MATH. SCAND. 57 (1985), 215–224

DIRECTIONAL DERIVATIVES AND ALMOST EVERYWHERE DIFFERENTIABILITY OF BICONVEX AND CONCAVE-CONVEX OPERATORS

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Abstract.

In this paper we establish some results about directional derivatives and almost everywhere differentiability of separately convex and concave-convex operators between topological vector spaces.

Introduction.

M. Valadier has extended to vector-valued convex operators the concept of directional derivatives and has given in [24] some properties of this notion. J. M. Borwein in [5] has established some other properties of directional derivatives of vector-valued convex operators. J. M. Borwein has also studied in [5] generic differentiability properties of a convex operator from an Asplund topological vector space into an ordered topological vector space (see also N. K. Kirov [12]). Many results about generic differentiability properties of real-valued convex functions were already known (see [1], [5], [16], [13] and the references therein). However to our knowledge the only results about directional derivatives of real-valued concave-convex functions have been established by R. T. Rockafellar in [19].

Our study is along the lines of existence of directional derivatives and of almost everywhere differentiability properties of biconvex and concave-convex operators with values in ordered topological vector spaces. It is divided in four sections. In the first section we recall some preliminary definitions which will be of use in the next parts. Section two is devoted to derivatives of convex operators. Section three studies the derivatives of biconvex and concave-convex operators and finally in section four we prove some results about almost everywhere differentiability of biconvex and concave-convex operators.

Received May 4, 1983; in revised form September 19, 1983.

1. Preliminaries.

In this paper E, F, and G denote (real separated) topological vector spaces. We assume that G_+ is a *closed pointed convex cone* in G ($sG_+ + tG_+ \subset G_+$ for all real numbers $s, t \geqq 0$ and $G_+ \cap -G_+ = \{0\}$) which induces an ordering in F by $z \leqq z'$ if $z' - z \in G_+$. So G is an ordered topological vector space.

A mapping f from a convex subset C of E into G is said to be a convex operator if

$$f(sx+ty) \leqq sf(x)+tf(y)$$

for all $x, y \in C$ and all nonnegative numbers s, t satisfying $s+t=1$. A mapping g from a convex subset $C \times D$ of $E \times F$ into G is a biconvex (respectively concave-convex) operator if for each $(x, y) \in C \times D$ the mapping $f(x, .)$ and $f(., y)$ (respectively $-f(., y)$) are convex.

In the sequel we shall always assume that G_+ is a *normal* cone that is there exists a base of neighbourhoods $\{W\}$ of zero in G such that

$$W = (W - G_+) \cap (W + G_+)\,.$$

Such neighbourhoods are said to be *full*. Usual ordered topological vector spaces are normal (see for example [18]).

2. Directional derivatives of convex operators.

Let f be a convex operator from an *open convex* subset C of E into G and let c be a point in C. Following Valadier [24] we shall put for each $u \in E$

$$f'(c; u) = \inf_{t>0} t^{-1}[f(c+tu) - f(c)]$$

when this *infimum* exists.

If g is any mapping from an open subset X_0 of a topological vector space X into another topological vector space Y, then for $a \in X_0$ and $u \in X$ we shall put

$$Dg(a; u) = \lim_{t \downarrow 0} t^{-1}[g(a+tu) - g(a)] \tag{2.1}$$

whenever this limit exists.

So if f is convex and G_+ is closed, then $f'(c; u)$ exists whenever $Df(c; u)$ exists and then $f'(c; u) = Df(c; u)$ as is easily seen by making use of the monotonicity of $t \mapsto t^{-1}[f(c+tu) - f(c)]$. Let us remark that the reverse is also true when G is *countably Dini* (see [5] and [17]) in the sense that every decreasing sequence in G with an infimum converges to that infimum.

2.1. PROPOSITION. *Let f be a continuous convex operator from an open convex* *subset C of E into G and let $c \in C$ and $u \in E$. If $Df(c;u)$ exists, then*

$$Df(c;u) = \lim_{\substack{x \to u \\ t \downarrow 0}} t^{-1}[f(c+tx)-f(c)] .$$

PROOF. Let W' be a full neighbourhood of zero in G and let W be an open circled neighbourhood of zero with $W+W+W \subset W'$. Choose a real number $s \in]0,1]$ such that

$$t^{-1}[f(c+tu)-f(c)] \in Df(c;u)+W \tag{2.2}$$

for every $t \in]0,s]$. By continuity of f on C there exists a circled neighbourhood X of zero in E such that

$$s^{-1}[f(c+sx)-f(c)] \in Df(c;u)+W$$

for every $x \in u+X$. So on the one hand we have by convexity of f and by the latter relation

$$\begin{aligned} t^{-1}[f(c+tx)-f(c)] &\in s^{-1}[f(c+sx)-f(c)]-G_+ \\ &\subset Df(c;u)+W-G_+ \end{aligned} \tag{2.3}$$

for all $x \in u+X$ and $t \in]0,s]$. On the other hand for each $x \in u+X$ and each $t \in]0,s]$ we have by relation (2.2) and by convexity of f once again

$$\begin{aligned} t^{-1}[f(c+tx)-f(c)] &= t^{-1}[f(c+tu)-f(c)] \\ &\quad +t^{-1}[f(c+tu+t(x-u))-f(c+tu)] \\ &\in Df(c;u)+W-t^{-1}[f(c+tu-t(x-u))-f(c+tu)]+G_+ . \end{aligned} \tag{2.4}$$

Moreover for each $x \in u+X$ and each $t \in]0,s]$ as $u-x \in X$ we have by relations (2.2) and (2.3)

$$\begin{aligned} &-t^{-1}[f(c+tu-t(x-u))-f(c+tu)] \\ &\quad = -t^{-1}[f(c+t(u+(u-x)))-f(c)]+t^{-1}[f(c+tu)-f(c)] \\ &\quad \in -Df(c;u)+W+G_+ +Df(c;u)+W \end{aligned}$$

and hence by relation (2.4)

$$\begin{aligned} t^{-1}[f(c+tx)-f(c)] &\in Df(c;u)+W+W+W+G_+ \\ &\subset Df(c;u)+W'+G_+ . \end{aligned}$$

Therefore making use of relation (2.3) again we obtain

$$t^{-1}[f(c+tx)-f(c)] \in Df(c;u)+(W'+G_+) \cap (W'-G_+) = Df(c;\, u)+W'$$

for every $x \in u+X$ and every $t \in]0,s]$ and the proof is complete.

REMARK. This proposition could also been proved by invoking the fact that g is locally Lipschitzian on C in the generalized sense given in [22]. However the above proof which does not make use of auxiliary results has seemed to us better in the context of this paper since it is in the line of the proof of proposition 3.1 and hence prepares the reader for the proof of this proposition.

3. Directional derivatives of biconvex and concave-convex operators.

In this section we shall study some properties of derivatives of binconvex and concave-convex operators.

3.1. PROPOSITION. *Let g be a concave-convex (respectively biconvex) operator from an open convex subset $C \times D$ of $E \times F$ into G and let $(c,d) \in C \times D$ and $(\bar{u},\bar{v}) \in E \times F$. Assume that g is continuous on a neighbourhood of (c,d) and that $Dg(c,d;\, \bar{u},0)$ and $Dg(c,d;\, 0,\bar{v})$ exist (respectively $Dg(c,d;\, -\bar{u},0)$ also exists with $Dg(c,d;\, -\bar{u},0) = -Dg(c,d;\, \bar{u},0)$). Then $Dg(c,d;\, u,v)$ exists. In fact*

$$Dg(c,d;\, \bar{u},\bar{v}) = Dg(c,d;\, \bar{u},0)+Dg(c,d;\, 0,\bar{v})$$

and

$$Dg(c,d;\, \bar{u},\bar{v}) = \lim_{\substack{(u,v)\to(\bar{u},\bar{v}) \\ t\downarrow 0}} t^{-1}[g(c+tu,d+tv)-g(c,d)]\,.$$

PROOF. Put $a:=Dg(c,d;\, \bar{u},0)$ and $b:=Dg(c,d;\, 0,\bar{v})$. Let W' be a full neighbourhood of zero in G. Choose an open circled neighbourhood W of zero in G with $W+W \subset W'$. By Proposition 2.1 there exist a real number $r>0$ and a neighbourhood $U' \times V'$ of zero in $E \times F$ such that g is continuous at $(c+r\bar{u},d)$ (respectively $(c-r\bar{u},d)$) and $(c,d+r\bar{v})$, and such that

$$t^{-1}[g(c+tu,d)-g(c,d)] \in a+W \tag{3.1}$$

and

$$t^{-1}[g(c,d+tv)-g(c,d)) \in b+W \tag{3.2}$$

(respectively relations (3.1) and (3.2) hold and $r^{-1}[g(c-r\bar{u},d)-g(c,d)] \in -a+W$) for all $u \in \bar{u}+U'$, $v \in \bar{v}+V'$ and $t \in]0,r]$. Moreover from relations

$$r^{-1}[g(c+r\bar{u},d)-g(c,d) \in a+W$$

$$(\text{respectively } r^{-1}[g(c-\bar{r},d)-g(c,d)] \in -a+W)$$

and

$$r^{-1}[g(c, d+r\bar{v})-g(c,d)] \in b+W$$

and from the continuity of g at $(c+r\bar{u}, d)$ (respectively $(c-r\bar{u}, d)$) and $(c, d+r\bar{v})$ it follows that there exist a real number $s \in]0, r]$ and a neighbourhood $U \times V \subset U' \times V'$ of zero in $E \times F$ such that

$$r^{-1}[g(c+ru, d+tv)-g(c, d+tv)] \in a+W \tag{3.3}$$

$$\text{(respectively } r^{-1}[g(c-ru, d+tv)-g(c, d+tv)] \in -a+W\text{)}$$

and

$$r^{-1}[g(c+tu, d+rv)-g(c+tu, d)] \in b+W \tag{3.4}$$

for all $t \in]0, s]$, $u \in \bar{u}+U$ and $v \in \bar{v}+V$. Therefore as g is concave-convex (respectively biconvex), for each $t \in]0, s]$, each $u \in \bar{u}+U$ and each $v \in \bar{v}+V$ we have on the one hand

$$t^{-1}[g(c+tu, d+tv)-g(c, d+tv)] \geqq r^{-1}[g(c+ru, d+tv)-g(c, d+tv)]$$

(respectively $t^{-1}[g(c+tu, d+tv)-g(c, d+tv)] \geqq -r^{-1}[g(c-ru, d+tv)-g(c, d+tv)]$) and hence by relation (3.3)

$$t^{-1}[g(c+tu, d+tv)-g(c, d+tv)] \in a+W+G_+ \tag{3.5}$$

and similarly

$$t^{-1}[g(c+tu, d+tv)-g(c+tu, d)] \leqq r^{-1}[g(c+tu, d+rv)-g(c+tu, d)]$$

and hence by relation (3.4)

$$t^{-1}[g(c+tu, d+tv)-g(c+tu, d)] \in b+W-G_+ \,. \tag{3.6}$$

So if we write

$$t^{-1}[g(c+tu, d+tv)-g(c,d)]$$
$$= t^{-1}[g(c+tu, d+tv)-g(c+tu, d)]+t^{-1}[g(c+tu, d)-g(c,d)]$$

and if we invoke relations (3.1) and (3.6) we see that

$$t^{-1}[g(c+tu, d+tv)-g(c,d)] \in a+b+W+W-G_+ \subset a+b+W'-G_+$$

for all $t \in]0, s]$, $u \in \bar{u}+U$ and $v \in \bar{v}+V$.

In the same way by writing

$$t^{-1}[g(c+tu, d+tv)-g(c,d)]$$
$$= t^{-1}[g(c+tu, d+tv)-g(c, d+tv)]+t^{-1}[g(c, d+tv)-g(c,d)]$$

and by making use of relations (3.2) and (3.5) we obtain

$$t^{-1}[g(c+tu, d+tv)-g(c,d)] \in a+b+W+W+G_+ \subset a+b+W'+G_+$$

for all $t \in]0,s]$, $u \in \bar{u}+U$ and $v \in \bar{v}+V$ and hence

$$t^{-1}[g(c+tu, d+tv)-g(c,d)] \in a+b+(W'+G_+) \cap (W'-G_+) = a+b+W' .$$

Therefore $Dg(c,d;\, \bar{u},\bar{v})$ exists

$$Dg(c,d;\, \bar{u},\bar{v}) = Dg(c,d;\, \bar{u},0)+Dg(c,d;\, 0,\bar{v})$$

and

$$Dg(c,d;\, \bar{u},\bar{v}) = \lim_{\substack{(u,v)\to(\bar{u},\bar{v})\\ t\downarrow 0}} t^{-1}[g(c+tu, d+tv)-g(c,d)] .$$

By slightly strengthening conditions on the positive cone G_+ we can derive from the above proposition some important consequences.

3.2. Definition. Following Penot [17] (see also [5]) we shall say that G is *countably Daniell* if every decreasing sequence in G with a lower bound has an infimum and converges to that infimum.

Many examples and properties of countably Daniell ordered topological vector spaces can be found in [17] and [5] (see also [22] and [24]).

Let us recall the following result (see Proposition 3.7 in [5] and Proposition 3 in [24]) which will allow us to state Corollary 3.4.

3.3. Proposition. *Let f be a continuous convex operator from an open convex subset C of E into G. Assume that G is countably Daniell. Then for each $c \in C$ the mapping $Df(c;\, .)$ exists on E and is a continuous positively homogeneous convex operator.*

3.4. Corollary. *Let g be a concave-convex operator from an open convex subset $C \times D$ of $E \times F$ into G which is continuous on a neighbourhood of $(c,d) \in C \times D$. Assume that G is countably Daniell. Then the mapping $Dg(c,d;\, .,.)$ exists on $E \times F$ and is a continuous positively homogeneous concave-convex operator. Moreover*

$$Dg(c,d;\, u,v) = Dg(c,d;\, u,0)+Dg(c,d;\, 0,v)$$

for every $(u,v) \in E \times F$.

Proof. This is a direct consequence of Proposition 3.1 and 3.3.

Of course with appropriate assumptions one can state similar results for biconvex operators.

Let us recall the following definition.

3.5. DEFINITION. A mapping g from an open subset X_0 of a topological vector spade X into a topological vector space Y is *Gateaux differentiable* at a point $a \in X_0$ if $Dg(a; .)$ exists and is a continuous linear operator from X into Y.

If moreover

$$\lim_{\substack{x \to u \\ t \downarrow 0}} t^{-1}[g(a+tx)-g(a)]$$

exists for every $u \in X$ one says that g is *Michal-Bastiani differentiable* or simply M-B-*differentiable* at a (see [15], [4], [3], [17]).

It is not difficult to see that if g is M-B-differentiable at a then g is *Hadamard differentiable* at a in the sense that for each compact subset K of X the limit in relation (2.1) exists uniformly with respect to $u \in K$.

Another important corollary of Proposition 3.1 can now be given.

3.6. COROLLARY. *Let g be a biconvex or concave-convex operator from an open convex subset $C \times D$ of $E \times F$ into G which is continuous on a neighbourhood of (c, d). Assume that g is partially Gateaux differentiable at (c, d) in the sense that $g(., d)$ and $g(c, .)$ are Gateaux differentiable at c and d respectively. Then g is* M-B-*differentiable at (c, d).*

PROOF. This is a direct conseqeunce of Proposition 3.1 and of the above definition.

4. Almost everywhere differentiability of biconvex and concave-convex operators.

The importance of strictly linear functionals in the study of differentiability of convex operators has been illustrated by J. M. Borwein in [5].

4.1. DEFINITION. A continuous linear functional l in the topological dual G' of G is said to be *strictly poisitive* if $\langle l, z\rangle > 0$ for all $z \in G_+ \setminus \{0\}$.

If G is locally convex and if the convex cone G_+ has a *convex base B, that is* $G_+ = \bigcup_{t \geqq 0} tB$ and $0 \notin \operatorname{cl} B$, then by separating 0 and $\operatorname{cl} B$ by the Hahn-Banach theorem one gets that G admits strictly positive linear functionals (see [5]).

Moreover if G is a *separable normed* vector space, then by Proposition 2.8 in [5], G admits a strictly positive linear functional.

4.2. PROPOSITION. *Let f be a continuous convex operator from an open convex* *subset C of E into G and let a be a point in C. Assume that G is countably Daniell and admits a strictly positive continuous linear functional l and assume that $l \circ f$ is Gateaux differentiable at a. Then f is* M-B-*differentiable at a.*

PROOF. This is a direct consequence of the preceding Propositions 2.1 and 3.3 and of Proposition 4.2 in [5].

4.3. COROLLARY. *Let g be a continuous biconvex or concave-convex operator from an open convex subset $C \times D$ of $E \times F$ into G and let (c, d) be a point in $C \times D$. Assume that G is countably Daniell and admits a strictly positive continuous linear functional l and assume $l \circ g$ is Gateaux differentiable at (c, d). Then g is* M-B-*differentiable at (c, d).*

PROOF. This is a direct consequence of Propositions 3.1 and 4.2.

Before stating our result about almost everywhere differentiability of biconvex or concave-convex operators let us recall the notion of Haar null sets introduced by J. P. R. Christensen.

4.4. DEFINITION. Let X be a *separable Fréchet* (locally convex) vector space and let N be an *universally measurable* subset of X, that is N belongs to the m-completion of the Borel tribe $\mathscr{B}(X)$ for each finite measure m on $\mathscr{B}(X)$. One says that N is *Haar null* in X (see [9]) if for every probability measure P on $\mathscr{B}(X)$

$$P(x+N) = 0 \quad \text{for every } x \in X .$$

This notion generalizes the one of Lebesgue negligible sets in $\mathbb{R}^n$ and has many important properties. For instance $X \setminus A$ is *topologically dense* in X whenever A is Haar null in X.

However the best result justifying the importance of Haar null sets is the following.

4.5. PROPOSITION (see Theorem 7.5 in [9]). *Let f be a function from an open subset A of a separable Fréchet space X into $\mathbb{R}$. Assume that f is locally lipschitz in the sense that for each $a \in A$ there exists a continuous semi norm p on X and a neighbourhood U of a in A such that*

$$|f(x)-f(y)| \leqq p(x-y) \quad \textit{for all } x, y \in U .$$

Then there exists a Haar null set $N \subset A$ in X such that f is Gateaux differentiable at each point in $A \setminus N$.

Theorem 7.5 in [9] is not stated exactly as above but it is easy to see that the proof given in [9] still holds.

We can now state our result about differentiability of biconvex or concave-convex operators.

4.6. PROPOSITION. *Assume that E and F are separable Fréchet spaces and that G is countably Daniell and admits a strictly positive continuous linear functional. If g is a continuous biconvex or concave-convex operator from an open convex subset* $C \times D$ *of* $E \times F$ *into G, then there exists a Haar null set* $N \subset C \times D$ *in* $E \times F$ *such that g is* M-B-*differentiable at each point of* $(C \times D) \setminus N$.

PROOF. Let l be a strictly positive continuous linear functional on G. Then $l \circ g$ is a real-valued continuous biconvex or concave-convex function on $C \times D$. Therefore for each point $(c, d) \in C \times D$ there exists an open convex neighbourhood $C_0 \times D_0 \subset C \times D$ of (c, d) in $E \times F$ and two real numbers m and M such that

$$m \leqq l \circ g(x, y) \leqq M \qquad \text{for all } (x, y) \in C_0 \times D_0$$

and hence by Proposition 3.5 in [11] the function $l \circ g$ is Lipschitz around (c, d). So $l \circ g$ is locally Lipschitz on $C \times D$ and by proposition 4.5 there exists a Haar null set $N \subset C \times D$ in $E \times F$ such that $l \circ g$ is Gateaux differentiable on $(C \times D) \setminus N$. Thus by Corollary 4.3, g is M-B-differentiable at each point in $(C \times D) \setminus N$, which completes the proof.

REMARK. If G is normed, then (as in [6], Theorem 3.1) it is not necessary by the remark following Proposition 4.1 to assume that Gadmits a strictly positive continuous linear functional since there exists a separable vector subspace $H \subset G$ with $\mathrm{cl}_G[f(C \times D)] \subset H$.

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