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SCALAR IRREDUCIBILITY OF EIGENSPACE REPRESENTATIONS ASSOCIATED TO A SYMMETRIC SPACE

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1. Introduction.

Let $X = G/K$ be a homogeneous manifold, K being a closed subgroup of the Lie group G. Denoting the algebra of all G-invariant differential operators on X by $\boldsymbol{D}(X)$, we consider for each homomorphism $\chi : \boldsymbol{D}(X) \to \mathrm{C}$ the joint eigenspace

$$\mathscr{E}_\chi(X) = \{f \in C^\infty(X) \mid Df = \chi(D)f,\ \forall D \in \boldsymbol{D}(X)\}$$

and the natural representation of G on $\mathscr{E}_\chi(X)$.

The problem of whether T_χ is (topologically) irreducible has been studied for different kinds of homogeneous spaces. See f.ex. [1], [2], and the survey paper [3] for symmetric spaces, and [4] and [5] for nilmanifolds. We shall here in the case of symmetric spaces treat the corresponding problem of scalar irreducibility:

A representation T of a group G on a topological vector space E is said to be scalarly irreducible, if the only continuous, linear operators $A : E \to E$ that satisfy $AT(g) = T(g)A$ for all $g \in G$, are scalar multiples of the identity operator on E. Such A are called intertwining operators.

The main result of this paper is that T_χ is scalarly irreducible for any χ on any symmetric space of the non-compact type [Theorem 2]. As a contrast T_χ is not always topologically irreducible: S. Helgason has in [1; Theorem 2.1 for the rank 1 case] and in [2; Theorem 9.1 in general] found a necessary and sufficient condition on χ for T_χ to be topologically irreducible.

Since we need results from Helgason's paper [2], this paper can be viewed as a supplement to the discussion of eigenspace representations in [2].

2. A general result.

Let G be a transitive Lie transformation group of a C^∞-manifold X. Let E be a closed subspace of $C^\infty(X)$, equipped with the topology from $C^\infty(X)$

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[i.e. uniform convergence of each derivative on compact sets]. We consider a representation T of G on E of the following form

$$[T(g)f](x) = \mu(g,x)f(g^{-1}x) \text{ for } f \in E,\ g \in G,\ x \in X,$$

where $\mu \in C^\infty(G \times X)$ is given. In case $\mu \equiv 1$ we say T is the natural representation of G on E.

1. PROPOSITION. *Let $K := \{g \in G \mid g \cdot x_0 = x_0\}$ where $x_0 \in X$. Let T be a representation of G on E as above. If $T_{|K}$ is a cyclic representation, then T is scalarly irreducible.*

PROOF. Let $f_0 \in E$ be a cyclic vector for $T_{|K}$. We shall show that there to each continuous, linear, intertwining operator $A: E \to E$ exists a $c \in \mathbf{C}$ such that

$$(Af)(x) - cf(x) = 0 \text{ for all } f \in E \text{ and all } x \in X.$$

Consider for $c \in \mathbf{C}$ the continuous linear functional $u: E \to \mathbf{C}$ given by

$$\langle u,f\rangle := (Af)(x_0) - cf(x_0) \text{ for } f \in E.$$

By the intertwining property of A it suffices to show that $u = 0$ for a suitable choice of the constant c. There are two cases to consider:

(α) $$f_0(x_0) = 0.$$

Here we have for any $k \in K$ that

$$(T(k)f_0)(x_0) = \mu(k,x_0)f_0(k^{-1}x_0) = \mu(k,x_0)f_0(x_0) = 0,$$

so by the cyclicity of $T_{|K}$ we get $f(x_0) = 0$ for all $f \in E$. But then $\langle u,f\rangle = 0 - 0 = 0$ for all $f \in E$.

(β) $$f_0(x_0) \neq 0.$$

Here we choose $c = (Af_0)(x_0)f_0(x_0)^{-1}$, so that $\langle u,f_0\rangle = 0$. Since

$$\langle u,T(k)f\rangle = \mu(k,x_0)\langle u,f\rangle \text{ for all } f \in E \text{ and } k \in K,$$

we have

$$\langle u,T(k)f_0\rangle = \mu(k,x_0)\langle u,f_0\rangle = 0,$$

so by (ii) we get

$$\langle u,f\rangle = 0 \text{ for all } f \in E.$$

3. Symmetric spaces of the non-compact type.

Combining Proposition 1 with results from [2] we get the following theorem on eigenspace representations associated to a symmetric space.

2. THEOREM. *Let G be a connected semi-simple Lie group with finite center, K a maximal compact subgroup of G, and $\mathbf{D}(G/K)$ the algebra of G-incariant differential operators on G/K.*

The natural representation of G on the joint eigenspace

$$\{f \in C^\infty(G/K) \mid Df = \chi(D)f,\ \forall D \in \mathbf{D}(G/K)\}$$

is scalarly irreducible for any homomorphism $\chi : \mathbf{D}(G/K) \to \mathbf{C}$.

Theorem 2 is an analogue to Theorem 9.1 of [2] which gives a necessary and sufficient condition on χ for the representation to be (topologically) irreducible.

PROOF. We use the (standard) notation from § 2 of [2] since our proof of Theorem 2 is based on that paper. In particular we let $X = G/K$ and parametrize the joint eigenspaces by $\mathfrak{a}_c^*$: According to [2; p. 208] the joint eigenspaces for $\mathbf{D}(G/K)$ have the form.

$$\mathscr{E}_\lambda(X) = \{f \in C^\infty(X) \mid Df = \Gamma(D)(i\lambda)f,\ \forall D \in \mathbf{D}(X)\},$$

where $\Gamma(D)$ is a certain polynomial function on $\mathfrak{a}_c^*$ and where λ runs through $\mathfrak{a}_c^*$.

We fix $\lambda \in \mathfrak{a}_c^*$ for the rest of the proof; we may and will assume that λ is simple (Cfr. p. 208 of [2]).

For $b \in B = K/M$ we introduce $e_b \in C^\infty(X)$ by

$$e_b(x) := \exp[(i\lambda + \rho)A(x,b)] \quad \text{for } x \in X.$$

It is known, Cfr. [1, p. 94], that $e_b \in \mathscr{E}_\lambda(X)$, and so the space

$$\mathscr{H}_\lambda := \{x \to \textstyle\int_B e_b(x)F(b)db \mid F \in L^2(B)\}$$

is contained in $\mathscr{E}_\lambda(X)$. Since λ is simple we get from the proof of Theorem 9.1 of [2] that $\mathscr{H}_\lambda$ is dense in $\mathscr{E}_\lambda(X)$, and so that span $\{e_b \mid b \in B\}$ is dense in $\mathscr{E}_\lambda(X)$.

The natural action of K on e_b is $k \cdot e_b = e_{kb}$ so we see that $T_{|K}$ is a cyclic representation of K on $\mathscr{E}_\lambda(X)$, any e_b being a cyclic vector.

All that remains is to refer to Proposition 1.

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