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MEANS AND CONVEX COMBINATIONS OF UNITARY OPERATORS

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Dedicated to Paul R. Halmos, a guiding spirit in operator theory

Abstract.

We prove that each element of the open unit ball of a C*-algebra $\mathfrak{A}$ is the mean of a finite number of unitary elements of $\mathfrak{A}$ – the closer to the surface, the more unitaries are needed in general. Each element of $\mathfrak{A}$ is a positive multiple of a sum of three unitary elements. If an element of $\mathfrak{A}$ is a convex combination of n unitary elements of $\mathfrak{A}$, it can be expressed as any convex combination "closer to the mean" than the original convex combination (of some other n unitary elements); in particular, it is a mean of n unitary elements.

1. Introduction.

We study the ways in which an element of a C*-algebra $\mathfrak{A}$ can be decomposed as a convex combination of elements of the unitary group $\mathscr{U}(\mathfrak{A})$ of $\mathfrak{A}$. We prove (Theorem 1) that each element S of $(\mathfrak{A})_1^0 (= \{A \in \mathfrak{A} : \|A\| < 1\})$ is a mean of a finite number n of unitary elements of $\mathfrak{A}$. Our estimate for n depends on the distance from A to the surface of the unit ball; the nearer A is to the surface, the larger n must be, in general. By considering the case where $\mathfrak{A}$ contains a non-unitary isometry (an element V such that $V^*V = I$ and $VV^* < I$, where I is the unit element of $\mathfrak{A}$), we show that the estimate for n in terms of the distance of A from the surface of the unit ball is "best possible" (cf. Proposition 3 and Remark 4). See [12] for related results.

In the last section, we use the convex decomposition in terms of unitaries to prove, again, one of the Gelfand-Neumark conjectures (cf. [4; Theorem 11]). In the two sections preceding the last, we study "asymmetric" convex combinations of unitary elements of $\mathfrak{A}$. If $a_1U_1 + \ldots + a_nU_n$ is such a combination, we show that the element it represents can be expressed as a convex combination of (other) unitary elements of $\mathfrak{A}$ with any set of coefficients "closer to the mean" than $a_1, \ldots, a_n$ (cf. Theorem 14). This result follows from an "asymmetric" convex decomposition (cf. Lemma 6)

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of a self-adjoint element of $(\mathfrak{A})_1$, the closed unit ball of $\mathfrak{A}$, extending the "mean" of "symmetric" (that is, invariant under permutation of the coefficients) decomposition introduced by Murray and von Neumann in [9; p. 239] and a (combinatorial) geometric analysis of the convex polyhedron in real n-space whose vertices are the points whose coordinates are a permutation of $a_1, \ldots, a_n$. Limits to the variation of coefficients of a convex combination are established in Proposition 20 if the element represented cannot be expressed as a combination of fewer unitary elements.

These results, indicating the ways in which an element of a C*-algebra can be expressed as a convex combination of unitary elements, constitute a refinement of the study initiated by Phelps [11] and followed by Russo-Dye [13]. In [13], Russo and Dye prove that the norm closure of $\operatorname{co}\mathscr{U}(\mathfrak{A})$, the convex hull of $\mathscr{U}(\mathfrak{A})$, is $(\mathfrak{A})_1$ for an arbitrary C*-algebra. (Phelps proves this for a commutative C*-algebra.) This result, known as the Russo-Dye theorem, has proved very useful. It provides one means of reducing the study of a non-normal element of a C*-algebra to that of normal (unitary) elements – and this device is reasonably sensitive to norm estimates. (The argument of the last section is an illustration of this process.)

Russo and Dye remark that "little is known about the preclosed convex hull," and go on to show that each element T in $\mathfrak{A}$ whose norm is less than $\frac{1}{2}$ is in $\operatorname{co}\mathscr{U}(\mathfrak{A})$. In [5], Harris proves a Russo-Dye theorem for more general Banach algebras with an involution. From among his results (in particular, his Corollary 1) one can read that each element of $(\mathfrak{A})_1^0$ is in the convex hull of $\mathscr{U}(\mathfrak{A})$ (indeed, in the convex hull of the "exponential" unitaries of $\mathfrak{A}$). (The second-named author takes this occasion to correct the inaccurate reference on p. 5 of [10], where the Harris proof of the Russo-Dye theorem is ascribed to Palmer.) A. G. Robertson proves, again, that $(\mathfrak{A})_1^0 \subsetneq \operatorname{co}\mathscr{U}(\mathfrak{A})$ in [12; Proposition 1]. He discusses the number of unitaries needed for such a decomposition and shows [12; Proposition 2] that each element of $(\mathfrak{A})_1^0$ is a mean of four unitary elements when the group of invertible (regular) elements of $\mathfrak{A}$ is norm dense in $\mathfrak{A}$. In Proposition 18 (combined with Corollary 15), we show that each element of $(\mathfrak{A})_1^0$ is a mean of three unitaries in $\mathfrak{A}$ if the regular elements are norm dense in $\mathfrak{A}$. The norm closure of the regular elements in a von Neumann algebra is determined in [1], and it is noted there that the finite von Neumann algebras are precisely the ones whose regular elements are norm dense. In the case of a finite von Neumann algebra $\mathscr{R}$, each element in $(\mathscr{R})_1^0$ is a mean of two unitaries (cf. Section 3).

The Russo-Dye theorem is an immediate consequence of the fact that

$(\mathfrak{A})_1^0 \subseteq \text{co}\,\mathscr{U}(\mathfrak{A})$; but both the Robertson and Harris proofs of this fact pass through the Russo-Dye theorem itself. In [2], Gardner observes that the argument of the Russo-Dye comment showing that the open ball of radius $\frac{1}{2}$ is contained in $\text{co}\,\mathscr{U}(\mathfrak{A})$ can be strengthened mildly and given a special ending to yield a very simple proof of the Russo-Dye theorem. This strengthening and another concluding argument is the basis for our proof of Theorem 1.

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2. Meaning unitaries.

We prove that each element of $(\mathfrak{A})_1^0$ is a mean of elements of $\mathscr{U}(\mathfrak{A})$ in the theorem that follows.

THEOREM 1. *If the element S of a* C*-*algebra $\mathfrak{A}$ has the property that $\|S\| < 1 - 2n^{-1}$ for some integer n greater than* 2, *then there are n unitary elements $U_1, \ldots, U_n$ in $\mathfrak{A}$ such that $S = n^{-1}(U_1 + U_2 + \ldots + U_n)$.*

PROOF. With T in $(\mathfrak{A})_1^0$ and V in $\mathscr{U}(\mathfrak{A})$,

$$(V+T)/2 = V(I + V^*T)/2 \quad \text{and} \quad \|V^*T\| = \|T\| < 1.$$

Thus $I + V^*T$ is invertible and $(V+T)/2$ is an invertible element of $(\mathfrak{A})_1^0$. Hence $(V+T)/2$ has a polar decomposition UH with U in $\mathscr{U}(\mathfrak{A})$ and H a (positive) self-adjoint element in $(\mathfrak{A})_1$. Now $H = (W_1 + W_2)/2$, where

$$W_1 = H + i(I - H^2)^{1/2}, \ W_2 = H - i(I - H^2)^{1/2}, \text{ and } W_1, W_2 \in \mathscr{U}(\mathfrak{A}).$$

It follows that $V + T = UW_1 + UW_2$. Thus, for each positive integer n, there are elements $U_1, \ldots, U_{n-1}$ and $V_1, \ldots, V_{n-1}$ $(= U_n)$ in $\mathscr{U}(\mathfrak{A})$ such that

$$\begin{aligned} V + (n-1)T &= U_1 + V_1 + (n-2)T \\ &= U_1 + U_2 + V_2 + (n-3)T = \ldots = U_1 + U_2 + \ldots + U_n. \end{aligned} \tag{1}$$

Under the assumption that $\|S\| < 1 - 2n^{-1}$ (and $n \geqq 3$), we have that

$$\|(n-1)^{-1}(nS - I)\| \leqq (n-1)^{-1}(n\|S\| + 1) < 1.$$

Thus, we may use $(n-1)^{-1}(nS - I)$ in place of T and I in place of V in (1). With these choices, we have $nS = \Sigma_{k=1}^n U_k$, with U_k in $\mathscr{U}(\mathfrak{A})$.

The Russo-Dye theorem is an immediate corollary, for each element A in $(\mathfrak{A})_1$ is a norm limit of $(1 - 3n^{-1})A$, and $(1 - 3n^{-1})A$ is a mean of n unitary elements of $\mathfrak{A}$. At the same time, we can apply Theorem 1 to produce a special representation of an element of a C*-algebra.

COROLLARY 2. *Each element of a C*-algebra is some positive multiple of a sum of three unitary elements.*

PROOF. If A is in the C*-algebra $\mathfrak{A}$ and $\|A\| < 1/3\ (=1-2/3)$, then A is a mean of three unitary elements of $\mathfrak{A}$, from Theorem 1. In any event, for each positive ε, $(3\|A\|+3\varepsilon)^{-1}A$ has norm less than 1/3 so that

$$A = (\|A\|+\varepsilon)(U_1+U_2+U_3)$$

for some U_1, U_2, U_3 in $\mathscr{U}(\mathfrak{A})$.

3. Minimal decompositions.

The fact that each element of $(\mathfrak{A})_1^0$ is the mean of a certain number of unitaries is an immediate consequence of Theorem 1 – with S in $(\mathfrak{A})_1^0$, choose n so large that $\|S\| < 1-2n^{-1}$ and then Theorem 1 assures us that S is the mean of n unitaries in $\mathfrak{A}$. In this section, we study the problem of "minimal" decompositions of elements as means and convex combinations of unitary elements. For this purpose, we define two numbers associated with an element T of a C*-algebra $\mathfrak{A}$.

$$u_m(T) = \min\{n: T = n^{-1}\sum_{j=1}^{n} U_j,\ U_j \in \mathscr{U}(\mathfrak{A})\}$$

$$u_c(T) = \min\{n: T = \sum_{j=1}^{n} a_jU_j,\ U_j \in \mathscr{U}(\mathfrak{A}), a_j > 0,\ \sum_{j=1}^{n} a_j = 1\}.$$

If T has no decomposition as a convex combination of elements of $\mathscr{U}(\mathfrak{A})$, we define $u_c(T)$ to be ∞. (We shall see, in Corollary 15, that $u_m(T) = u_c(T)$.) The context will make clear which C*-algebra is involved in the decomposition so that we have omitted the algebra from the notation "$u_m(T)$" and "$u_c(T)$". Of course, $u_c(T) = u_m(T) = \infty$ when $\|T\| > 1$, and Theorem 1 tells us that $u_m(T) \leqq n$ when $\|T\| < 1-2n^{-1}$. If $\mathscr{R}$ is a finite von Neumann algebra, then as we noted in Section 1, $u_m(T) \leqq 2$ for every T in $(\mathscr{R})_1$. To see this, observe that each T in $(\mathscr{R})_1$ has the form UH, where $U \in \mathscr{U}(\mathscr{R})$ and H is a self-adjoint operator in $(\mathscr{R})_1$. As in the proof of Theorem 1, $H = (W_1+W_2)/2$ $(W_1, W_2 \in \mathscr{U}(\mathscr{R}))$, and $T = (UW_1+UW_2)/2$. If V is a non-unitary isometry in a C*-algebra $\mathfrak{A}$, then $u_c(V) = \infty$ from [7; Theorem 1]. Also, $u_c(T) = \infty$ if T is the function $T(z) = z$ on the closed unit disk $\boldsymbol{D}$ in $\mathbf{C}$, where T is regarded as an element of the C*-algebra $C(\boldsymbol{D})$. For the proof of this, note that if $T = a_1U_1 + \ldots + a_nU_n$ for some unitary elements (functions) in $C(\boldsymbol{D})$, then $z = a_1U_1(z) + \ldots + a_nU_n(z)$ for each z in $\boldsymbol{D}$. Since the boundary of $\boldsymbol{D}$ consists of extreme points for $\boldsymbol{D}$ and $|U_k(z)| = 1$ for each k and each z in $\boldsymbol{D}$, $U_k(z) = z$ for each k and each z such that $|z| = 1$. Hence each U_k provides a retraction of $\boldsymbol{D}$ onto its boundary-contradicting the fact that there is no such retraction. Thus $C(\boldsymbol{D})$ supplies us with an example of a C*-algebra in which some element of the unit ball has no convex decomposition in terms of unitary elements while the von Neumann algebra generated by $C(\boldsymbol{D})$ (in each representation) is finite (and so each element of the unit ball of this von Neumann algebra is the mean of two unitary elements).

In the next proposition, we make use of the fact that the spectrum of a non-unitary isometry V is $\boldsymbol{D}$, the closed unit disk. This may be proved by noting that $e_0 + \lambda e_1 + \lambda^2 e_2 + \dots$ is an eigenvector for V^* corresponding to the eigenvalue λ, where $|\lambda| < 1$, e_0 is a unit vector orthogonal to the range of V, and $e_k = V^k e_0$. In this case, $V^* e_0 = 0$ and $V^* e_k = e_{k-1}$ for k in $\{1,2,\dots\}$. Thus

$$V^*\left(\sum_{k=0}^{\infty} \lambda^k e_k\right) = \sum_{k=0}^{\infty} \lambda^{k+1} e_k = \lambda\left(\sum_{k=0}^{\infty} \lambda^k e_k\right).$$

PROPOSITION 3. *If V is a non-unitary isometry and $U_1,\dots,U_n$ are unitary elements in some* C*-*algebra, then* $\| V - n^{-1}(U_1 + \dots + U_n)\| \geqq 2n^{-1}$.

PROOF. Assume the contrary. Then

$$\begin{aligned} \| U_1^* V - n^{-1} I \| &= \| V - n^{-1} U_1 \| < 2n^{-1} + \| n^{-1}(U_2 + \dots + U_n)\| \\ &\leqq 2n^{-1} + (n-1)n^{-1} = 1 + n^{-1}. \end{aligned}$$

But $U_1^* V$ is a non-unitary isometry, and $-1 - n^{-1} \in \operatorname{sp}(U_1^* V - n^{-1} I)$, so that $1 + n^{-1} \leqq \| U_1^* V - n^{-1} I \|$; a contradiction. From this, we have that $\| V - n^{-1}(U_1 + \dots + U_n)\| \geqq 2n^{-1}$.

REMARK 4. Suppose $\mathfrak{A}$ is a C*-algebra containing a non-unitary isometry V (as is the case, for example, when $\mathfrak{A}$ is an infinite von Neumann algebra), and let S_n be $a_n V$, where

$$1 - 2(n-1)^{-1} < a_n < 1 - 2n^{-1}. \tag{2}$$

Then $\| S_n \| = a_n < 1 - 2n^{-1}$, and from Theorem 1, S_n is a mean of n unitary elements. Thus $u_m(S_n) \leqq n$. Suppose $S_n = r^{-1}(U_1 + \dots + U_r)$ with $U_1,\dots,U_r$ in $\mathscr{U}(\mathfrak{A})$. Then

$$1 - a_n = \| V - S_n \| = \| V - r^{-1}(U_1 + \dots + U_r)\| \geqq 2r^{-1},$$

from Proposition 3. Hence $r \geqq 2(1 - a_n)^{-1} > n - 1$ from (2). Since r is an integer $r \geqq n$. It follows that $u_m(S_n) \geqq n$, whence $u_m(S_n) = n$. Thus $u_m(S_n) \to \infty$ as $n \to \infty$. At the same time, we see that the representation of S_n as a mean of n unitary elements, constructed in the proof of Theorem 1, is "best possible."

In the next proposition, we give an "approximate" characterization of the set of elements S of a C*-algebra for which $u_m(S) \leqq n$.

PROPOSITION 5. *Let S be an element of a C*-algebra $\mathfrak{A}$ and d_n be the distance from nS to $\mathscr{U}(\mathfrak{A})$. If $d_n < n-1$, then $u_m(S) \leqq n$. If $u_m(S) \leqq n$, then $d_n \leqq n-1$.*

PROOF. Suppose $d_n < n-1$. Then $\|nS - U\| < n-1$ for some U in $\mathscr{U}(\mathfrak{A})$. Hence $(n-1)^{-1}(nU^*S - I) \in (\mathfrak{A})_1^0$. If we replace T in (1) by $(n-1)^{-1}(nU^*S - I)$ and V by I (see the proof of Theorem 1), we see that U^*S is the mean of n elements of $\mathscr{U}(\mathfrak{A})$. Thus $u_m(S) \leqq n$.

Suppose $u_m(S) \leqq n$. Then $S = r^{-1}(U_1 + \ldots + U_r)$ for some r in $\{1, \ldots, n\}$. Thus $\|S\| \leqq 1$. Moreover,

$$\|rS - U_1\| = \|U_2 + \ldots + U_r\| \leqq r-1.$$

Hence

$$\|nS - U_1\| \leqq \|rS - U_1\| + \|(n-r)S\| \leqq n-1,$$

and $d_n \leqq n-1$.

4. Toward the mean.

We show (see Theorem 14) that each convex combination of n unitary elements of a C*-algebra can be expressed as every other convex combination of n unitary elements of the algebra for which the coefficients lie "nearer to the mean." In particular (see Corollary 15), each convex combination of n unitary elements of a C*-algebra is a mean of n unitary elements of the algebra. It follows that $u_m(T) = u_c(T)$ for each T in the algebra. One of the principal tools we use in proving these results is the following extension of the classical decomposition (due to Murray and von Neumann [9; p. 239]) of a self-adjoint operator in the unit ball of a C*-algebra as a mean of two unitary elements of the algebra.

LEMMA 6. *Let A be a self-adjoint element of a C*-algebra $\mathfrak{A}$ and let a be a real number in $[0, 1/2]$. Let $\mathscr{S}_a$ be $[-1,1] \setminus ((2a-1), (1-2a))$. Then* $\mathrm{sp}(A) \subsetneq \mathscr{S}_a$ *if and only if $A = aU_1 + (1-a)U_2$ for some unitary elements U_1 and U_2 of $\mathfrak{A}$.*

PROOF. If $A = aU_1 + (1-a)U_2$ with U_1 and U_2 in $\mathscr{U}(\mathfrak{A})$, then $\|A\| \leqq 1$. Suppose $\lambda \in \mathrm{sp}(A)$. Choose a pure state ρ of $\mathfrak{A}$ that is definite on A and for which $\rho(A) = \lambda$. (See [8; Exercises 4.6.16 and 4.6.31].) Then

$$\begin{aligned}|\lambda| &\geqq |\rho(U_2^*)|\,|\rho(A)| = |\rho(U_2^*A)| = |\rho(aU_2^*U_1 + (1-a)I)| \\ &\geqq 1 - a - a|\rho(U_2^*U_1)| \geqq 1 - 2a.\end{aligned}$$

Hence $\mathrm{sp}(A) \subsetneq \mathscr{S}_a$.

Suppose, now, that $\mathrm{sp}(A) \subsetneq \mathscr{S}_a$. We shall construct two continuous functions f_1 and f_2 defined on $\mathscr{S}_a$, taking complex values of modulus 1, and having the property that $af_1(t) + (1-a)f_2(t) = t$ for each t in $\mathscr{S}_a$. Once we have the functions f_1 and f_2, we can define unitary elements U_1 and U_2 in $\mathfrak{A}$ as $f_1(A)$ and $f_2(A)$, respectively; and $aU_1 + (1-a)U_2 = A$.

Let C_+ be the set of complex numbers of modulus 1 with non-negative imaginary part. Let $\theta(t)$ be the (unique) element of C_+ with real part t, where $t \in [-1,1]$. Then $t \to |1 + a(1-a)^{-1}\theta(t)|$ is a continuous (increasing) one-to-one mapping (hence, homeomorphism) of $[-1,1]$ onto $[(1-2a)(1-a)^{-1}, (1-a)^{-1}]$. Let g be the inverse mapping to this homeomorphism. If $t \in \mathscr{S}_a$, then

$$(1-a)^{-1} \geqq |t(1-a)^{-1}| \geqq (1-2a)(1-a)^{-1}.$$

Thus $\theta(g(|t(1-a)^{-1}|))(=\xi(t))$ is defined for t in $\mathscr{S}_a$, and ξ is a continuous mapping of $\mathscr{S}_a$ into C_+. By definition of g and θ, $|1 + a(1-a)^{-1}\xi(t)| = |t(1-a)^{-1}|$.

When $a = 1/2$, we can choose $t + i(1-t^2)^{1/2}$ for $f_1(t)$ and $t - i(1-t^2)^{1/2}$ for $f_2(t)$. (In this case, $\mathscr{S}_a = [-1,1]$.) When $a \neq 1/2$, we have that $t(1-a)^{-1}(1 + a(1-a)^{-1}\xi(t))^{-1}$ is a complex number $f_2(t)$ of modulus 1 and f_2 is continuous on $\mathscr{S}_a$. Hence $f_2(t)\xi(t)$ is a complex number $f_1(t)$ of modulus 1 and f_1 is continuous on $\mathscr{S}_a$. Moreover, $af_1(t) + (1-a)f_2(t) = t$ for t in $\mathscr{S}_a$.

Remark 7. The functions f_1, f_2, θ, and g occurring in the proof of Lemma 6 can, of course, be expressed in terms of elementary functions. Doing that, we find that the unitaries U_1 and U_2 from Lemma 6 have the form

$$U_1 = B + i(1-a)D \quad\text{and}\quad U_2 = C - iaD,$$

where B, C, and D are the self-adjoint elements of $\mathfrak{A}$ given by

$$\begin{aligned} B &= \tfrac{1}{2}a^{-1}(A - (1-2a)A^{-1}), \\ C &= \tfrac{1}{2}(1-a)^{-1}(A + (1-2a)A^{-1}), \\ D &= (1-a)^{-1}(I - B^2)^{1/2} = a^{-1}(I - C^2)^{1/2}. \end{aligned}$$

Here $(1-2a)A^{-1}$ should be interpreted as 0 when $a = 1/2$, and when $a = 0$ (so that A is a symmetry), the formula for B should be interpreted as 0.

We retain the notation $\mathscr{S}_a$ for $[-1,1] \setminus (-(1-2a), (1-2a))$ and deduce the following characterization of $a\mathscr{U}(\mathfrak{A}) + (1-a)\mathscr{U}(\mathfrak{A})$ with the aid of Lemma 6.

Proposition 8. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathscr{U}(\mathfrak{A})$ be its unitary group. If $0 \leqq a < 1/2$, then*

$$\{UH : U \in \mathcal{U}(\mathfrak{A}), H = H^* \in \mathfrak{A}, \operatorname{sp} H \subsetneq \mathcal{S}_a\}$$
$$= a\mathcal{U}(\mathfrak{A}) + (1-a)\mathcal{U}(\mathfrak{A}).$$

PROOF. Suppose $T = UH$, where $U \in \mathcal{U}(\mathfrak{A})$ and H is a self-adjoint element of $\mathfrak{A}$ with spectrum in $\mathcal{S}_a$. Then $H = aU_1 + (1-a)U_2$ by Lemma 6. Hence

$$T = aUU_1 + (1-a)UU_2 \in a\mathcal{U}(\mathfrak{A}) + (1-a)\mathcal{U}(\mathfrak{A}).$$

Suppose $S = aV_1 + (1-a)V_2$ with V_1 and V_2 in $\mathcal{U}(\mathfrak{A})$. Then

$$V_2^* S = (1-a)(a(1-a)^{-1} V_2^* V_1 + I),$$

and $a(1-a)^{-1} < 1$ since $0 \leqq a < 1/2$. Thus $V_2^* S$ and S are invertible and $S = U(S^*S)^{1/2}$ with U in $\mathcal{U}(\mathfrak{A})$. Finally,

$$(S^*S)^{1/2} = U^*S = aU^*V_1 + (1-a)U^*V_2,$$

whence $\operatorname{sp}(S^*S)^{1/2} \subsetneq \mathcal{S}_a$ by Lemma 6.

REMARK 9. The first half of the proof of Proposition 8 is valid also in the limiting case $a = \frac{1}{2}$, so that

$$\{UH : U \in \mathcal{U}(\mathfrak{A}), H = H^* \in \mathfrak{A}, \operatorname{sp}(H) \subset [-1,1]\} \subset \tfrac{1}{2}(\mathcal{U}(\mathfrak{A}) + \mathcal{U}(\mathfrak{A})).$$

The two sets above are equal, when $\mathfrak{A}$ is a von Neumann algebra. Indeed, if $S = \frac{1}{2}(U_1 + U_2)$, then $U_1^* S (= \frac{1}{2}(I + U_1^* U_2))$ is a normal element of $\mathfrak{A}$ and generates an abelian von Neumann subalgebra $\mathcal{C}$ of $\mathfrak{A}$. The polar decomposition VH of U_1^*S has its components V and H in $\mathcal{C}$. In particular, $V^*V = VV^*$. Hence $I - V^*V + V$ is a unitary operator W in $\mathfrak{A}$ and $U_1^*S = WH$. Thus $S = U_1WH$ with U_1W in $\mathcal{U}(\mathfrak{A})$ and $0 \leqq H \leqq I$.

COROLLARY 10. *If* $0 \leqq a \leqq b \leqq 1/2$, $\mathfrak{A}$ *is a* C*-*algebra and* $\mathcal{U}(\mathfrak{A})$ *is its unitary group, then*

$$a\mathcal{U}(\mathfrak{A}) + (1-a)\mathcal{U}(\mathfrak{A}) \subsetneq b\mathcal{U}(\mathfrak{A}) + (1-b)\mathcal{U}(\mathfrak{A}).$$

PROOF. If $a < b < 1/2$, then $\mathcal{S}_a \subset \mathcal{S}_b$ and the inclusion follows from Proposition 8. If $a < b = 1/2$ the inclusion follows from Proposition 8 in conjunction with Remark 9.

COROLLARY 11. *Let* T *be an element of the unit ball of a* C*-*algebra* $\mathfrak{A}$, *whose distance to* $\mathcal{U}(\mathfrak{A})$ *is* $2a$ *or less, where* $a < 1/2$. *In this case, we have that* $T \in a\mathcal{U}(\mathfrak{A}) + (1-a)\mathcal{U}(\mathfrak{A})$.

PROOF. If $a < b < 1/2$, there is a U in $\mathcal{U}(\mathfrak{A})$ such that $\|T - U\| < 2b < 1$. Thus $\|I - U^*T\| < 1$ so that U^*T and T are invertible. Hence $T = VH$, where $V \in \mathscr{U}(\mathfrak{A})$ and $H = (T^*T)^{1/2} \in \mathfrak{A}$. If λ is in sp H, there is a pure state ρ of $\mathfrak{A}$ definite on H such that $\rho(H) = \lambda$ and $\rho(AH) = \rho(A)\rho(H)$ for each A in $\mathfrak{A}$. (See [8; Exercises 4.6.16 and 4.6.31] or [10; 4.3.10].) Thus

$$|1-\rho(U^*V)\lambda| = |\rho(I-U^*VH)| \leqq \|I-U^*VH\| = \|U-T\| < 2b$$

and

$$1-2b \leqq \lambda|\rho(U^*V)| \leqq \lambda.$$

Thus sp $H \subseteq [1-2b,1]$. Since this inclusion holds for each b greater than a, sp $H \subseteq [1-2a,1]$. From Lemma 6, H and, hence, T are elements of $a\mathscr{U}(\mathfrak{A}) + (1-a)\mathscr{U}(\mathfrak{A})$.

COROLLARY 12. *If a_1 and a_2 are non-negative real numbers and $\mathfrak{A}$ is a C*-algebra, then*

$$a_1\mathscr{U}(\mathfrak{A}) + a_2\mathscr{U}(\mathfrak{A}) \subseteq b_1\mathscr{U}(\mathfrak{A}) + b_2\mathscr{U}(\mathfrak{A})$$

provided (b_1,b_2) lies on the line segment joining (a_1,a_2) to (a_2,a_1) in R^2.

PROOF. If $a_1 = a_2$, then $b_1 = b_2 = a_1 = a_2$ and there is nothing to prove. Since the hypotheses are symmetric in a_1 and a_2, we may assume that $a_1 < a_2$. By assumption, b_1 and b_2 lie in $[a_1,a_2]$. Let s be a_1+a_2 $(\geqq a_2 > 0)$. It will suffice to show that

$$s^{-1}a_1\mathscr{U}(\mathfrak{A}) + s^{-1}a_2\mathscr{U}(\mathfrak{A}) \subseteq s^{-1}b_1\mathscr{U}(\mathfrak{A}) + s^{-1}b_2\mathscr{U}(\mathfrak{A}).$$

Of course, $(s^{-1}b_1,s^{-1}b_2)$ lies on the line segment joining $(s^{-1}a_1,s^{-1}a_2)$ to $(s^{-1}a_2,s^{-1}a_1)$. Thus, we may assume that $0 \leqq a_1 = a < 1/2$, $a_2 = 1-a$, and b_1,b_2 lie in $[a,1-a]$. With these assumptions, $b_1+b_2 = 1$, and we may also assume that $b = b_1 \leqq 1/2 \leqq b_2$. In this case, we have that $0 \leqq a \leqq b \leqq 1/2$, and from Corollary 10,

$$a\mathscr{U}(\mathfrak{A}) + (1-a)\mathscr{U}(\mathfrak{A}) \subseteq b\mathscr{U}(\mathfrak{A}) + (1-b)\mathscr{U}(\mathfrak{A}).$$

If an element S in a C*-algebra $\mathfrak{A}$ is a convex combination with coefficients $a_1,\ldots,a_n$ of unitary elements of $\mathfrak{A}$, then the same is, of course, true for these coefficients in each rearrangement, $a_{\pi(1)},\ldots,a_{\pi(n)}$, where π is in the group Σ_n of permutations of $\{1,\ldots,n\}$. In Theorem 14, we shall show that each point $(b_1,\ldots,b_n)$ in the convex hull of

$$\{(a_{\pi(1)},\ldots,a_{\pi(n)}): \pi \in \Sigma_n\}$$

provides a set of coefficients for a decomposition of S as a convex combination of unitary elements of $\mathfrak{A}$. Our proof of Theorem 14 is based on information about the geometry of that convex hull. The information is contained in the lemma that follows.

LEMMA 13. *Let* $(a_1,\dots,a_n)$ *be a point of* R^n *such that* $a_1 \geqq a_2 \geqq \dots \geqq a_n \geqq 0$ *and* $a_1+\dots+a_n=1$. *Let* $\mathscr{K}$ *be the convex hull of* $\{(a_{\pi(1)},\dots,a_{\pi(n)}):\pi\in\Sigma_n\}$ $(=\mathscr{S})$. *Then a point of* R^n *lies in* $\mathscr{K}$ *if and only if it has the form* $(b_{\pi(1)},\dots,b_{\pi(n)})$ *for some* π *in* Σ_n *and* $(b_1,\dots,b_n)$ *satisfies the conditions*

(i) $b_1 \geqq b_2 \geqq \dots \geqq b_n \geqq 0$, $b_1+\dots+b_n=1$;

(ii) $b_1 \leqq a_1$, $b_1+b_2 \leqq a_1+a_2,\dots,$ $b_1+\dots+b_{n-1} \leqq a_1+\dots+a_{n-1}$.

When $(b_1,\dots,b_n)$ *satisfies* (i) *and* (ii), *there are points* $(a_1^{(1)},\dots,a_n^{(1)}),\dots,(a_1^{(n)},\dots,a_n^{(n)})$ *in* $\mathscr{K}$ *such that*

(iii) $(a_1^{(1)},\dots,a_n^{(1)})=(a_1,\dots,a_n)$, $(a_1^{(n)},\dots,a_n^{(n)})=(b_1,\dots,b_n)$;

(iv) $(a_1^{(k+1)},\dots,a_n^{(k+1)})=t(a_1^{(k)},\dots,a_n^{(k)})+(1-t)(a_{\tau(1)}^{(k)},\dots,a_{\tau(n)}^{(k)})$

for all k *in* $\{1,\dots,n-1\}$, *some* t *in* $[0,1]$, *and some transposition* τ *in* Σ_n *(both* t *and* τ *depend on* k*).*

PROOF. Let φ_k be the linear functional on R^n such that

$$\varphi_k(r_1,\dots,r_n)=r_1+\dots+r_k.$$

Then φ_k attains its maximum on $\mathscr{K}$ at some extreme point of $\mathscr{K}$. Now the extreme points of $\mathscr{K}$ are each in $\mathscr{S}$ and the maximum value of φ_k on $\mathscr{S}$ is $a_1+\dots+a_k$ (since $a_1 \geqq a_2 \geqq \dots \geqq a_n$). Thus, each point of $\mathscr{K}$ satisfies (ii), and of course, some permutation of the coordinates arranges those coordinates in decreasing order. Finally, the coordinates of a point of $\mathscr{K}$ add to 1 and are non-negative since that is true of the coordinates of each point of $\mathscr{S}$ and $\mathscr{K}$ is the convex hull of $\mathscr{S}$.

Suppose, now, that $(b_1,\dots,b_n)$ satisfies (i) and (ii). When we establish (iii) and (iv), it will follow that $(b_1,\dots,b_n)$ lies in $\mathscr{K}$ since $(a_1^{(1)},\dots,a_n^{(1)})$ does, each point arising from a point of $\mathscr{K}$ by permuting the coordinates of that point lies in $\mathscr{K}$, and $\mathscr{K}$ is convex. We begin the construction of the points $(a_1^{(k)},\dots,a_n^{(k)})$ by taking $(a_1^{(1)},\dots,a_n^{(1)})$ to be $(a_1,\dots,a_n)$ (as we must). Suppose we have constructed $(a_1^{(1)},\dots,a_n^{(1)}),\dots,(a_1^{(j)},\dots,a_n^{(j)})$ such that (iv) is satisfied for k in $\{1,\dots,j-1\}$, such that

(v) $b_1 \leqq a_1^{(k)}$, $b_1+b_2 \leqq a_1^{(k)}+a_2^{(k)},\dots,$ $b_1+\dots+b_{n-1} \leqq a_1^{(k)}+\dots+a_{n-1}^{(k)}$

for each k in $\{1,\dots,j\}$, and such that

(vi) $b_1=a_1^{(k)},\dots,b_{k-1}=a_{k-1}^{(k)}$

for each k in $\{2,\ldots,j\}$. Note that (v) becomes our assumption (ii) with 1 in place of k (by choice of $(a_1^{(1)},\ldots,a_n^{(1)})$). From (v) and (vi)

$$b_1+\ldots+b_j \leqq a_1^{(j)}+\ldots+a_{j-1}^{(j)}+a_j^{(j)} = b_1+\ldots+b_{j-1}+a_j^{(j)}.$$

Hence $b_j \leqq a_j^{(j)}$. Now

$$1-b_n = b_1+\ldots+b_{n-1} \leqq a_1^{(j)}+\ldots+a_{n-1}^{(j)} = 1-a_n^{(j)},$$

so that $a_n^{(j)} \leqq b_n \leqq b_j$. Let m be the smallest number in $\{j+1,\ldots,n\}$ such that $a_m^{(j)} \leqq b_j$. Then

(3) $$b_{j+1} \leqq b_j < a_{j+1}^{(j)},\ldots,b_{m-1} \leqq b_j < a_{m-1}^{(j)}.$$

Since $a_m^{(j)} \leqq b_j \leqq a_j^{(j)}$, there is a t in $[0,1]$ such that $b_j = ta_j^{(j)}+(1-t)a_m^{(j)}$. Let τ be the transposition that interchanges j and m, and let $(a_1^{(j+1)},\ldots,a_n^{(j+1)})$ be

$$t(a_1^{(j)},\ldots,a_n^{(j)})+(1-t)(a_{\tau(1)}^{(j)},\ldots,a_{\tau(n)}^{(j)}).$$

Then

$$(a_1^{(j+1)},\ldots,a_n^{(j+1)}) = (b_1,\ldots,b_j,a_{j+1}^{(j)},\ldots,a_{m-1}^{(j)},a_j^{(j)}+a_m^{(j)}-b_j,a_{m+1}^{(j)},\ldots,a_n^{(j)}),$$

and $(a_1^{(j+1)},\ldots,a_n^{(j+1)})$ satisfies (vi). By definition, $(a_1^{(j+1)},\ldots,a_n^{(j+1)})$ satisfies (iv). We show that $(a_1^{(j+1)},\ldots,a_n^{(j+1)})$ satisfies (v). If $1 \leqq p \leqq j$, then

$$b_1+\ldots+b_p = a_1^{(j+1)}+\ldots+a_p^{(j+1)}.$$

If $j+1 \leqq p \leqq m-1$, then from (3),

$$\begin{aligned} b_1+\ldots+b_j+b_{j+1}+\ldots+b_p &\leqq b_1+\ldots+b_j+a_{j+1}^{(j)}+\ldots+a_p^{(j)} \\ &= a_1^{(j+1)}+\ldots+a_p^{(j+1)}. \end{aligned}$$

Finally, if $m \leqq p \leqq n-1$, then from (v) (for k equal to j),

$$b_1+\ldots+b_p \leqq a_1^{(j)}+\ldots+a_p^{(j)} = a_1^{(j+1)}+\ldots+a_p^{(j+1)}.$$

This completes the construction of $(a_1^{(1)},\ldots,a_n^{(1)}),\ldots,(a_1^{(n)},\ldots,a_n^{(n)})$.

In the preceding proof, we remark that each extreme point of $\mathscr{K}$ lies in $\mathscr{S}$. Noting that the points of $\mathscr{S}$ are equidistant from $(n^{-1},\ldots,n^{-1})$, we have that *each* point of $\mathscr{S}$ is an extreme point of $\mathscr{K}$.

THEOREM 14. *Let $\mathfrak{A}$ be a C*-algebra and $(a_1,\ldots,a_n)$ be a point in R^n such that each a_k is non-negative. Then*

$$a_1\mathscr{U}(\mathfrak{A})+\ldots+a_n\mathscr{U}(\mathfrak{A}) \subseteq b_1\mathscr{U}(\mathfrak{A})+\ldots+b_n\mathscr{U}(\mathfrak{A})$$

if $(b_1,\ldots,b_n)\in \mathrm{co}\{(a_{\pi(1)},\ldots,a_{\pi(n)}):\pi\in\Sigma_n\}$, where Σ_n is the group of permutations of $\{1,\ldots,n\}$.

PROOF. If $a_1 = \ldots = a_n = 0$, then $b_1 = \ldots = b_n = 0$ and there is nothing to prove. We may assume that $0 < a_1 + \ldots + a_n = s$. It will suffice to prove that

$$s^{-1}a_1\mathscr{U}(\mathfrak{A}) + \ldots + s^{-1}a_n\mathscr{U}(\mathfrak{A}) \subseteq s^{-1}b_1\mathscr{U}(\mathfrak{A}) + \ldots + s^{-1}b_n\mathscr{U}(\mathfrak{A}).$$

We may assume that $a_1 + \ldots + a_n = 1$ and hence that

$$\mathrm{co}\{(a_{\pi(1)},\ldots,a_{\pi(n)}) : \pi \in \Sigma_n\}$$

is a subset of the $(n-1)$-simplex σ consisting of those points $(r_1,\ldots,r_n)$ in R^n such that $0 \leqq r_j$ for each j and $r_1 + \ldots + r_n = 1$.

The assertion we are proving can be rephrased as follows: If the operator S in $\mathfrak{A}$ is a convex combination $a_1U_1 + \ldots + a_nU_n$ of unitary elements $U_1,\ldots,U_n$ of $\mathfrak{A}$ and

$$(b_1,\ldots,b_n) \in \mathrm{co}\{(a_{\pi(1)},\ldots,a_{\pi(n)}) : \pi \in \Sigma_n\},$$

then there are unitary elements $V_1,\ldots,V_n$ in $\mathfrak{A}$ such that $S = b_1V_1 + \ldots + b_nV_n$. We call $(b_1,\ldots,b_n)$ a *representing point* (in σ) for S when such unitary elements exist. Of course, each $(b_{\pi(1)},\ldots,b_{\pi(n)})$ is a representing point when $(b_1,\ldots,b_n)$ is; and from Corollary 12, each point of $\mathrm{co}\{(b_1,\ldots,b_n),\ (b_{\pi(1)},\ldots,b_{\pi(n)})\}$ is a representing point when π is a transposition.

We may suppose that $a_1 \geqq a_2 \geqq \ldots \geqq a_n$ and that $b_1 \geqq b_2 \geqq \ldots \geqq b_n$. In this case, since

$$(b_1,\ldots,b_n) \in \mathrm{co}\{(a_{\pi(1)},\ldots,a_{\pi(n)}) : \pi \in \Sigma_n\},$$

it follows from Lemma 13 that there are n points

$$(a_1^{(1)},\ldots,a_n^{(1)}),\ldots,(a_1^{(n)},\ldots,a_n^{(n)})$$

such that

$$(a_1^{(1)},\ldots,a_n^{(1)}) = (a_1,\ldots,a_n),\ (a_1^{(n)},\ldots,a_n^{(n)}) = (b_1,\ldots,b_n),$$

and

$$(a_1^{(k+1)},\ldots,a_n^{(k+1)}) = t(a_1^{(k)},\ldots,a_n^{(k)}) + (1-t)(a_{\tau(1)}^{(k)},\ldots,a_{\tau(n)}^{(k)})$$

for some t in $[0,1]$ and some transposition τ in Σ_n. From the preceding paragraph, we have that $(a_1^{(k)},\ldots,a_n^{(k)})$ is a representing point for each k in $\{1,\ldots,n\}$ since $(a_1^{(1)},\ldots,a_n^{(1)})$ $(=(a_1,\ldots,a_n))$ is. Thus $(b_1,\ldots,b_n)$ is a representing point.

COROLLARY 15. *Each convex combination of unitary elements of a* C*-*algebra is a mean of the same number of unitary elements of the algebra.*

PROOF. If $a_1,\dots,a_n$ are non-negative real numbers with sum 1, then

$$\frac{1}{n!}\sum_{\pi\in\Sigma_n}(a_{\pi(1)},\dots,a_{\pi(n)}) = \frac{1}{n!}((n-1)!,\dots,(n-1)!) = \left(\frac{1}{n},\dots,\frac{1}{n}\right).$$

It follows from Theorem 14 that

$$a_1\mathscr{U}(\mathfrak{A})+\dots+a_n\mathscr{U}(\mathfrak{A}) \subseteq \frac{1}{n}\mathscr{U}(\mathfrak{A})+\dots+\frac{1}{n}\mathscr{U}(\mathfrak{A}),$$

for each C*-algebra $\mathfrak{A}$.

5. Asymmetric decompositions.

In this section, we study the extent to which variation in the coefficients of a convex combination of unitary elements of a C*-algebra permits that combination to be expressed as a convex combination of fewer unitary elements of the algebra (see Proposition 20). We begin with a result (Lemma 16) related to Corollary 11. With $\mathfrak{A}$ a C*-algebra, it will be convenient to introduce the notation $\mathrm{co}_n\mathscr{U}(\mathfrak{A})$ for the set

$$\{a_1U_1+\dots+a_nU_n : U_1,\dots,U_n\in\mathscr{U}(\mathfrak{A}), 0\leqq a_j, a_1+\dots+a_n=1\}.$$

LEMMA 16. *Let $\mathfrak{A}$ be a C*-algebra and S be an element of $\mathfrak{A}$ such that $\|S\|\leqq 1-\varepsilon$, where $0<\varepsilon<(n+1)^{-1}$. If S has distance to $\mathrm{co}_n\mathscr{U}(\mathfrak{A})$ less than $\varepsilon^2(1-\varepsilon)^{-1}$, then there are unitary elements $U_1,\dots,U_{n+1}$ in $\mathscr{U}(\mathfrak{A})$ such that*

$$S = a_1U_1+\dots+a_nU_n+\varepsilon U_{n+1},$$

where $a_k\geqq 0$ and $a_1+\dots+a_n+\varepsilon=1$.

PROOF. By assumption, we can find unitary elements $V_1,\dots,V_n$ in $\mathscr{U}(\mathfrak{A})$ such that

$$\|S-(b_1V_1+\dots+b_nV_n)\| < \varepsilon^2(1-\varepsilon)^{-1}$$

for some non-negative real numbers $b_1,\dots,b_n$ with sum 1. Let T be

$$b^{-1}[S-(1-\varepsilon)(b_1V_1+\dots+b_{n-1}V_{n-1})],$$

where

$$b=\varepsilon+(1-\varepsilon)b_n(=b_n+\varepsilon(1-b_n)\leqq 1).$$

Then

$$\begin{aligned}\|T\| &\leqq b^{-1}[(1-\varepsilon)\|S-(b_1V_1+\dots+b_nV_n)\|+(1-\varepsilon)b_n+\varepsilon\|S\|]\\ &\leqq b^{-1}[\varepsilon^2+(1-\varepsilon)b_n+\varepsilon(1-\varepsilon)]=1,\end{aligned}$$

and, assuming as we may that $n^{-1} \leqq b_n$, we have

$$\begin{aligned}\|T - V_n\| &\leqq \|T - b^{-1}(1-\varepsilon)b_n V_n\| + \|b^{-1}(1-\varepsilon)b_n V_n - V_n\| \\ &\leqq b^{-1}[(1-\varepsilon)\|S - (b_1 V_1 + \ldots + b_n V_n)\| + \varepsilon\|S\|] \\ &\quad + 1 - b^{-1}(1-\varepsilon)b_n \\ &\leqq b^{-1}[\varepsilon^2 + \varepsilon(1-\varepsilon)] + 1 - b^{-1}(1-\varepsilon)b_n \\ &= 2\varepsilon b^{-1} \leqq 2n(n-1+\varepsilon^{-1})^{-1} < 1.\end{aligned}$$

From Corollary 11, there are unitary elements U_n and U_{n+1} in $\mathfrak{A}$ such that

$$T = (1-\varepsilon b^{-1})U_n + \varepsilon b^{-1} U_{n+1}.$$

Thus

$$\begin{aligned}S &= (1-\varepsilon)b_1 V_1 + \ldots + (1-\varepsilon)b_{n-1}V_{n-1} + bT \\ &= a_1 U_1 + \ldots + a_n U_n + \varepsilon U_{n+1},\end{aligned}$$

where $U_1 = V_1, \ldots, U_{n-1} = V_{n-1}$, $a_1 = (1-\varepsilon)b_1, \ldots, a_{n-1} = (1-\varepsilon)b_{n-1}$, $a_n = b - \varepsilon = (1-\varepsilon)b_n$.

As an easy consequence of Lemma 16, the proposition that follows characterizes the (norm-)closure of $\mathrm{co}_n\mathscr{U}(\mathfrak{A})$ in $(\mathfrak{A})_1^0$. Toward this end, we define $\mathrm{co}_{n+}\mathscr{U}(\mathfrak{A})$ as the set of elements S in $\mathfrak{A}$ with the property that for each positive ε there is a convex decomposition $a_1 U_1 + \ldots + a_{n+1}U_{n+1}$ of S with $U_1, \ldots, U_{n+1}$ in $\mathscr{U}(\mathfrak{A})$ and a_{n+1} less than ε.

PROPOSITION 17. *If $\mathfrak{A}$ is a C*-algebra and n is an integer greater than* 1, *then*

$$(\mathfrak{A})_1^0 \cap (\mathrm{co}_n\mathscr{U}(\mathfrak{A}))^= = (\mathfrak{A})_1^0 \cap \mathrm{co}_{n+}\mathscr{U}(\mathfrak{A}).$$

PROOF. If $S \in (\mathfrak{A})_1^0 \cap \mathrm{co}_{n+}\mathscr{U}(\mathfrak{A})$ and a positive ε is given, there are elements $U_1, \ldots, U_{n+1}$ in $\mathscr{U}(\mathfrak{A})$ and non-negative real numbers $a_1, \ldots, a_{n+1}$ with sum 1 such that $a_{n+1} < \varepsilon$ and $S = a_1 U_1 + \ldots + a_{n+1}U_{n+1}$. We have

$$\|S - (a_1 U_1 + \ldots + a_{n-1}U_{n-1} + (a_n + a_{n+1})U_n\| \leqq 2a_{n+1} < 2\varepsilon.$$

Hence $S \in (\mathfrak{A})_1^0 \cap (\mathrm{co}_n\mathscr{U}(\mathfrak{A}))^=$.

Suppose, now, that $T \in (\mathfrak{A})_1^0 \cap (\mathrm{co}_n\mathscr{U}(\mathfrak{A}))^=$. For ε small enough, $\|T\| < 1 - \varepsilon$. Since T has distance 0 to $\mathrm{co}_n\mathscr{U}(\mathfrak{A})$, Lemma 16 applies, and $T \in \mathrm{co}_{n+}\mathscr{U}(\mathfrak{A})$.

Applying Proposition 17 to the case where n is 2, we obtain a characterization of those C*-algebras $\mathfrak{A}$ with dense sets of invertible elements as those in which $(\mathfrak{A})_1^0 \subseteq \mathrm{co}_{2+}\mathscr{U}(\mathfrak{A})$ (cf. [12; Section 3], where Robertson conjectures that the C*-algebras $\mathfrak{A}$ with dense sets of invertible elements are precisely those in which $\mathrm{co}\mathscr{U}(\mathfrak{A}) = (\mathfrak{A})_1$).

PROPOSITION 18. *Let $\mathfrak{A}$ be a C*-algebra and let $\mathfrak{A}_{\text{inv}}$ be its group of invertible elements. The following are equivalent.*

(i) *$\mathfrak{A}_{\text{inv}}$ is dense in $\mathfrak{A}$;*
(ii) *$\frac{1}{2}(\mathscr{U}(\mathfrak{A} + \mathscr{U}(\mathfrak{A}))$ is dense in $(\mathfrak{A})_1$;*
(iii) *$(\mathfrak{A})_1^0 \subseteq \text{co}_{2+}\mathscr{U}(\mathfrak{A})$.*

PROOF. (i) → (ii). If $S \in (\mathfrak{A})_1$, there is a sequence $\{S_n\}$ in $\mathfrak{A}_{\text{inv}}$ tending to S. Let a_n be $(\max\{1, \|S_n\|\})^{-1}$. Then $\{a_nS_n\}$ tends to S and $a_nS_n \in (\mathfrak{A})_1 \cap \mathfrak{A}_{\text{inv}}$. As noted (for example, in the proof of Theorem 1), each

$$a_nS_n \in \tfrac{1}{2}(\mathscr{U}(\mathfrak{A}) + \mathscr{U}(\mathfrak{A})).$$

Hence $\frac{1}{2}(\mathscr{U}(\mathfrak{A}) + \mathscr{U}(\mathfrak{A}))$ is dense in $(\mathfrak{A})_1$.

(ii) → (iii). By assumption, $(\mathfrak{A})_1^0 = (\mathfrak{A})_1^0 \cap (\text{co}_2\mathscr{U}(\mathfrak{A}))^=$. From Proposition 17,

$$(\mathfrak{A})_1^0 \cap (\text{co}_2\mathscr{U}(\mathfrak{A}))^= = (\mathfrak{A})_1^0 \cap \text{co}_{2+}\mathscr{U}(\mathfrak{A}).$$

(iii) → (i). It will suffice to show that each element of $(\mathfrak{A})_1^0$ is in $\mathfrak{A}_{\text{inv}}^=$. Let a positive ε ($\leqq 1/3$) be given. By assumption, each S in $(\mathfrak{A})_1^0$ has the form $a_1U_1 + a_2U_2 + a_3U_3$, where $U_1, U_2, U_3 \in \mathscr{U}(\mathfrak{A})$, $0 \leqq a_1 \leqq a_2$, $0 \leqq a_3 < \varepsilon$ ($\leqq 1/3$), and $a_1 + a_2 + a_3 = 1$. If $a_1 = 0$, then

$$S = a_2U_2(I + a_2^{-1}a_3U_2^*U_3) \in \mathfrak{A}_{\text{inv}},$$

since $1/3 < a_2$. If $a_1 > 0$, let ε' be $\min\{\varepsilon, a_1/2\}$ and let S_0 be $(a_1 - \varepsilon')U_1 + (a_2 + a_3 + \varepsilon')U_2$. Then $\|S - S_0\| \leqq 2(\varepsilon' + a_3) < 4\varepsilon$ and

$$S_0 = (a_2 + a_3 + \varepsilon')U_2[(a_2 + a_3 + \varepsilon')^{-1}(a_1 - \varepsilon')U_2^*U_1 + I] \in \mathfrak{A}_{\text{inv}}.$$

REMARK 19. It is not, in general, possible to replace $2+$ by 2 in condition (iii) of Proposition 18. We illustrate this with the C*-algebra $\mathfrak{A}$ of convergent complex sequences (so that $\mathfrak{A} = C(\mathsf{N} \cup \{\infty\})$). The invertible elements of $\mathfrak{A}$ are dense in $\mathfrak{A}$, but the element S in $(\mathfrak{A})_1^0$, defined by $S(\infty) = 0$ and

$$S(n) = (2n)^{-1}e^{\frac{1}{2}i\pi n} \quad (n \in \mathsf{N})$$

cannot be expressed in the form $\frac{1}{2}(U + V)$ with U and V in $\mathscr{U}(\mathfrak{A})$. Indeed, if $S = \frac{1}{2}(U + V)$ with U and V in $\mathscr{U}(\mathfrak{A})$, then, letting $U(n)$ be $u(n) + iu'(n)$ and $V(n)$ be $v(n) + iv'(n)$ with $u(n)$, $u'(n)$, $v(n)$, $v'(n)$ real, we have

$$\tfrac{1}{2}[u(2n) + v(2n)] + \tfrac{1}{2}i[u'(2n) + v'(2n)] = \tfrac{1}{2}[U(2n) + V(2n)] = S(2n) = \frac{(-1)^n}{4n}.$$

Thus $u'(2n) = -v'(2n)$ and $u(2n) + v(2n) = (2n)^{-1}(-1)^n$. Since U and V are unitary elements of $\mathfrak{A}$,

$$1 = u(2n)^2 + u'(2n)^2 = v(2n)^2 + v'(2n)^2.$$

But $u'(2n)^2 = v'(2n)^2$ so that $u(2n)^2 = v(2n)^2$. Since $u(2n)+v(2n) = (2n)^{-1}(-1)^n \neq 0$,

$$u(2n) = v(2n) = \frac{(-1)^n}{4n} \to 0 = u(\infty).$$

At the same time,

$$\tfrac{1}{2}[u(2n-1)+v(2n-1)] + \tfrac{1}{2}i[u'(2n-1)+v'(2n-1)] = \frac{(-1)^{n+1}i}{4n-2},$$

and now $u'(2n-1) = v'(2n-1) = (4n-2)^{-1}(-1)^{n+1}$. Thus

$$|u(2n-1)| = (1-u'(2n-1)^2)^{1/2} \to 1 = |u(\infty)|$$

– a contradiction.

PROPOSITION 20. *Let $\mathfrak{A}$ be a C*-algebra and T be an element of $\mathfrak{A}$ such that $u_c(T) = n \geqq 3$. If $a_1U_1 + \ldots + a_nU_n = T$, where $a_1,\ldots,a_n$ are non-negative real numbers with sum 1 and $U_1,\ldots,U_n \in \mathcal{U}(\mathfrak{A})$, then*

(i) $a_i \leqq a_j + a_k$ *provided* $j \neq k$;

(ii) $(n-1)^{-1} \leqq a_j + a_k$ *provided* $j \neq k$;

(iii) $a_j \leqq 2(n+1)^{-1}$ *for all* j.

PROOF. By renumbering, we may assume that $a_1 \leqq a_2 \leqq \ldots \leqq a_n$. If $a_n > a_1 + a_2$, then

$$a_1U_1 + a_2U_2 + a_nU_n = a_nU_n(a_n^{-1}U_n^*(a_1U_1+a_2U_2)+I).$$

Hence, $(a_1+a_2+a_n)^{-1}(a_1U_1+a_2U_2+a_nU_n) \in \mathfrak{A}_{\text{inv}} \cap (\mathfrak{A})_1$ and

$$a_1U_1 + a_2U_2 + a_nU_n = \tfrac{1}{2}(a_1+a_2+a_n)(V_1+V_2)$$

for some V_1 and V_2 in $\mathcal{U}(\mathfrak{A})$. This provides a convex decomposition of T in terms of $n-1$ elements of $\mathcal{U}(\mathfrak{A})$-contradicting the assumption that $u_c(T) = n$. Thus

$$a_i \leqq a_n \leqq a_1 + a_2 \leqq a_j + a_k$$

provided $j \neq k$. This proves (i).

When $j \neq k$, (ii) follows from (i) and

$$1 = (a_1+a_2) + a_3 + \ldots + a_n \leqq (n-1)(a_1+a_2) \leqq (n-1)(a_j+a_k).$$

We show that $a_n \leqq 2(n+1)^{-1}$, whence $a_j \leqq a_n \leqq 2(n+1)^{-1}$ for all j, and (iii) follows. From (i), we have

$$\begin{aligned}(n-1)a_n &\leqq (a_1+a_2) + (a_2+a_3) + \ldots + (a_{n-2}+a_{n-1}) + (a_{n-1}+a_1) \\ &= 2(a_1+\ldots+a_{n-1}) = 2(1-a_n),\end{aligned}$$

whence $(n+1)a_n \leqq 2$, as desired.

6. An application.

Gelfand and Neumark [3] list six conditions for a Banach algebra with a unit and a * operation (involution) to be isometrically *isomorphic to a C*-algebra of operators acting on a Hilbert space. They conjectured that the last two conditions are redundant. The fact that the * operation is an isometry, the fifth condition, is proved in [4]. In fact, a full proof of the Gelfand-Neumark theorem, without assuming the last two conditions, is given in [4; Theorem 11] and an account of the work on the Gelfand-Neumark conjecture follows Remark 10 of [4]. If we join the proof of [4; Theorem 11] on p. 555, where it has just been established that invertible elements of the B*-algebra $\mathfrak{A}$ have a polar decomposition with components in $\mathfrak{A}$, unitary elements of $\mathfrak{A}$ have norm 1, and the self-adjoint elements of $\mathfrak{A}$ form a norm-closed set, then the proof given in Section 2 applies to this B*-algebra $\mathfrak{A}$, and $(\mathfrak{A})_1^0 \subseteq \mathrm{co}\,\mathscr{U}(\mathfrak{A})$. Suppose $T \in (\mathfrak{A})_1^0$ and $T = \Sigma_{j=1}^n a_j U_j$, where $U_j \in \mathscr{U}(\mathfrak{A})$, $a_j > 0$, and $\Sigma_{j=1}^n a_j = 1$. Then

$$T^* = \Sigma_{j=1}^n a_j U_j^* \text{ and } \|T^*\| \leqq \Sigma_{j=1}^n a_j = 1.$$

If S in $\mathfrak{A}$ is such that $\|S\| < \|S^*\|$, choose b such that $\|S\| < b < \|S^*\|$ and let T_0 be $b^{-1}S$. Then $T_0 \in (\mathfrak{A})_1^0$, but $1 < b^{-1}\|S^*\| = \|T_0^*\|$ – contradicting what we have just proved. Thus $\|S^*\| \leqq \|S\|$ for each S in $\mathfrak{A}$, and $\|(S^*)^*\| \leqq \|S^*\|$. It follows that $\|S\| = \|S^*\|$ for each S in $\mathfrak{A}$.

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