MATH. SCAND. 57 (1985), 225–235

NOTE ON A CONGRUENCE FOR p-ADIC L-FUNCTIONS

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1. Introduction.

For a fixed prime p, let $L_p(s,\chi)$ denote the Kubota-Leopoldt p-adic L-function attached to a primitive Dirichlet character χ. If the so-called first factor θ of χ is nonprincipal, then $L_p(s,\chi)$ possesses a well known power series expression, say $f(X,\theta)$, due to Iwasawa [4]. In this note we consider a congruence relation between $f(X,\theta)$ and $f(X,\theta\psi)$, where the character ψ is of p-power order.

Let $\mathbf{Q}_p$ denote the field of p-adic numbers and let K be a finite extension of $\mathbf{Q}_p$ containing the values of the characters θ and $\theta\psi$. Then $f(X,\theta)$ and $f(X,\theta\psi)$ belong to the power series algebra $\Lambda = \mathfrak{o}[[X]]$, where $\mathfrak{o}$ is the ring of integers of K. Let $\mathfrak{p}$ stand for the maximal ideal of $\mathfrak{o}$. It was recently proved by Gras [2] that there is a congruence mod $\mathfrak{p}\Lambda$ between $f(X,\theta)$ and $f(X,\theta\psi)$ (previously, some results in the same direction had been obtained by the author [9] and Ferrero [1]). However, the proof in [2] is based on a new construction of p-adic L-functions. We shall prove the congruence directly, by using Iwasawa's original construction of $f(X,\theta)$. We note that the proof brings some new aspects to the original theory of $f(X,\theta)$ that are perhaps of independent interest.

The congruence has an application to the Iwasawa λ-invariants. Indeed, one immediately deduces an explicit relation between the λ-invariants of $f(X,\theta)$ and $f(X,\theta\psi)$ (formulated by Gras in [2, Proposition V.3]) and, from this, the so-called Riemann–Hurwitz genus formula between the λ^--invariants of an imaginary abelian field and its p-extension (see [3], [6], [7], [8], [11]). We shall briefly discuss this application at the end of the paper.

2. The group algebra elements behind the power series.

Put $q = p$ if $p > 2$ and $q = 4$ if $p = 2$. If the conductor f_χ of a character χ is not divisible by qp, then χ is said to be of the first kind. Let C_1 denote the set of all such characters χ (all characters are assumed primitive).

Making a slight change to the previous notation we will consider the

Received August 28, 1984.

Iwasawa power series attached to a character of the form $\theta\omega$, where ω is the Teichmüller character mod q and θ is an odd character ($\neq \omega^{-1}$) in C_1. Let $f_\theta = d$ or dq, where d is prime to p.

In the definition of $f(X,\theta\omega)$ (see [4], [12]), a crucial role is played by an element $\xi_n(\theta)$ of the group algebra $R_n = \mathfrak{o}[\Gamma_n]$ ($n = 0,1,\ldots$), where Γ_n is the multiplicative group of those residue classes $a + dqp^n\mathbf{Z}$ for which $a \equiv 1 \pmod{dq}$. Denote by $\gamma_n(a)$ the image of $a + dqp^n\mathbf{Z}$ under the canonical projection $(\mathbf{Z}/dqp^n\mathbf{Z})^\times \to \Gamma_n$. Then

$$\xi_n(\theta) = -\frac{1}{2dqp^n}\sum_a a\theta(a)\gamma_n(a)^{-1} \quad (0 < a < dqp^n,\ (a,dp) = 1). \tag{1}$$

Now observe that Γ_n is a cyclic group of order p^n, generated by $\gamma_n(1 + b_\theta q)$, where b_θ is any integer satisfying the conditions

$$(b_\theta,p) = 1,\ (1 + b_\theta q,\ dp) = 1. \tag{2}$$

It follows that we can write $\xi_n(\theta)$ in the form

$$\xi_n(\theta) = \sum_{k=0}^{p^n-1} c_k(\theta)\gamma_n(1 + b_\theta q)^{-k} \tag{3}$$

with

$$c_k(\theta) = c_k(n,\theta) = -\frac{1}{2dqp^n}\sum_a a\theta(a) \in \mathfrak{o} \tag{4}$$

($k = 0,\ldots,p^n - 1$), where the sum is extended over the values of a for which $0 < a < dqp^n$, $(a,dp) = 1$ and $\gamma_n(a) = \gamma_n(1 + b_\theta q)^k$.

Now set $\Phi_n = (1 + X)^{p^n} - 1$. The power series $f(X,\theta\omega) \in \Lambda = \mathfrak{o}[[X]]$ is defined by the $\mathfrak{o}$-algebra isomorphisms

$$\Lambda/\Phi_n\Lambda \to R_n,\ 1 + X + \Phi_n\Lambda \to \gamma_n(1 + b_\theta q)$$

as follows:

$$f(X,\theta\omega) + \Phi_n\Lambda \to \xi_n(\theta).$$

Thus, by (3),

$$f(X,\theta\omega) \equiv \sum_{k=0}^{p^n-1} c_k(\theta)(1 + X)^{p^n-k} \pmod{\Phi_n\Lambda}. \tag{5}$$

We observe that $f(X,\theta\omega)$ depends upon the choice of b_θ; when wishing to emphasize this dependence we say that $f(X,\theta\omega)$ *belongs to* a particular value of this parameter. (A natural choice would be $b_\theta = d$. This was fixed in [4] and [12] but does not suffice for the present purpose.)

For a p-adic integer α and for $n \geqq 0$, let $s_n(\alpha) \in \mathbf{Z}$ denote the unique number such that $0 \leqq s_n(\alpha) < qp^n$ and $s_n(\alpha) \equiv \alpha \pmod{qp^n}$. If $p > 2$, let V denote the group of $(p-1)$st roots of 1, and if $p = 2$, let $V = \{\pm 1\}$. For $k \in \mathbf{Z}$, let $\alpha_k = \alpha_k(n,\theta)$ be any p-adic integer such that

$$\alpha_k \equiv (1 + b_\theta q)^k \pmod{qp^n}. \tag{6}$$

By using these notations we can express $c_k(n,\theta)$ in the following form (see [12, p. 122]): If $d > 1$, then

$$c_k(n,\theta) = -\frac{1}{d} \sum_{\pm\eta} \sum_{i=0}^{d-1} i\theta(s_n(\eta\alpha_k) + iqp^n), \tag{7}$$

where, as always in the sequel, $\sum_{\pm\eta}$ denotes summation over any system of representatives η of V mod ± 1. If $d = 1$, that is, $p > 2$ and $\theta = \omega^u$ with u odd, $u \not\equiv -1 \pmod{p-1}$, then

$$c_k(n,\theta) = -\frac{1}{p^{n+1}} \sum_{\pm\eta} s_n(\eta\alpha_k)\eta^u + \frac{1}{2} \sum_{\pm\eta} \eta^u. \tag{8}$$

Both formulas are valid for all $k = 0, \ldots, p^n - 1$.

We now extend (7) and (8) to all $k \in \mathbf{Z}$ by taking them as definitions of $c_k(n,\theta)$ for the other k. Then it is readily seen that $c_k(n,\theta) = c_h(n,\theta)$ whenever $k \equiv h \pmod{p^n}$.

For further reference, let us finally record a formula needed in the proof of (7) (e.g., [12, p. 121]): if $d > 1$ then, for all $z \in \mathbf{Z}$,

$$\sum_{i=0}^{d-1} \theta(z + iqp^n) = 0. \tag{9}$$

3. Congruences for the group algebra elements; case $p > 2$.

In this section we suppose that $p > 2$. Let l be a prime $\equiv 1 \pmod{p}$, and let ψ be a nonprincipal character mod l having a p-power order. As $p > 2$, we see that ψ is even and so $\xi_n(\theta\psi)$ is defined, unless $\theta\psi = \omega^{-1}$.

PROPOSITION 1. *Suppose that $\theta_l\psi \neq 1$, where θ_l denotes the l-component of θ. Choose a common value b for the parameters b_θ and $b_{\theta\psi}$. Then, for $n = 0,1,\ldots$ and for all $k \in \mathbf{Z}$,*

$$c_k(n,\theta\psi) \equiv c_k(n,\theta) - \theta(l)c_{k-t}(n,\theta) \pmod{\mathfrak{p}},$$

where $t = t(n)$ is the unique rational integer such that

$$0 \leqq t < p^n, \quad l \equiv (1 + bp)^t \pmod{p^{n+1}}. \tag{10}$$

PROOF. The assumption $\theta_l\psi \neq 1$ implies, in particular, that $l \mid f_{\theta\psi}$. Hence $\theta\psi \neq \omega^{-1}$.

Let us keep n and k fixed. First assume that $l | f_\theta$. Then $f_{\theta\psi} = f_\theta = d$ or dp with $d > 1$. Since the order of ψ is a p-power, $\psi(a) \equiv 1 \pmod{\mathfrak{p}}$ unless a is divisible by l. Thus we find, because $\theta_l \neq \psi^{-1}$, that $(\theta\psi)(a) = \theta(a)\psi(a) \equiv \theta(a) \pmod{\mathfrak{p}}$ for all $a \in \mathbf{Z}$. Consequently, by (7), $c_k(\theta\psi) \equiv c_k(\theta) \pmod{\mathfrak{p}}$ which proves the assertion.

Secondly let $(l, f_\theta) = 1$. Now $f_{\theta\psi} = ld$ or ldp with $(l,d) = 1$, and by using the fact that $\alpha_k l \equiv (1+bp)^{k+t} \pmod{p^{n+1}}$ we can write

$$c_{k+t}(\theta\psi) = -\frac{1}{ld} \sum_{\pm\eta} \sum_{i=0}^{ld-1} i\theta(s_n(\eta\alpha_k l) + ip^{n+1})\psi(s_n(\eta\alpha_k l) + ip^{n+1}).$$

To get rid of ψ, observe that among the numbers $s_n(\eta\alpha_k l) + ip^{n+1}$ $(i = 0, \ldots, ld-1)$ those divisible by l are precisely $s_n(\eta\alpha_k l) + (i_\eta + jl)p^{n+1}$ with $j = 0, \ldots, d-1$, where i_η is defined by

$$s_n(\eta\alpha_k l) + i_\eta p^{n+1} = l s_n(\eta\alpha_k), \ 0 \leqq i_\eta \leqq l-1.$$

It follows that

$$\begin{aligned} dc_{k+t}(\theta\psi) \equiv &- \sum_{\pm\eta} \sum_{i=0}^{ld-1} i\theta(s_n(\eta\alpha_k l) + ip^{n+1}) \\ &+ \sum_{\pm\eta} \sum_{j=0}^{d-1} (i_\eta + jl)\theta(l s_n(\eta\alpha_k) + jlp^{n+1}) \pmod{\mathfrak{p}}. \end{aligned}$$

To reformulate this congruence, note that

$$\begin{aligned} \sum_{i=0}^{ld-1} i\theta(z + ip^{n+1}) &= \sum_{i=0}^{l-1} \sum_{j=0}^{d-1} (j + id)\theta(z + jp^{n+1}) \\ &\equiv \sum_{j=0}^{d-1} j\theta(z + jp^{n+1}) \pmod{\mathfrak{p}} \end{aligned}$$

for all $z \in \mathbf{Z}$. If $d > 1$, we therefore infer, on recalling (9), that

$$dc_{k+t}(\theta\psi) \equiv dc_{k+t}(\theta) - \theta(l) dc_k(\theta) \pmod{\mathfrak{p}}.$$

Thus the proposition is established in this case.

Now let $d = 1$. Then $\theta = \omega^u$ and our congruence reduces to

$$c_{k+t}(\theta\psi) \equiv \theta(l) \sum_{\pm\eta} i_\eta \theta(s_n(\eta\alpha_k)) \equiv \sum_{\pm\eta} i_\eta \eta^u \pmod{\mathfrak{p}}$$

or

$$c_{k+t}(\omega^u\psi) \equiv -\frac{1}{p^{n+1}} \sum_{\pm\eta} (s_n(\eta\alpha_k l) - l s_n(\eta\alpha_k))\eta^u \pmod{\mathfrak{p}}. \tag{11}$$

Combined with (8) this yields

$$c_{k+t}(\omega^u\psi) \equiv c_{k+t}(\omega^u) - lc_k(\omega^u) \pmod{\mathfrak{p}}.$$

Thus the proof is complete.

The following supplement to Proposition 1 deals with the case in which $\xi_n(\theta\psi)$ is defined but $\xi_n(\theta)$ is not. Here the parameter $b_{\theta\psi}$ may be chosen to fulfil only the original conditions (see (2)).

For $z \in \mathbf{Z}$, denote by $\mathrm{ord}_p(z)$ the exponent of p in the prime decomposition of z.

PROPOSITION 2. *Let $m = \mathrm{ord}_p((l-1)/p)$. For $n = 0,\dots,m$ and for all $k \in \mathbf{Z}$,*

$$c_k(n,\omega^{-1}\psi) \equiv \frac{1-l}{2p^{n+1}} \pmod{\mathfrak{p}}.$$

PROOF. This is in fact proved in [1, pp. 20–21]. It can also be verified easily by using above computations: since now $s_n(\eta\alpha_k l) = s_n(\eta\alpha_k)$, one finds in analogy of (11) that

$$c_{k+t}(\omega^{-1}\psi) \equiv -\frac{1-l}{p^{n+1}} \sum_{\pm\eta} s_n(\eta\alpha_k)\eta^{-1} \equiv -\frac{1-l}{p^{n+1}} \cdot \frac{p-1}{2} \pmod{\mathfrak{p}}.$$

This proves the claim.

By applying Proposition 1 to $\xi_n(\theta)$ and $\xi_n(\theta\psi)$ (see (3)) we obtain the following result.

PROPOSITION 3. *If ψ is a character with $f_\psi = l$ and with order a power of p, and if $\theta_l\psi \neq 1$, then for $n \geqq 0$,*

$$\xi_n(\theta\psi) \equiv (1-\theta(l)\gamma_n(1+bp)^{-t})\xi_n(\theta) \pmod{\mathfrak{p}R_n}, \tag{12}$$

where $b = b_\theta = b_{\theta\psi}$ and t is defined by (10).

This proposition actually gives a congruence mod $\mathfrak{p}R_n$ between $\xi_n(\theta)$ and $\xi_n(\theta\psi)$ for *any* character $\psi \in C_1$ whose order is a power of p (provided that $\theta\psi \neq \omega^{-1}$). Firstly, if $f_\psi = l$ and $\theta_l\psi = 1$, then we apply Proposition 3 with $\theta\psi$ and ψ^{-1} in place of θ and ψ, respectively, to get

$$(1-(\theta\psi)(l)\gamma_n(1+bp)^{-t})\xi_n(\theta\psi) \equiv \xi_n(\theta) \pmod{\mathfrak{p}R_n}. \tag{13}$$

Secondly, if f_ψ is not a prime, we can split ψ as $\psi = \psi_1 \dots \psi_r$, where the conductors of ψ_i are distinct primes. Then, on using (12) or (13) successively for each ψ_i we arrive at the desired congruence.

4. Case $p = 2$.

Now let $p = 2$. Consider an even nonprincipal character $\psi \in C_1$, having order a power of 2. As above, let $\psi = \psi_1 \dots \psi_r$ be the canonical decomposition of ψ into components ψ_i modulo prime powers. The conductor of each ψ_i is, in fact, either 4 or an odd prime l_i. If ψ_i is even, then $f_{\psi_i} = l_i \equiv 1 \pmod 4$ and between $\xi_n(\theta)$ and $\xi_n(\theta\psi_i)$ there is a congruence analogous to (12) or (13), as we shall see. But if ψ_i is odd, then $\xi_n(\theta\psi_i)$ is not defined. Therefore, to derive a congruence between $\xi_n(\theta)$ and $\xi_n(\theta\psi)$ we must extend the previous considerations as follows.

Let C_1' denote the set obtained from C_1 on excluding the principal character and ω. For any even character θ in C_1' and for all $n \geqq 0$ and $k \in \mathbf{Z}$, let us *define* $c_k(\theta) = c_k(n,\theta)$ by the equation (cf. (7))

$$c_k(n,\theta) = -\frac{1}{d} \sum_{i=0}^{d-1} i\theta(s_n(\alpha_k) + 2^{n+2}i); \tag{14}$$

then define $\xi_n(\theta)$ by (3) and $f(X,\theta\omega)$ by (5). It follows that $\xi_n(\theta) \in R_n$ and that (14), (3) and (5) hold true whenever $\theta \in C_1'$ (but note that (1) and (4) do not hold, in general, in case of even θ). The following lemma states two simple properties of $c_k(\theta)$.

LEMMA. *Let $\theta \in C_1'$ and let b_θ be fixed. If $n \geqq 0$ and k is any rational integer, then*

(a) $c_k(n,\omega\theta) = c_k(n,\theta)$,

(b) $c_k(n,\theta) \equiv \sum_{i=0}^{d-1} i\theta(s_n(-\alpha_k) + 2^{n+2}i) \pmod p$.

PROOF. (a) If $f_\theta = d$, then $f_{\omega\theta} = 4d$ and the assertion follows from (14) since $\omega(s_n(\alpha_k) + 2^{n+2}i) = \omega(\alpha_k) = 1$. If $f_\theta = 4d$, then write $\theta = \omega\theta'$ and note that $c_k(\omega\theta') = c_k(\theta')$.

(b) As $\theta(-1)\theta(s_n(-\alpha_k) + 2^{n+2}i) = \theta(s_n(\alpha_k) - 2^{n+2}(i+1))$, we get, on putting $j = d - i - 1$,

$$\theta(-1) \sum_{i=0}^{d-1} i\theta(s_n(-\alpha_k) + 2^{n+2}i) = -\sum_{j=0}^{d-1} (j - d + 1)\theta(s_n(\alpha_k) + 2^{n+2}j).$$

By (14) and (9) (which holds for even θ as well), this gives the assertion.

Now let l be an odd prime and put $l^* = (-1)^{(l-1)/2}l$. Let ψ be a nonprincipal character mod l having a 2-power order. The following Propositions 4–6 correspond to Propositions 1–3 in the present case.

PROPOSITION 4. *Let $\theta \in C_1'$ and suppose that $\theta_1\psi \neq 1$. Choose a common value b for b_θ and $b_{\theta\psi}$. If $n \geqq 0$ and k is any rational integer, then*

$$c_k(n,\theta\psi) \equiv c_k(n,\theta) - \theta(l)c_{k-t}(n,\theta) \pmod{\mathfrak{p}},$$

where $t = t(n)$ is the unique rational integer such that

(15) $$0 \leqq t < 2^n, \; l^* \equiv (1+4b)^t \pmod{2^{n+2}}.$$

PROOF. The proof is similar to that of Proposition 1 and its details are omitted. In case $(l,f_\theta)=1$ observe that l divides $s_n(\alpha_k l^*)+2^{n+2}i$ $(i=0,\dots,ld-1)$ if and only if $i=i_0+jl$, where $0\leqq j\leqq d-1$ and i_0 is defined by

(16) $$s_n(\alpha_k l^*)+2^{n+2}i_0 = l s_n((-1)^{(l-1)/2}\alpha_k), \; 0\leqq i_0\leqq l-1.$$

When transforming the expression of $c_{k+t}(\theta\psi)$ one has to use both (14) and Lemma (b).

PROPOSITION 5. *Let $m = \operatorname{ord}_2((l^*-1)/4)$. For $n=0,\dots,m$ and for all $k\in \mathbf{Z}$,*

$$c_k(n,\omega\psi) = c_k(n,\psi) \equiv \frac{1-l^*}{2^{n+2}} \pmod{\mathfrak{p}}.$$

PROOF. For the values of n and k in question,

$$c_k(\psi) \equiv \sum_{i=0}^{l-1} i\psi(s_n(\alpha_k)+2^{n+2}i) \equiv \sum_{i=0}^{l-1} i+i_0 \pmod{\mathfrak{p}},$$

where (cf. (16))

$$2^{n+2}i_0 = l s_n((-1)^{(l-1)/2}\alpha_k) - s_n(\alpha_k).$$

This gives easily the claimed congruence for $c_k(\psi)$. Lemma (a) completes the proof.

PROPOSITION 6. *As above, let ψ be a character with $f_\psi = l$ and with order a power of 2. Let $\theta\in C_1'$ and suppose that $\theta_l\psi \neq 1$. For all $n\geqq 0$,*

$$\xi_n(\theta\omega) \equiv (1-\theta(l)\gamma_n(1+4b)^{-t})\xi_n(\theta) \pmod{\mathfrak{p}R_n},$$

where $b = b_\theta = b_{\theta\psi}$ and t is defined by (15).

As in case $p>2$, we actually get a congruence for more general characters $\psi=\psi_1\dots\psi_r$ (see the remark after Proposition 3). Note, in particular, that if ψ has a factor $\psi_i=\omega$, this factor can be ignored by Lemma (a).

5. Congruences for the power series.

The definition of $f(X,\theta\omega)$ together with Propositions 3 and 6 immediately yields the following result.

THEOREM 1. *Let θ be an odd character of the first kind and let ψ be a character with $f_\psi = l$, a prime, and with a p-power order; suppose that $\theta\omega$ and $\theta\psi\omega$ are nonprincipal. Then the power series $f(X,\theta\omega)$ and $f(X,\theta\psi\omega)$ belonging to a common parameter b satisfy*

$$f(X,\theta\psi\omega) \equiv h(X,\theta,l)f(X,\theta\omega) \quad (\mathrm{mod}\ \mathfrak{p}\Lambda), \text{ if } l | f_{\theta\psi},$$
$$h(X,\theta\psi,l)f(X,\theta\psi\omega) \equiv f(X,\theta\omega) \quad (\mathrm{mod}\ \mathfrak{p}\Lambda), \textit{ otherwise},$$

where $h(X,\chi,l) = 1 - \chi(l)(1+X)^{-\tau}$, $\tau = \tau(l)$ being the p-adic integer defined by $(1+bq)^\tau = \omega(l)l$.

We remind that here $f(X,\theta\psi\omega)$ does not represent a p-adic L-function if ψ is odd (a case which can occur only for $p = 2$).

If ψ is an arbitrary character in C_1 with a p-power order, then an iterated use of Theorem 1 gives a congruence of the form

$$h'(X)f(X,\theta\psi\omega) \equiv h''(X)f(X,\theta\omega) \ (\mathrm{mod}\ \mathfrak{p}\Lambda),$$

where $h'(X)$ and $h''(X)$ are products of certain power series $h(X,\chi,l)$. More precisely, if ψ has the canonical decomposition $\psi = \psi_1 \ldots \psi_r$, then each $\psi_i \neq \omega$ gives rise to a factor $h(X,\chi,l)$ in $h'(X)$ or $h''(X)$ (l a prime factor of f_ψ).

In case $\theta\omega = 1$, Propositions 2 and 5 yield the following supplement to Theorem 1.

THEOREM 2. *If ψ is the character of Theorem 1, then*

$$f(X,\psi) \equiv cX^{p^m - 1} \ (\mathrm{mod}\ \mathfrak{p}\Lambda + X^{p^m}\Lambda),$$

where $m = \mathrm{ord}_p((\omega(l)l - 1)/q)$ and $c = (1-\omega(l)l)/2p^{m+1}$.

Again, if $\psi = \psi_1 \ldots \psi_r$ with $r > 1$, then we may combine Theorems 1 and 2 to get a congruence of the form

$$f(X,\psi) \equiv h_0(X)g(X) \ (\mathrm{mod}\ \mathfrak{p}\Lambda),$$

where $h_0(X)$ is the product of certain $h(X,\chi,l)$ and $g(X)$ satisfies the congruence of Theorem 2.

6. Application to λ-invariants.

Denote by λ_θ the λ-invariant of the power series $f(X,\theta\omega)$, and put $\lambda_\theta = -1$ for $\theta = \omega^{-1}$. As a corollary of the previous results, we obtain a relation between λ_θ and $\lambda_{\theta\psi}$ whenever $\psi \in C_1$ is an even character of p-power order. This relation is due to Gras [2]. We will formulate it in a slightly different form which gives some direct information about the structure of the so-called Riemann–Hurwitz genus formula for the λ^--invariants of abelian fields.

Let W denote the set of all roots of 1 with a p-power order. Note that the λ-invariant of $h(X,\chi,l)$ equals p^m or 0 according to whether $\chi(l) \in W$ or not (m is defined in Theorem 2). Hence we may formulate our corollary as follows.

COROLLARY. *Let θ and ψ be characters of the first kind. Suppose that θ is odd and that ψ is even and of p-power order. Let $l_1, \dots, l_r$ be the odd prime factors of f_ψ; denote by θ_i and ψ_i the l_i-components of θ and ψ, respectively ($i = 1, \dots, r$). Then*

$$\lambda_{\theta\psi} = \lambda_\theta + \sum_{i=1}^{r} p^{m_i} \varepsilon_\theta(\psi_i),$$

where, for all i, $m_i = \operatorname{ord}_p((\omega(l_i)l_i - 1)/q)$ *and*

$$\begin{aligned} \varepsilon_\theta(\psi_i) &= +1, \quad \text{if } \theta(l_i) \in W, \\ &= -1, \quad \text{if } \theta_i = \psi_i^{-1} \text{ and } (\theta\psi_i)(l_i) \in W, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

REMARKS. (i) By the Ferrero–Washington theorem, the μ-invariant μ_θ of $f(X,\theta\omega)$ is zero. If we do not assume this theorem, then Theorem 1 tells us that $\mu_\theta = 0$ if and only if $\mu_{\theta\psi} = 0$, where θ and ψ satisfy the conditions of the above corollary. A different proof for this result has been given earlier by Iwasawa (see [5, Theorem 3]).

(ii) Ribet [10] has proved the corollary in the special case $p > 2$, $\theta = \omega^{-1}$.

Now let F be an imaginary abelian field. Denote by F_∞ the cyclotomic $\mathbf{Z}_p$-extension of F. When considering λ_F^-, the λ^--invariant of F_∞/F, we may suppose without loss of generality that the conductor of F is not divisible by qp. Then the character group X of F consists of characters of the first kind, and

$$\lambda_F^- - \delta_F = \sum_{\theta \in X^-} \lambda_\theta,$$

where X^- denotes the set of odd characters in X and $\delta_F = 1$ of 0 according to whether or not F contains a primitive qth root of 1.

Let E be an extension of F with $[E:F] = p$. Suppose that qp does not divide its conductor. We can write the character group of E in the form

$$Y = \{\theta\psi^k : \theta \in X,\ 0 \leqq k \leqq p-1\},$$

where ψ is a fixed even character in Y having a p-power order and the property that, in the notation of the corollary, its l_i-components ψ_i do not belong to X $(i = 1, \ldots, r)$. Then the corollary implies the following relation between λ_F^- and λ_E^-:

$$\lambda_E^- - \delta_E = \sum_{\theta \in X^-} \sum_{k=0}^{p-1} \lambda_{\theta\psi^k}$$

$$= p(\lambda_F^- - \delta_F) + \sum_{i=1}^{r} p^{m_i} \sum_{\theta \in X^-} \sum_{k=1}^{p-1} \varepsilon_\theta(\psi_i^k).$$

To deduce the "genus formula" from this, one has to show that for $i = 1, \ldots, r$,

$$p^{m_i} \sum_{\theta \in X^-} \sum_{k=1}^{p-1} \varepsilon_\theta(\psi_i^k) = g_i(e_i - 1) - g_i^+(e_i^+ - 1),$$

where e_i denotes the ramification index of l_i in E_∞/F_∞, g_i is the number of prime factors of l_i in E_∞, and e_i^+, g_i^+ are the corresponding symbols for the maximal real subfields of E_∞ and F_∞. This is a straightforward application of the known relations between the decomposition laws and character groups of abelian fields.

The formula thus obtained can be easily generalized to any p-extension E/F of imaginary abelian fields (see, e.g., [7]).

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