat{G})$ we have*

$$\alpha_f(x) = \int h(p) u_p dp,$$

where $h = f(-\cdot)\|\hat{x}(-\cdot)\|\theta_x \in L^1(\hat{G})$.

PROOF. Using Fubini's Theorem and the fact that both $\hat{f}$ and x are integrable, we get

$$\int f(s)\alpha_s(x)ds = \int \hat{f}(p) \int \langle s,p\rangle \alpha_s(x) ds dp$$
$$= \int \hat{f}(p)\hat{x}(-p)dp.$$

By ergodicity of α, $p \to v_p = \hat{x}(p)\|\hat{x}(p)\|^{-1}$ defines a new Borel cross-section on supp $\hat{x}$. Also, on the same set, we have another Borel-measurable function θ_x defined by $\theta_x(p) = u_p v_p^*$. We get

$$\alpha_f(x) = \int \hat{f}(-p)\|\hat{x}(-p)\|\theta_x(p) u_p dp$$

and since $p \to \|x(-p)\|$ is bounded and continuous by the proof of [4; Lemma 1.13],

$$h(p) = \hat{f}(-p)\|\hat{x}(-p)\|\theta_x(p)$$

is an $L^1(\hat{G})$-function, satisfying the conditions.

2.5. PROPOSITION. $\varphi(\mathcal{M}^{ic}) = C^*_{r,\omega}(\hat{G})$.

PROOF. It is sufficient to prove that $\mathcal{M}^{ic}$ is the norm-completion of

$$\{\int f(p) u_p dp \mid f \in L^1(\hat{G})\}.$$

For the inclusion $\subsetneq$, take $x \in \mathcal{M}^c \cap \mu_\alpha$ and let $\{f_i\}_{i\in I}$ be an approximate unit of $L^1(G)$ in the sense of [8; Theorem 33.11]. Since the functions $\{f_i\}_{i\in I}$ have integral 1 and decreasing compact supports, and since $x \in \mathcal{M}^c$, we have

$$\|\int f_i(s)\alpha_s(x)ds - x\| \to 0.$$

On the other hand, $\hat{f}_i \in L^1(\hat{G})$ and $x \in \mu_\alpha$, so that by Lemma 2.4 there exist functions $h_i \in L^1(\hat{G})$ such that

$$\int f_i(s)\alpha_s(x)ds = \int h_i(p) u_p dp.$$

For the second inclusion, we examine the *-algebra

$$B = \{\int (f *_\omega g)(p) u_p dp \mid f, g \in L^1(\hat{G}) \cap L^2(\hat{G})\}.$$

From [4; Lemma 1.10] it is obvious that $B \subset \mathcal{M}^{ic}$. Now, take any nest $\{K_n\}_{n\in\mathbb{N}_0}$ of compact neighbourhoods of 0 in $\hat{G}$ with Haar-measures $\{m(K_n)\}_{n\in\mathbb{N}_0}$, and define

$$E_n(\mu,p) = \frac{n}{2m(K_n)}\chi_{K_n}(p)\chi_{[e^{-i/n},e^{i/n}]}(\mu),$$

$\mu \in \boldsymbol{T}$ and $p \in \hat{G}$. Then, $\{E_n\}_{n\in\mathrm{N}_0}$ is an approximate unit in $L^1(\boldsymbol{T}\times_\omega \hat{G})$ and, using Lemma 2.3, one can check that the same holds for $\{e_n = \Omega(E_n)\}_{n\in\mathrm{N}_0}$ in $L^1(\hat{G})$. For every $f \in L^1(\hat{G}) \cap L^2(\hat{G})$ we have

$$\begin{aligned}\| \int (f *_\omega e_n - f)(p)u_p dp \| &= \| \lambda^\omega(f *_\omega e_n) - f \| \\ &\leqq \| f *_\omega e_n - f \|_1 \to 0,\end{aligned}$$

so that

$$\begin{aligned}\bar{B}^{\|\cdot\|} &= \{ \int f(p)u_p dp \,|\, f\in L^1(\hat{G}) \cap L^2(\hat{G})\}^{-\|\cdot\|} \\ &= \int f(p)u_p dp \,|\, f\in L^1(\hat{G})\}^{-\|\cdot\|}.\end{aligned}$$

Finally, we prove that every faithful, I-E*-system $(\mathscr{A}, G, \beta)$ is of the form $(C^*_{r,\omega}(\hat{G}), G, \operatorname{ad} v)$. Half the result is obtained in the following lemma.

2.6. Lemma. *There exists a 2-cocycle $\omega \in Z_b^2(\hat{G}, \boldsymbol{T})$, so that $(\mathscr{A}, G, \beta)$ is covariantly isomorphic to a C*-subsystem of $(C^*_{r,\omega}(\hat{G}), G, \operatorname{ad} v)$.*

Proof. By Lemma 2.1 we have the W*-system $(\mathscr{M}, G, \operatorname{ad} U)$ associated to $(\mathscr{A}, G, \beta)$. Obviously $\mathscr{A} \subset \mathscr{M}^{ic}$, so that by Proposition 2.5, $\varphi(\mathscr{A})$ is a C*-subalgebra of $C^*_{r,\omega}(\hat{G})$.

It remains to show that $\varphi: \mathscr{A} \to C^*_{r,\omega}(\hat{G})$ is onto, or equivalently, that for every $f\in L^1(\hat{G})$, $\int_{\hat{G}} f(p)u_p dp$ is in $\mathscr{A}$.

Denote

$$H = \{p\in\hat{G} \,|\, \text{there exists a compact neighbourhood } K_p \text{ of } p \text{ in } \hat{G}, \text{ so that } \int_{K_p} f(q)u_q dq \in \mathscr{A} \text{ for all } f\in L^1(\hat{G})\}.$$

We will show that $H = \hat{G}$.

2.7. Lemma. *Let $p \in \hat{G}$, for which there exist $f\in L^1(\hat{G})$, a compact neighbourhood K_p of p in $\hat{G}$ and $c > 0$, so that $|f(K_p)| \subset]c, +\infty[$ and*

$$\int_{\hat{G}} f(q)u_q dq \in \mathscr{A},$$

then $p \in H$.

Proof. Let $h \in K^1(G)$, then

$$\int_{\hat{G}} \hat{h}(-q) f(q)u_q dq = \int_G h(s)\beta_s(\int_{\hat{G}} f(q)u_q dq)ds$$

is in $\mathscr{A}$.

Now, $\{\hat{h} \,|\, h\in K^1(G)\}$ is dense in $L^1(\hat{G})$, thus for each $g\in L^1(\hat{G})$ there is a net $\{h_i\}_{i\in I}$ in $K^1(G)$, so that

$$\hat{h}_i \to (f^{-1}.g.\chi_{K_p})(-\cdot) \text{ in } L^1(\hat{G}).$$

Then, since $|f^{-1}| < 1/c$ on K_p,

$$\int_{K_p} \hat{h}_i(-q)f(q)u_q dq \to \int_{K_p} g(p)u_p dq \text{ in } \mathscr{A}.$$

For the following lemma, we need a Borel measurable cross-section $p \to u_p$ which is continuous at a given element p_1 of $\hat{G}$, and for which $q \to u_q^*$ is continuous at another point $p_2 \neq p_1$ in $\hat{G}$ as well. Such a map can be constructed by the same technique as we used to obtain the cross-section in the proof of Lemma 2.2.

2.8. Lemma. *H is a subgroup of $\hat{G}$.*

Proof. Let $x \in \mu_\beta \cap \mathscr{A}_0^+$ and $f \in K^1(G)$ with $\hat{f}(0) \neq 0$. Then from Lemma 2.4 we have

$$\int_{\hat{G}} \hat{f}(-p)\|\hat{x}(-p)\|\theta_x(p)u_p dp = \int_G f(s)\beta_s(x)ds \in \mathscr{A}.$$

Moreover, $x > 0$ implies $\hat{x}(0) \neq 0$ and by continuity of $p \to \hat{f}(p)\|\hat{x}(p)\|$ there must be a compact neighbourhood K_0 of 0 and a constant $c > 0$, such that

$$|\hat{f}(-p)|.\|\hat{x}(-p)\| > c, \text{ for all } p \in K_0.$$

Thus $0 \in H$.

Next, if $f \in L^1(\hat{G})$, $K_p \subset \hat{G}$ and $c > 0$ satisfy the conditions of Lemma 2.7 at $p \in \hat{G}$, then the same is true for $q \to f(-q)\omega(q,-q)^{-1}$, $K_{-p} = \{-q \mid q \in K_p\}$ and c at $-p$. So $-p \in H$ along with p.

Proving that $p_1 + p_2 \in H$, for $p_1, p_2 \in H \setminus \{0\}$ is a lot harder. For each $f,g \in L^1(\hat{G})$, we have that

$$\int_{\hat{G}} (\chi_{K_{p_1}} \cdot f) *_\omega (\chi_{K_{p_2}} \cdot g)(p)u_p dp = \int_{K_{p_1}} f(p)u_p dp \,.\, \int_{K_{p_2}} g(p)u_p dp \in \mathscr{A},$$

so that by Lemma 2.7 it will be sufficient to find $f,g \in L^1(\hat{G})$, a compact neighbourhood $K_{p_1+p_2}$ of $p_1 + p_2$ and $c > 0$, such that

$$|(\chi_{K_{p_1}} \cdot f) *_\omega (\chi_{K_{p_2}} \cdot g)(p)| > c, \text{ for } p \in K_{p_1+p_2}.$$

First we show that there exist $f,g \in L^1(\hat{G})$ for which $|(\chi_{K_{p_1}} \cdot f) *_\omega (\chi_{K_{p_2}} \cdot g)|$ is continuous at $p_1 + p_2$. Since $p_1 \neq p_1 + p_2$ we know that there is a Borel measurable cross-section $p \to v_p$, continuous at p_1 and so that $p \to v_p^*$ is continuous at $p_1 + p_2$. Let $\omega'(p,q) = v_p v_q v_{p+q}^*$, then by taking q in a small enough neighbourhood V_{p_2} of p_2, $p \to \omega'(p-q,q)$ will be continuous at $p_1 + p_2$. Furthermore, there is a function $h \in L^\infty(\hat{G},T)$ satisfying

$$\omega(p,q) = h(p)h(q)h(p+q)^{-1}\omega'(p,q).$$

We get

$$|(\chi_{K_{p_1}} \cdot f) *_\omega (\chi_{K_{p_2}} \cdot g)(p)| =$$
$$= |\int_{\hat{G}} (\chi_{K_{p_1}} . f . h)(p-q) . (\chi_{K_{p_2}} \cdot g . h)(q) . \omega'(p-q,q) dq|.$$

We now take $g \in L^1(\hat{G})$ with $\operatorname{supp} g \subset V_{p_2}$ and $f \in L^1(\hat{G})$ so that $\chi_{K_{p_1}} . f . h$ is continuous at $p_1 + p_2$.

The 2-variable function we now have under the integral is a continuous p-function at $p_1 + p_2$, for every q, and is bounded as a q-function by a fixed integrable function, for every p. So, the Dominated Convergence Theorem applies and $|(\chi_{K_{p_1}} . f) *_\omega (\chi_{K_{p_2}} \cdot g)|$ is continuous at $p_1 + p_2$.

It remains to show that for some f and g satisfying the conditions we already imposed, we have

$$(\chi_{K_{p_1}} \cdot f) *_\omega (\chi_{K_{p_2}} \cdot g)(p_1 + p_2) \neq 0.$$

This can be done by taking f and g such that

$$(\chi_{K_{p_2}} \cdot g . h)(q) = \omega'(p_1 + p_2 - q, q)$$

for $q \in \operatorname{supp} \chi_{K_{p_2}} . g . h$ and $f . h > 0$. Then,

$$|(\chi_{K_{p_1}} \cdot f) *_\omega (\chi_{K_{p_2}} \cdot g)(p_1 + p_2) = \int_{\operatorname{supp} g \cap K_{p_2}} (f . h)(p_1 + p_2 - q) dq > 0.$$

Observe that from its definition, H is obviously open. We also have:

2.9. LEMMA. *H is dense in $\hat{G}$.*

PROOF. We investigate the set $\{\beta_f(x) | \hat{f} \in L^1(G), \hat{f} \in L^1(\hat{G})$ and $x \in \mu_\beta\}$. In the proof of Proposition 2.5 we saw that this is a dense set in A. Also, in the proof of Lemma 2.4 we obtained for these elements

$$\beta_f(x) = \int \hat{f}(-p) \| \hat{x}(-p) \| \theta_x(p) u_p dp.$$

Observe that every p in the open support of the function $q \to \hat{f}(-q)$. $\| \hat{x}(-q) \|$. $\theta_x(q)$ is in H, since $q \to f(-q) \| \hat{x}(-q) \|$ is continuous and $|\theta_x(q)| = 1$ for all $q \in \hat{G}$.

We now assume that H is not dense in $\hat{G}$, then there exists an $s_0 \in G$ so that $\langle s_0, p \rangle = 1$ for all $p \in H$. So, by the remarks above, $\beta_{s_0}(\beta_f(x)) = \beta_f(x)$ for every element from that dense set. But then, $\beta_{s_0} = 1$, which contradicts the faithfulness of β.

Combining the Lemmas 2.8 and 2.9 with the fact that H is open, we get $H = \hat{G}$. With this conclusion the classification theorem is within reach.

2.10. THEOREM. *Let $(\mathscr{A}, G, \beta)$ be a C*-dynamical system with an* *integrable, ergodic and faithful action β and a second countable, abelian group G, then there exists a 2-cocycle $\omega \in Z_b^2(\hat{G}, \boldsymbol{T})$ so that*

$$(\mathscr{A}, G, \beta) \cong (C^*_{r,\omega}(\hat{G}), G, \operatorname{ad} v).$$

PROOF. Let $f \in L^1(\hat{G})$ and take $h \in C_c(\hat{G})$, so that $\| f - h \|_1 < \varepsilon$, $\varepsilon > 0$. For each $p \in \operatorname{supp} h$ we have a compact neighbourhood K_p, so that

$$\int_{K_p} g(q) u_q d_q \in \mathscr{A}, \text{ for all } g \in L^1(\hat{G}).$$

Since $\operatorname{supp} h$ is compact, there exist $\{p_1, p_2, \ldots, p_n\}$ in $\hat{G}$ with $\operatorname{supp} h \subset \bigcup_{i=1}^n K_{p_i}$. Then

$$\int h(q) u_q dq = \sum_{i=1}^n \int_{K_{p_i}} h(q) \cdot \prod_{j=1}^{i=1} (1 - \chi_{p_j})(q) \cdot u_q dq,$$

so that $\int h(q) u_q dq \in \mathscr{A}$ and $\int f(q) u_q dq$ as well.

2.11. EXAMPLE. An immediate consequence of the above theorem and Example 1.4 is that for each $p \in \hat{G}$, $G \times_{\tilde{p}} C(\boldsymbol{T})$ is isomorphic to a twisted, reduced group C*-algebra on $G \times \mathrm{Z}$. The question then rises as to how the connection between a character $p \in \hat{G}$ and the associated $\omega \in Z_b^2(G \times \mathrm{Z}, \boldsymbol{T})$,

$$G \times_{\tilde{p}} C(\boldsymbol{T}) \cong C^*_{r,\omega}(G \times \mathrm{Z}),$$

can be expressed.

To see this, we examine the W*-system $(G \times_{\tilde{p}} L^\infty(\boldsymbol{T}), \hat{G} \times \boldsymbol{T}, \alpha)$, where $\alpha = \operatorname{ad}(v \otimes T)$. Arguments similar to the ones of Example 1.4 show that α is faithful and ergodic. Also, α is integrable, since β is integrable,

$$(G \times_{\tilde{p}} C(\boldsymbol{T}))'' = G \times_{\tilde{p}} L^\infty(\boldsymbol{T}) \text{ and } \beta = \alpha|_{G \times_{\tilde{p}} C(\boldsymbol{T})}.$$

Now, let $\xi_0 : \boldsymbol{T} \to \boldsymbol{T}$, $\mu \to \mu$ and

$$u_{(s,n)} = \pi(\xi_0)^{-n} \lambda_s, \quad s \in G \text{ and } n \in \mathrm{Z},$$

then $u_{(s,n)}$ is a unitary eigenoperator in $G \times_p L^\infty(\boldsymbol{T})$ and

$$\alpha_{(q,u)}(u_{(s,n)}) = \langle s, q \rangle \mu^n \cdot u_{(s,n)},$$

for all $q \in \hat{G}$, $\mu \in \boldsymbol{T}$. Therefore, the 2-cocycle associated to $(G \times_{\tilde{p}} L^\infty(\boldsymbol{T}), \hat{G} \times \boldsymbol{T}, \alpha)$ is given by

$$\omega((s,n),(t,m)) = (\langle t, p \rangle)^n,$$

and if we can show that $(G \times_{\tilde{p}} L^\infty(\boldsymbol{T}))^{ic} = G \times_{\tilde{p}} C(\boldsymbol{T})$, then by Proposition 2.5, ω is also the 2-cocycle associated to $G \times_{\tilde{p}} C(\boldsymbol{T})$.

$$G \times_{\tilde{p}} C(\boldsymbol{T}) \subsetneq (G \times_{\tilde{p}} L^\infty(\boldsymbol{T}))^{ic}$$

is clear, since the operators $\pi(\xi)\lambda_f$, $\xi \in C(\boldsymbol{T})$, $f \in K^1(G)$, are α-norm-continuous, β-integrable (thus, α-integrable), and norm-dense in $G \times_{\tilde{p}} C(\boldsymbol{T})$. To prove the second inclusion, by Proposition 2.5 it will be sufficient to show that

$$\int \sum_{n \in \mathbf{Z}} h(s,n) u_{(s,n)} ds \in G \times_{\tilde{p}} C(\boldsymbol{T}),$$

for every $h \in L^1(G \times \mathbf{Z})$. Obviously, it is also enough that this holds for $h \in C_c(\hat{G} \times \mathbf{Z})$, so that it remains to show that

$$\int h(s) u_{(s,n)} ds \in G \times_{\tilde{p}} C(\boldsymbol{T}),$$

for $h \in C_c(G)$. This is the case, since

$$\int h(s) \pi(\xi_0)^{-n} \lambda_s ds = \pi(\xi_0)^{-n} \lambda_h.$$

Observe that with the construction of Example 1.4 we do not obtain every twisted, reduced group C*-algebra over $G \times \mathbf{Z}$. With an easy computation one verifies that the anti-symmetric bi-characters on $G \times \mathbf{Z}$ are of the form

$$\chi((s,n),(t,m)) = \psi(s,t) \,.\, \langle ms - nt, p \rangle,$$

with ψ an anti-symmetric bi-character on G and $p \in \hat{G}$. Therefore, by [9; p. 29],

$$H_b^2(G \times \mathbf{Z}, \boldsymbol{T}) = \{\omega((s,n),(t,m)) = \omega_1(s,t) \langle t,p \rangle^n \,|\, \omega_1 \in H_b^2(G,\mathbf{Z}), p \in \hat{G}\},$$

implying that only the systems $(C^*_{r,\omega}(G \times \mathbf{Z}), \operatorname{ad} v)$ for which ω_1 is trivial can be obtained through a construction of the type $(G \times_{\tilde{p}} C(\boldsymbol{T}), \beta)$.

Let us conclude with a remark on the unitary eigenoperators. The representation λ^ω of $L^1(\hat{G})$ and the projective representation $p \to \lambda_p m_{\omega(\cdot,p)}$ of $\hat{G}$ are closely related. As we know from [4], $\mathscr{M}_\omega(\hat{G})$ is not only the σ-weak completion of the *-algebra $\lambda^\omega(L^1_\omega(\hat{G}))$, but also of the linear span of the operators $\lambda_p \mathrm{m}_{\omega(\cdot,p)}$.

In the C*-case, and for a compact group G, it still does not matter whether we start off with the linear span or with $\lambda^\omega(L^1_\omega(\hat{G}))$. In both cases the norm-completion is $C^*_{r,\omega}(\hat{G})$. For a non-compact group, however, by Lemma 1.3, an integrable $C^*_{r,\omega}(\hat{G})$ never contains any of the $\lambda_p \mathrm{m}_{\omega(\cdot,p)}$'s. Conversely, the C*-algebra $C^*\{\lambda_p m_{\omega(\cdot,p)} | p \in \hat{G}\}$ neither contains any of the operators $\lambda^\omega(f)$, $f \in L^1(\hat{G})$, different from 0. This can easily be seen from the abelian case, where $C^*\{\lambda_p\} \cong$ A.P.(G) under the isomorphism implemented by Fourier transform, $C^*_{r,\omega}(\hat{G}) \cong C_0(G)$ under the same isomorphism and $C_0(G) \cap$ A.P.$(G) = \{0\}$. In fact, the basic reason for $C^*\{u_p\}$ not containing $\int f(p) u_p dp$, is the lack of continuity on $p \to u_p$.

All this seems to indicate that next to the faithful, I-E. C*-G-systems, we get a second class of C*-systems over G which admits classification by means of $H_b^2(\hat{G},\boldsymbol{T})$. Namely, the faithful and ergodic systems of the form $(C^*\{u_p \mid p \in \hat{G}\},\beta)$, where $p \to u_p$ is a projective representation of $\hat{G}$, satisfying $\beta_s(u_p) = \langle s,p\rangle u_p$, for each $s \in G$ and $p \in \hat{G}$.

However, with the techniques of [6; Lemma 3.1], it is not hard to see that the action β of $(C^*\{u_p\}, G,\beta)$ can be extended to G_b, the Bohr-compactification of G, in a continuous way by putting

$$\beta_{\tilde{s}}(u_p) = \langle \tilde{s},p\rangle u_p, \quad \tilde{s} \in G_b \text{ and } p \in \hat{G}.$$

This means that the systems we obtain are the C*-analogue of a special case of the almost periodic W*-dynamical systems described in [10]. Their special feature is that the associated pure point spectrum $\mathrm{Sp}_d(\beta)$ for these systems is $\hat{G}$ itself. Therefore, by [10; Theorem 7.4], they admit complete classification by means of $\chi^2(\hat{G}_{\text{disc.}},\boldsymbol{T})$. For each of the systems, there is a 2-cocycle ω in $Z^2(\hat{G},\boldsymbol{T})$, so that $(C^*\{u_p\},G,\beta)$ is covariantly isomorphic to $(C^*_{r,\omega}(\hat{G}_{\text{disc.}}), G, \operatorname{ad} v)$.

REFERENCES

1. N. Bourbaki, *Eléments de mathématique*, Fasc. XXXV, Livre 6: *Integration.* Chapitre 9: *Intégration sur les espaces topologiques séparés*, (Actualités Sci. Indust. 1343), Hermann, Paris, 1969.
2. A. Connes and M. Takesaki, *The flow of weights on a factor of type* III, Tohoku Math. J. 29 (1977), 473–573.
3. M. De Brabanter and H. H. Zettl, *C*-algebras associated with rotation groups and characters*, Manuscripta Math., to appear.
4. D. De Schreye, *Integrable, ergodic actions of abelian groups on von Neumann algebras*, Math. Scand. 53 (1983), 265–280.
5. C. M. Edwards and J. T. Lewis, *Twisted group algebras* I, Comm. Math. Phys. 13 (1969), 110–130.
6. G. Henrard, *Duality and fixed point theorem for almost periodic* C*-*crossed products*, Preprint, Leuven, 1983.
7. G. Henrard, *Fixed point theorems for* C*-*crossed products with an amenable group*, Preprint, Leuven, 1983.
8. E. Hewitt and K. A. Ross, *Abstract harmonic analysis* I (Grundlehren Math. Wiss. 115), Springer-Verlag, Berlin-Heidelberg-New York, 1963.
9. A. Kleppner, *Multipliers on abelian groups*, Math. Ann. 158 (1965), 11–34.
10. D. Olesen, G. K. Pedersen, and M. Takesaki, *Ergodic actions of compact abelian groups*, J. Operator Theory 3 (1980), 237–269.
11. G. K. Pedersen, C**-algebras and their automorphism groups* (London Mathematical Society Monographs 14) Academic Press, London, New York, San Francisco, 1979.

12. M. A. Rieffel, *Irrational rotation* C*-*algebras*, in *Abstracts*, p. 135. International Congress of Mathematicians, Helsinki, 1978.
13. A. J. Wassermann, *Automorphic actions of compact groups on operator algebras*, Thesis, Pennsylvania University, 1981.
14. G. Zeller-Meier, *Produits croisés d'une* C*-*algèbre par une groupe d'automorphismes*, J. Math. Pures Appl. 47 (1968), 101–239.
15. H. H. Zettl, *On compact abelian* W*-*dynamical systems* I, Preprint, Saarbrücken, 1982.

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