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A GENERAL DIFFERENTIATION THEOREM FOR n-DIMENSIONAL ADDITIVE PROCESSES

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Abstract.

We prove the differentiation theorem for n-dimensional additive processes with respect to a semi-group of L_1 contractions.

This generalizes the one-parameter theorems ([4], [14], [11]), the n-parameter local ergodic theorem [11] and the positive contractions case [3].

1. Introduction.

Let L_1 denote the usual space of equivalence classes of *complex* valued integrable functions on a σ-fine measure space $(X, \mathscr{F}, \mu)$. Let $n \geqq 1$ be an integer.

1.1. Denoting $P_n = (\mathsf{R}_+ - \{0\})^n$, we consider a strongly continuous semi-group $T = (T_t)_{t \in P_n}$ of linear L_1-contractions. We do not assume the continuity of T at 0 and as it is said in ([15, p. 552]) the behaviour of T near 0 can be extremely pathological. See also ([2, p. 550]).

1.2. Let $\mathscr{I}_n$ (respectively λ_n) be the class of all intervals of R_+^n (respectively the Lebesgue measure on R^n). We recall the

Definition ([5], [3]). A set function $F: \mathscr{I}_n \to L_1$ will be called a bounded additive process with respect to T if it satisfies the following conditions:

1.3. $\sup\{\|F(I)/\lambda_n(I)\| \mid I \in \mathscr{I}_n, \lambda_n(I) > 0\} = \gamma(F) < \infty$.

1.4. $T_u F(1) = F(u+I)$ for all $u \in P_n$ and $I \in \mathscr{I}_n$.

1.5. If $I_1, \ldots, I_k \in \mathscr{I}_n$ are pairwise disjoint and if $I = \bigcup_{i=1}^k I_i \in \mathscr{I}_n$, then $F(I) = \sum_{i=1}^k F(1_i)$.

The main result of this paper is the following

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1.6. THEOREM. *Let T and F be as in* 1.1 *and* 1.2. *Then*

$$\lim_{\alpha \to 0^+} \alpha^{-n} F([0, \alpha]^n) \quad \textit{exists a.e.}$$

It is well-known that 1.6 contains the Lebesgue differentiation theorems and the local ergodic ones (see [5], [3]). In fact 1.6 generalizes additive theorems recently proved in the following cases:

For positive contractions.

$n=1$, see M. A. Akcoglu-U. Krengel [5], D. Feyel [13], [14],

$n \geqq 1$, T continuous at 0 and $F(I)=\int_I T_t f dt$ for some fixed $f \in L_1$, see T. R. Terrell [17],

$n \geqq 1$, see M. A. Akcoglu-A. del Junco [3].

For general contractions.

$n=1$ and real L_1 spaces, see M. A. Akcoglu-M. Falkowitz [4],

$n=1$, T continuous at 0 and complex L_1 spaces, see D. Feyel [14],

$n=1$ and complex L_1 spaces [see 11],

$n \geqq 1$, T continuous at 0 and $F(I)=\int_I T_t f\, dt$ for some fixed $f \in L_1$ [see 11].

We also mention that the maximal inequality of M. A. Akcoglu [1] yields a more general version in L_p $(1<p<\infty)$: $\lim (\alpha_1 \ldots \alpha_n)^{-1} F([0, (\alpha_1, \ldots, \alpha_n)])$ exists a.e. as the $\alpha_i \to 0^+$ independently [see 10] (the T_t being necessarily positive [see 7], whereas the L_∞ theorem holds for positive contractions (see M. Lin [16]) and also for general contractions [see 12]

REMARK. All the limits above are taken through any countable set or for representatives of F (see [5], [3]).

1.7. Let us indicate briefly the various steps of the proof of 1.6.

We first reduce the problem to the case T continuous at 0. Then using the technique of the sub-semi-groups of Dunford-Schwartz ([9, VIII.7.12-15]) and the arguments of M. A. Akcoglu-A. del Junco [3], we define almost additive processes to obtain the Akcoglu-del Junco inequality ([3, 3.5]) for general contractions. The rest of the proof is a generalization of the one-dimensional nice proof of M. A. Akcoglu-M. Falkowitz [4] in dimension n and for complex L_1 spaces; this avoids the difficulties concerning the singular processes [see 3].

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2. The continuity of T at 0.

First recall the following inequality:

2.1. THEOREM ([11, 4.2]). *Let T be as in Section 1.1. Then there exists a constant $c_n > 1$ and a one-dimensional strongly continuous semi-group $U = (U_t)_{t>0}$ of L_1 positive contractions such that:*

$$\forall f \in L_1,\ \forall \alpha > 0,\ \alpha^{-n} \int_{[0,\alpha]^n} |T_t f|\, dt \leqq c_n \bar{\alpha}^{-1} \int_0^{\bar{\alpha}} U_t |f|\, dt\,,$$

where $\bar{\alpha} = \alpha^{2^{-K}}$ if $2^{k-1} < n \leqq 2^k$.

2.2. Let C be the initially conservative part of U (see M. A. Akcoglu-R. V. Chacon [2]), that is

$$C = \left\{ \int_0^1 U_t f_0\, dt > 0 \right\},$$

where $f_0 \in L_1$ and $f_0 > 0$ a.e. If $D = X \setminus C$ then the strong-continuity of U implies $1_D U_t = 0$ for all $t > 0$.

In the case of positive contractions the following is due to Akcoglu-Chacon-del Junco ([2], [3]).

2.3. THEOREM. *Let T be as in Section* 1.1. *Then $1_D T_t = 0$,*

$$R_0 = \text{strong-}\lim_{t\to 0} T_{t|L_1(C)}$$

exists, and

$$|R_0| = \text{strong-}\lim_{t\to 0^+} U_{t|L_1(C)}\,.$$

2.4. COROLLARY. *Let T and F be as in* 1.1 *and* 1.2. *Then $1_D F_I = 0$ and $R_0 F_I = F_I$ for $I \in \mathscr{I}_n$.* (See [3, Section 2.2]).

PROOF OF THEOREM 2.3. Since T is continuous on P_n, 2.1. implies that $1_D T_t = 0$. For any $x > 0$ let

$$M_x f = x^{-n} \int_{[0,x]^n} T_t f\, dt, \quad f \in L_1\,.$$

First consider the (strongly and weakly) closed space

$$H = \{ f \in L_1(C) \mid \text{norm-}\lim_{x\to 0} M_x f \text{ exists} \}\,.$$

The inequality

$$\|T_t M_y - M_y\| \leqq y^{-n}\lambda_n(([0,y]^n + t)\Delta[0,y]^n),$$

where Δ stands for the symetrical difference, clearly implies the following assertions:

(2.5) $$\text{strong-lim}\, T_t M_y = M_y \quad \text{as } t \to 0$$

(2.6) $$\text{strong-lim}\, M_x M_y = M_y \quad \text{as } x \to 0^+ .$$

Let $f \in L_1(C)$. It is known that strong-$\lim_{t\to 0^+} U_t f$ exists (see [2] and also a direct proof in [10]). Then 2.1. shows that there exists a sequence $x_i \to 0^+$ such that $f^* = w-\lim_i M_{x_i} f$ exists. Since $M_{x_i} f \in H$, $f^* \in H$. But

$$M_x f^* = w-\lim_i M_x M_{x_i} f = w-\lim_i M_{x_i} M_x f = M_x f$$

by (2.6). Thus $f \in H$ and $H = L_1(C)$. Let

$$R_0 f = \text{norm-}\lim_{x\to 0^+} M_x f, \quad f \in L_1(C) .$$

By (2.6) we get $R_0 M_x = M_x$ and $R_0^2 = R_0$. Furthermore, the equalities

$$T_t M_x f - T_t f = M_x T_t f - T_t f = x^{-n} \int_{[0,x]^n} (T_{x+t} f - T_t f)\, ds$$

and the continuity of T at t imply

$$T_t R_0 f = R_0 T_t f = T_t f \quad \text{for any } f \in L_1(C) .$$

Finally considering the closed space

$$K = \{f \in L_1(C) \mid \text{norm-}\lim_{t\to 0} T_t f = R_0 f\}$$

we see that for all $f \in L_1(C)$, $M_x f \in K$ (2.5) and thus $R_0 f \in K$, that is

$$\text{norm-}\lim_{t\to 0} T_t R_0 f = R_0^2 f \quad \text{and} \quad \text{norm-}\lim_{t\to 0} T_t f = R_0 f .$$

The last point of the theorem is proved in [11, Proof of Theorem 4.1].

3. Reduction of the dimension.

The following generalizes an inequality of M. A. Akcoglu-A. del Junco [3, 3.5].

3.1. THEOREM. *Let T and F be as in* 1.1 *and* 1.2. *Let c_n and U be as in* 2.1. *Then there exists a one-dimensional bounded additive process G with respect to U such that*:

$$\forall \alpha > 0,\ \alpha^{-n}|F_{[0,\alpha]^n}| \leqq c_n \bar{\alpha}^{-1} G_{[0,\bar{\alpha}]}\,,$$

where $\bar{\alpha} = \alpha^{2^{-k}}$ *if* $2^{k-1} < n \leqq 2^k$.

To prove 3.1 we will assume without loss of generality that $n = 2^k$ for an integer $k \geqq 0$ (see [3]).

3.2. *The sub-semi-groups of Dunford-Schwartz* [9]. Recall the following construction due to Dunford-Schwartz [9, VII.7.12-18].

For each $x > 0$ and $y \in \mathbf{R}$, let

$$\varrho_x(y) = \frac{x}{2\sqrt{\pi y^3}} e^{-(x^2/4y)} \text{ if } y>0 \quad \text{and} \quad \varrho_x(y) = 0 \text{ if } y \leqq 0\,.$$

If $k \geqq 1$ let

$$\Phi_t(u) = \prod_{i=1}^{m} \varrho_{t_i}(u_{2i-1}) \varrho_{t_i}(u_{2i})\,,$$

where $m = n/2$, $t \in P_m$ and $u \in \mathbf{R}^n$.

Starting with the semi-group $T = (T_t)_{t \in P_n}$, we let $T_t^0 = |T_t|$ so that $T^0 = (T_t^0)_{t \in P_n}$ is a 2^k-dimensional strongly continuous sub-semi-group of L_1-positive contractions: $T_{t+s}^0 \leqq T_t^0 T_s^0$ and

$$\operatorname*{strong-lim}_{\substack{s \to 0 \\ s \in P_n}} T_{t+s}^0 = T_t^0\,.$$

If $k \geqq 1$ let

$$T_t^1 f = \int_{\mathbf{R}_+^n} \Phi_t(u) T_u^0 f\, du, \qquad t \in P_m\,.$$

Then $T^1 = (T_t^1)_{t \in P_m}$ is a 2^{k-1}-parameter sub-semi-group.

Define similarly the sub-semi-groups T^j for $j = 0, \ldots, k$ so that T^k is a one-parameter sub-semi-group. Also recall that the semi-group U given by 2.1 satisfies $T_t^k \leqq U_t$ for all $t > 0$ [see 11].

3.3. Almost additive processes. Denoting

$$L_1^+ = \{f \in L_1 \mid f \text{ is real valued and positive}\}\,,$$

we make the following

DEFINITION. Let $P = (P_t)_{t \in P_n}$ be a sub-semi-group of positive operators on L_1. A set function $F: \mathscr{I}_n \to L_1^+$ will be called an almost additive process with respect to P if it satisfies the following conditions.

3.4. $\sup\{\|F(I)/\lambda_n(I)\| \mid I \in \mathscr{I}_n, \lambda_n(I)>0\}=\gamma(F)<\infty$.

3.5. $P_u F(I) \geqq F(u+I)$ for all $u \in P_n$ and $I \in \mathscr{I}_n$.

3.6. If $I_1,\ldots,I_k \in \mathscr{I}_n$ are pairwise disjoint and if $I=\bigcup_{i=1}^k I_i \in \mathscr{I}_n$, then $F(I) =\sum_{i=1}^k F(I_i)$.

EXAMPLE. If $f \in L_1^+$ then $F_I=\int_I T_t^0 f\,dt$ is almost additive with respect to T^0.

3.7. LEMMA. *Let T and F be as in* 1.1 *and* 1.2, *and let T^0 be as in* 3.2. *Then there exists an almost additive process with respect to T^0, say F^0, such that $|F_I| \leqq F_I^0$ and $\gamma(F)=\gamma(F^0)$.*

PROOF OF 3.7. Let $I \in \mathscr{I}_n$ and let $\mathscr{P}_I$ denote the class of all partitions of I into pairwise disjoint intervals. If $X, Y \in \mathscr{P}_I$, then we will write $X<Y$, if Y refines X so that $(\mathscr{P}_I, <)$ is a directed set. Next, if $X=(I_1,\ldots,I_k) \in \mathscr{P}_I$, let

$$\bar{F}_X = \sum_{i=1}^{k} |F(I_i)| \,.$$

It is then clear that $\|\bar{F}_X\| \leqq \gamma(F)\lambda_n(I)$ and that $X<Y$ implies $\bar{F}_X \leqq \bar{F}_Y$. Therefore if we put

$$F_I^0 = \sup_{X \in \mathscr{P}_I} \bar{F}_X = \lim_{\mathscr{P}_I} \uparrow \bar{F}_X \,,$$

then $\gamma(F^0)=\gamma(F)$ and $T_u^0 F_I^0 \geqq F_{u+I}^0$. Finally, note that it suffices to prove 3.6, when $k=2$. For this, observe that if $X \in \mathscr{P}_I$, $Y \in \mathscr{P}_J$, and $I \cup J \in \mathscr{I}_n$ then $X \cup Y \in \mathscr{P}_{I\cup J}$ and thus $F_{I\cup J}^0 \geqq F_I^0+F_J^0$. Conversely if $X \in \mathscr{P}_{I\cup J}$ then, with obvious notations, $X \cap I \in \mathscr{P}_I$ and $X \cap J \in \mathscr{P}_J$ and thus $F_{I\cup J}^0 \leqq F_I^0+F_J^0$. This completes the proof of the lemma.

PROOF OF THEOREM 3.1. Recall that $n=2^k$. If $k \geqq 1$ then, following Akcoglu-del Junco [3] we let $F_I^1=\int_I f_t^0\,dt$, where

$$f_t^0 = \int_{\mathbf{R}^n} \Phi_t(v) F^0(dv), \qquad t \in P_m \,,$$

f_t^0 being well-defined by Lemma 3.7. It is then easy to see that F^1 is an almost additive process with respect to T^1 and that $\gamma(F^1) \leqq \gamma(F^0)$. Further there is a constant d_n such that

$$\alpha^{-n} F_{[0,\alpha]^n}^0 \leqq d_n \sqrt{\alpha}^{\,-m} F_{[0,\sqrt{\alpha}]^m}^1 \,,$$

see Sections 3.4 and 3.5 in [3]. Define similarly the almost additive process F^j for $j=0,\ldots,k$. F^k is then a one-dimensional almost additive process with respect to T^k and

$$\alpha^{-n}F^0_{[0,\alpha]^n} \leqq c_n\bar{\alpha}^{-1}F^k_{[0,\bar{\alpha}]}\,.$$

Next, for any $x>0$, let $G^k_x = F^k_{[0,x]}$ so that

$$G^k_{x+y} \leqq G^k_x + T^k_x G^k_y \leqq G^k_x + U_x G^k_y\,.$$

By the proof Lemma 4.1 in [5], we see that there is an additive process $(G_t)_{t>0}$ with respect to the semi-group $(U_t)_{t>0}$, in the sense of [5], such that $G^k_x \leqq G_x$ for all positive dyadic x, and such that $\gamma(G)=\gamma(G^k)\leqq\gamma(F)$. Since G^k and $(G_t)_{t>0}$ are continuous processes, we also have $G^k_x \leqq G_x$ for all $x>0$. Finally it suffices to put $G_{[a,b]} = U_a G_{b-a}$ for all $I=[a,b] \in \mathscr{I}_1$ $(a<b)$, to obtain the additive process which verifies the inequality 3.1.

4. Proof of the main result 1.6.

We are now in a position to adapt the one-dimensional proof of M. A. Akcoglu-M. Falkowitz to the n-dimensional case. The nice truncation argument in [4] will be slightly modified as we deal with complex valued functions.

Let T and F be as in Section 1. Let U, C, D, R_0 be as in Section 2, and finally let G be given by Theorem 3.1.

For any $x>0$ we put

$$f_x = x^{-n}F_{[0,x]^n}, \qquad M_x = x^{-n}\int_{[0,x]^n} T_t\,dt$$

(in the strong topology), and

$$g_x = \bar{x}^{-1}G_{[0,\bar{x}]}\,,$$

so that f_x and $g_x \in L_1(C)$ (by 2.4).

By M. A. Akcoglu-U. Krengel's theorem [5], we know that $g=\lim_{x\to 0} g_x$ exists a.e. and since $|R_0|g_x = g_x$ (by 2.3 and 2.4), we have $|R_0|g=g$.

Now, let $x_i>0$ be a sequence such that $x_i \to 0^+$. Since g_{x_i}, $g \in L_1^+$, $g_i = g_{x_i} \wedge g$ is well-defined and $g_i \to g$ a.e. and thus in L_1-norm. On the other hand let

$$f_i = f_{x_i}1_{\{|f_{x_i}|\leqq c_n g_i\}} + \frac{c_n g_i f_{x_i}}{|f_{x_i}|}1_{\{|f_{x_i}|>c_n g_i\}}$$

so that $|f_i|\leqq c_n g_i$. Since g_i converges in L_1-norm, by passing to a subsequence, if necessary, we may assume that $f=w\text{-}\lim_i f_i$ exists (see [9, p. 292]). Further, on the set $\{|f_{x_i}|\leqq c_n g_i\}$ we have $|f_{x_i}-f_i|=0$, and on the set $\{|f_{x_i}|>c_n g_i\}$, we have

$$\begin{aligned}|f_{x_i}-f_i| &= \big||f_{x_i}|-c_n g_i\big| = |f_{x_i}|-c_n g_i\\ &\leqq c_n g_{x_i} - c_n g_i \qquad \text{(by Theorem 3.1)}\,.\end{aligned}$$

So, $|f_{x_i}-f_i| \leqq c_n(g_{x_i}-g_i)$ holds a.e.

We then obtain for all $x>0$,

$$|f_x - M_x f| = |R_0 f_x - M_x f| = |\text{strong-}\lim_i M_x f_{x_i} - M_x f|$$

(by 2.3 and Lemma 3.2 in [3])

$$= |w\text{-}\lim_i M_x(f_{x_i} - f_i)| \quad \text{(by definition of } f)$$

$$\leqq w\text{-}\lim_i c_n \bar{x}^{-1} \int_0^{\bar{x}} U_t |f_{x_i} - f_i|\, dt \quad \text{(by Theorem 2.1)}$$

$$\leqq w\text{-}\lim_i c_n^2 \bar{x}^{-1} \int_0^{\bar{x}} U_t (g_{x_i} - g_i)\, dt$$

$$= c_n^2 \left(|R_0| g_x - \bar{x}^{-1} \int_0^{\bar{x}} U_t g\, dt \right) \quad \text{(by Lemma 3.2 in [3] and definition of } g_i)$$

$$= c_n^2 \left(g_x - \bar{x}^{-1} \int_0^{\bar{x}} U_t g\, dt \right).$$

Now, since $g \in L_1(C)$, the last member tends a.e. to $c_n^2(g - |R_0| g) = 0$ as $x \to 0^+$. But as $f \in L_1(C)$, 2.3 and the n-parameter local ergodic theorem [see 11] applied in $L_1(C)$ yield

$$\lim_{x \to 0} M_x f = R_0 f \quad \text{a.e.}$$

We then have

$$\lim_{x \to 0} f_x = R_0 f \quad \text{a.e.}$$

The proof is completed.

REFERENCES

1. M. A. Akcoglu, *A pointwise ergodic theorem in L_p-spaces*, Canad. J. Math. 27 (1975), 1075–1082.
2. M. A. Akcoglu and R. V. Chacon, *A local ratio theorem*, Canad. J. Math. 22 (1970), 545–552.
3. M. A. Akcoglu and A. del Junco, *Differentiation of n-dimensional additive processes*, Canad. J. Math. 33 (1981), 749–768.
4. M. A. Akcoglu and M. Falkowitz, *A general local ergodic theorem in L_1*, Pacific J. Math., to appear.
5. M. A. Akcoglu and U. Krengel, *A differentiation theorem for additive processes*, Math. Z. 163 (1978), 199–210.
6. M. A. Akcoglu and U. Krengel, *A differentiation theorem in L_p*, Math. Z. 169 (1979), 31–40.
7. M. A. Akcoglu and U. Krengel, *Two examples of local ergodic divergence*, Israel J. Math. 33 (1979), 225–230.
8. A. Brunel, *Théorème ergodique ponctuel pour un semi-groupe commutatif finiment engendré de contractions de L_1*, Ann. Inst. H. Poincaré, 9 (1973), 327–343.

9. N. Dunford and J. T. Schwartz, *Linear operators*, Part I; *General theory* (Pure and Appl. Math. 7), Interscience Publ. Inc., New York, 1958.
10. R. Emilion, *Continuity at* 0 *of semi-groups on* L_1 *and differentiation of additive processes*, to appear.
11. R. Emilion, *Additive and superadditive local theorems*, to appear.
12. R. Emilion, *Semi-groups in* L_∞ *and local ergodic theorem*, to appear.
13. D. Feyel, *Compléments sur la convergence presque sûre des familles résolues.* (Séminaire de théorie du potentiel 4, Paris, 1977–1978) eds. R. Gerard, J.-P. Ramis (Lecture Notes in Math. 713), pp. 51–55. Springer-Verlag, Berlin - Heidelberg - New York, 1979.
14. D. Feyel, *Sur une classe remarquable de processus abéliens*, Math. Z., to appear.
15. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, (Amer. Math. Soc. Colloq. Oubl. 31), American Mathematical Society, Providence, R.I., 1957.
16. M. Lin, *On local ergodic convergence of semi-groups and additive processes.* Israel J. Math. 42 (1982), 300–308.
17. R. T. Terrell, *Local ergodic theorems for n-parameter semi-groups of operators*, in *Contributions to ergodic theory and probability*, (Proc. Ohio State University, 1970), ed. L. Sucheston, (Lecture notes in Math. 160), pp. 262–278. Springer-Verlag, Berlin - Heidelberg - New York, 1970.

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