MATH. SCAND. 57 (1985), 293–345

DIFFERENTIAL OPERATORS CANONICALLY ASSOCIATED TO A CONFORMAL STRUCTURE

THOMAS P. BRANSON*

Abstract.

Let (M,g) be a pseudo-Riemannian manifold of dimension n. Explicit formulas are given for (1) a fourth-order operator $D_{4,k}$ on k-forms in M, $n \neq 1,2,4$, (2) a sixth-order operator D_6 on functions, $n \neq 1,2,4$; and (3) a sixth-order operator on $(n-6)/2$-forms, $n = 6,8,10,\ldots$; which are covariant under conformal deformation and transformation of g (subject, in the case of D_6, to a curvature constraint). Various associated nonlinear conformally covariant operators are also computed. Previously known conformally covariant linear operators include the modified Laplace/wave operator $D_2 = \Box + (n-2)K/4(n-1)$ on functions (K = scalar curvature, $n \neq 1$); the operator giving the Maxwell equations (n even); a second-order operator $D_{2,k}$ on forms of arbitrary order k ($n \neq 1,2$) introduced by the present author, which specializes to D_2 when $k = 0$ and to Maxwell when $k = (n-2)/2$; and a fourth-order operator D_4 on functions ($n \neq 1,2$) introduced by S. Paneitz.

The old and new results are applied to the problem of finding representations of the conformal transformation group of (M,g) which are unitary, or at least admit invariant complex inner products. Explicit differential geometric formulas for such inner products are given on the representation spaces determined by D_2, Maxwell, $D_{2,k}$, and D_4; the natural setting here seems to be that of Lorentz manifolds.

In the setting of compact Riemannian manifolds, all these operators produce numerical invariants of conformal structure, and D_4 produces an analogue of the Yamabe problem. This is used to get an inequality obstruction to the possibility of finding an Einstein metric in the conformal class of a given Riemannian metric.

---

* Supported by a NATO Postdoctoral Fellowship at Odense Universitet, and by NSF grant #MCS 8108814 (A01) at the Institute for Advanced Study.

Received September 12, 1984.

0. Introduction and contents.

The theory of conformal transformation and deformation of pseudo-Riemannian metrics has long been important in geometry and physics. In Riemannian geometry, conformal deformation of the metric provides a natural way to vary geometric structures in a way which depends on only one arbitrary function. The best-known application of this idea is to the *Yamabe problem* of finding a Riemannian metric on a compact manifold with prescribed scalar curvature [25,1]. In physics, where one has a Lorentz rather than Riemannian geometry, the importance of conformal structure was recognized much earlier, and here conformality is probably a more fundamental notion. Conformal changes of metric and conformal transformations leave invariant the set of null geodesics, i.e., the paths of massless particles. For this reason, the conformal group of a spacetime has been proposed as a fundamental physical symmetry group [9, 3, 10, 22, 23].

More recently, conformal structure has been important in the problem of deciding whether two strictly pseudoconvex domains $\mathbf{C}^n$ are biholomorphic. In [11], Fefferman defines a Lorentz metric on the manifold $C = S^1 \times \partial U$ for a general strictly pseudoconvex domain U, in such a way that a biholomorphism between domains induces a conformal diffeomorphism between the assiciated Lorentz manifolds. General invariants of conformal structure, realized in C, are then biholomorphic invariants of U. The light rays (null geodesics) in C play a basic role in this circle of ideas, and there is probably a strong connection with the physical problems which first motivated the study of conformality.

This paper is about invariants of conformal structure, specifically, differential operators which are *conformally covariant* (or *conformally quasi-invariant*) in the same sense as the modified Laplace/wave operator

$$D_2 = \Box + \frac{n-2}{4(n-1)}K, \quad K = \text{scalar curvature}; \; n \neq 1 \tag{0.1}$$

(n = dimension), and the operator giving the Maxwell equations (n even). Our main results are explicit formulas for various general fourth- and sixth-order conformally covariant differential operators on functions and differential forms. The search for these operators was inspired by four things: (1) the result of Jakobsen and Vergne [14] that in the special case of four-dimensional Minkowski space, nonnegative powers $\Box^p$ of the d'Alembertian are covariant under the 15-parameter group of conformal transformations; (2) the application by Ørsted [19] of the conformal covariance laws for D_2 and the Maxwell equations to the representation theory of SO $(2,n)$; (3) the discovery by the present author [5] of a general second-order conformally covariant operator $D_{2,k}$ on differential forms of arbitrary order k, for $n \neq 1,2$ (specializing to D_2 when $k = 0$, and to the Maxwell operator on "vector potentials" when $k = (n-2)/2$); and (4) the discovery by S. Paneitz [20] of a general fourth-order conformally covariant operator D_4 on functions, $n \neq 1,2$, with leading term $\Box^2$. To get the present results without excessive calculation, somewhat more sophisticated computational machinery than that used to get $D_{2,k}$ and D_4 is required, and this is developed here.

Specifically, the new linear operators are (a) a fourth-order operator $D_{4,k}$ on k-forms, $n \neq 1,2,4$; (b) a sixth-order operator D_6 on functions when $n \neq 1,2,4$ and a certain curvature obstruction vanishes; (c) a sixth-order operator on $(n-6)/2$-forms, $n = 6,8,10,\ldots$. Like D_2, the Maxwell operator, $D_{2,k}$, and D_4, all these operators are polynomial in the covariant derivative and the Riemann curvature tensor.

All of these operators are also "invariants of conformal structure", or "operators canonically associated to a conformal structure". This is because their conformal covariance laws, in the setting of ordinary functions and forms, imply the existence of operators on function or form *densities* (sometimes called *weighted* functions and forms) which, though appearing to depend on the pseudo-Riemannian metric g, actually depend only on the conformal class of g.

From the point of view of the classification problem for conformally covariant operators, the dimension and curvature constraints above seem to reveal some general trends. We do not get a conformally covariant D_2 for manifolds of dimension 1, or a $D_{2,k}$ or D_4 for manifolds of dimension 1 or 2. For $D_{4,k}$ and D_6, this list of *critical dimensions* expands to $n = 1,2,4$. One might expect more critical dimensions to appear for higher-order operators. Actually, the calculation which gives $D_{4,k}$ *does* give an operator in dimension 4, but here $D_{4,k}$ collapses to a zeroth-order differential operator, an action of a 2-tensor known classically as the *Bach tensor*. The covariance of D_6 under the conformal change $g \to \Omega^{-2}g$, $0 < \Omega \in C^\infty(M)$, is subject not only to $n \neq 1,2,4$, but also to the curvature condition

$$(0.2) \qquad r^{\beta}{}_{\alpha}(\nabla_\lambda r^{\alpha}{}_{\beta} - 2\nabla_\beta r^{\alpha}{}_{\lambda}) = 0, \; r = \text{Ricci tensor}$$

(vanishing of a one-form), which must be satisfied for all metrics in some differentiable path of metrics conformally related to the original g; this path beginning with g and ending with $\Omega^{-2}g$. For manifolds (M,g) satisfying (0.2), D_6 will be covariant under the subgroup of the group

$\mathscr{C}(M,g)$ of conformal transformations generated by the identity component of $\mathscr{C}(M,g)$ and the isometries. It may be that a manifold symmetric enough to have a conformal group of nonzero dimension is *automatically* conformal to a manifold satisfying (0.2). At any rate, D_6 seems to represent a point at which the theories of conformal *deformation* and conformal *transformation* diverge.

From the conformal covariance laws for the above *linear* differential operators, one can deduce the conformal covariance of various *nonlinear* operators, analogous to the well-known operators on functions

$$\text{(0.3)} \qquad \begin{aligned} &\varphi \to D_2\varphi \pm |\varphi|^{4/(n-2)}\varphi, \quad && n \neq 1,2, \\ &\varphi \to \Box\varphi \pm |d\varphi|^2\varphi, && n = 2, \end{aligned}$$

and to various hybrids of these. The operator (0.3) appears in the theory of the Yamabe problem of prescribing K; in the same way, a nonlinear operator based on D_4,

$$\varphi \to D_4\varphi \pm |\varphi|^{8/(n-4)}\varphi, \ n \neq 1,2,4,$$

appears in a Yamabe prescription problem for a function which is a linear combination of $\Box K$, K^2, and $\|r\|^2 = r^\beta{}_\alpha r^\alpha{}_\beta$.

Linear conformally covariant operators D yield *intertwining* operators for representations of the conformal group of a given pseudo-Riemannian manifold (M,g); basically, such a D intertwines one representation on the null space $\mathscr{N}(D)$ with another on the range $\mathscr{R}(D)$. It is natural to ask when these representations are unitary, or at least admit invariant complex inner products (which may not be positive definite). We show that invariant inner products can be produced in remarkable generality; that is, by differential geometric formulas, without passing to examples. In the case of $\mathscr{R}(D)$, the formula is almost completely general. For $\mathscr{N}(D)$. we assume that (M,g) is a Lorentz manifold, and impose certain topological conditions on its spacelike surfaces. In a large class of examples, these inner products can be shown to be nondegenerate; in special cases, they are known to be unitary.

The paper is organized as follows. In Section 1.a, we standardize differential geometric notation, and record some identities which will be useful later on. In Sections 1.b and 1.c, we develop the computational machinery necessary to calculate the high-order conformally covariant operators. The highlight here is a theorem which states, roughly, that to check covariance under conformal deformations $g \to \Omega^{-2}g$ of the metric, $0 < \Omega \in C^\infty(M)$, it is sufficient to work only "to first order in derivatives of $\log \Omega$". In Section 1.d, we apply this machinery to give short proofs of the conformal covariance of the known operators D_2, Maxwell, $D_{2,k}$, and D_4. In Sections 2. a–c, we derive formulas for $D_{4,k}$, D_6, and $D_{6,(n-6)/2}$ directly from the requirement that they be conformally covariant "to first order", and at the same time, prove their covariance. Along the way, more computational machinery, applicable to conformal deformation laws for operators on forms, is developed. In Section 2.d, we catalogue the various nonlinear operators whose conformal covariance can be derived from that of the linear operators.

In Section 3, we explore the applications to group representation theory described above. In Sections 4.a, we apply the results of Sections 1.d and 2 to get global conformal invariants of compact Riemannian manifolds. Along these lines, we use the D_4-Yamabe problem to obtain an inequality obstruction to the prescribability of an Einstein metric by conformal deformation of a given metric. Section 4.b explains how conformally covariant operators on ordinary tensor fields lead to operators on tensor densities which depend only on conformal structure. Section 4.c is about a relation, discovered by Paneitz, of D_4 to the gauge theory of the Maxwell equations. Section 4.d briefly describes results obtained jointly with Ørsted on an interaction of conformal covariance with the Minakshisundaram–Pleijel expansion for the heat kernel.

The author would like to thank Irving Segal for teaching him the importance of conformality in mathematics and physics, and Bent Ørsted for many suggestions and enlightening conversations throughout the course of this work. Thanks are also due to Harold Donnelly for educating the author on various topics related to this work over the last few years, and to Odense Universitet and the Institute for Advanced Study for their hospitality.

1. Conformal covariance.

a. *Notation. Some identities.* Throughout this paper, (M,g) will be a pseudo-Riemannian (ΨR) manifold; that is, a C^∞ manifold with a C^∞ metric tensor $g = (g_{\alpha\beta})$ which is nondegenerate, but not necessarily positive definite. n will always denote the dimension of M.

From g we get a unique symmetric ΨR affine connection ∇, and the associated Riemann, Ricci, and scalar curvatures $R = (R^\alpha{}_{\beta\lambda\mu})$, $r = (r_{\beta\mu})$, and K respectively. (See, for example, [13, § 1].) We use the sign conventions in which the Ricci and scalar curvatures of standard spheres are positive: in local coordinates (x^α), with $\partial_\alpha = \partial/\partial x^\alpha$,

$$R^{\alpha}{}_{\beta\lambda\mu}\partial_\alpha = (\nabla_\lambda\nabla_\mu - \nabla_\mu\nabla_\lambda)\partial_\beta$$
$$r_{\beta\mu} = R^{\alpha}{}_{\beta\alpha\mu}$$
$$K = r^{\mu}{}_{\mu}.$$

Here and below, we employ the summation convention, and raise and lower indices using the metric tensor and its inverse $g^1 = (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$.

From g^1 we get natural metrics g^k on the exterior bundles $\Lambda^k(M)$ (see, e.g., [5]), and thus a formal adjoint δ for the exterior derivative d. If E is a local normalized orientation, i.e., an n-form defined on some open set U in M with $g^n(E,E) = \pm 1$, then δ is uniquely determined locally by the requirement that

$$\int_U g^k(\varphi, d\psi)E = \int_U g^{k-1}(\delta\varphi, \psi)E$$

whenever either the k-form φ or the $(k-1)$-form ψ has compact support in U. Since orientation-reversal has no effect on this definition, the orientability of M is not an issue. More generally, one has unique, locally determined formal adjoints for linear differential operators between ΨR bundles on a ΨR manifold.

Exterior multiplication $\varphi \to \eta \wedge \varphi$ of forms by a one-form η will be denoted $\varepsilon(\eta)$. Exterior multiplication by a vector field X is just exterior multiplication by the g-associated one-form $(X_\alpha) = (g_{\alpha\beta}X^\beta)$. *Interior multiplication* of forms by a vector field X will be denoted $\iota(X)$:

$$(\iota(X)\varphi)(Y_1, \ldots, Y_{k-1}) = \varphi(X, Y_1, \ldots, Y_{k-1}),$$

where the Y_j are vector fields. Interior multiplication by a one-form is just interior multiplication by the g-associated vector field $(\eta^\alpha) = (g^{\alpha\beta}\eta_\beta)$. $\iota(\eta)$ is the *pointwise* adjoint of $\varepsilon(\eta)$: if φ is a k-form and ψ a $(k-1)$-form, then

$$g^k(\varphi, \varepsilon(\eta)\psi) = g^{k-1}(\iota(\eta)\varphi, \psi).$$

In classical notation, if $\varphi = (1/k!)\varphi_{\alpha_1\ldots\alpha_k} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}$ is a k-form, then

$$(1.1)\qquad (d\varphi)_{\alpha_1\ldots\alpha_{k+1}} = \sum_{s=1}^{k+1} (-1)^{s-1}\nabla_{\alpha_s}\varphi_{\alpha_1\ldots\hat{\alpha}_s\ldots\alpha_{k+1}}$$

$$(1.2)\qquad (\delta\varphi)_{\alpha_1\ldots\alpha_{k-1}} = -\nabla^\lambda\varphi_{\lambda\alpha_1\ldots\alpha_{k-1}}$$

$$(1.3)\qquad (\varepsilon(\eta)\varphi)_{\alpha_1\ldots\alpha_{k+1}} = \sum_{s=1}^{k+1} (-1)^{s-1}\eta_{\alpha_s}\varphi_{\alpha_1\ldots\hat{\alpha}_s\ldots\alpha_{k+1}}$$

$$(1.4)\qquad (\iota(X)\varphi)_{\alpha_1\ldots\alpha_{k-1}} = X^\lambda\varphi_{\lambda\alpha_1\ldots\alpha_{k-1}}.$$

Formulas (1.1)–(1.4) are valid in *any* local frame. Recall that in classical notation, an expression like $\nabla_\lambda \varphi_{\alpha\beta}$ really means $(\nabla\varphi)_{\lambda\alpha\beta}$.

The following notational convention will be used extensively in what follows.

DEFINITION 1.1. If $(A^\alpha{}_\beta)$ is any rule which assigns to each local frame a list of n^2 functions, and if $\varphi = (\varphi^{\lambda_1 \dots \lambda_s}{}_{\mu_1 \dots \mu_t})$ is a tensor field, we abbreviate by $A^\bullet{}_\bullet \# \varphi$, or sometimes $A \# \varphi$, the rule which assigns to a given local frame $\mathscr{F}$ the list

$$(A^\bullet{}_\bullet \# \varphi)^{\lambda_1 \dots \lambda_s}{}_{\mu_1 \dots \mu_t} = -\sum_{i=1}^{s} A^{\lambda_i}{}_\beta \, \varphi^{\lambda_1 \dots \underset{\substack{\uparrow \\ \lambda_i \text{ place}}}{\beta} \dots \lambda_s}{}_{\mu_1 \dots \mu_t} +$$

$$+ \sum_{j=1}^{t} A^\alpha{}_{\mu_j} \, \varphi^{\lambda_1 \dots \lambda_s}{}_{\mu_1 \dots \underset{\substack{\uparrow \\ \mu_j \text{ place}}}{\alpha} \dots \mu_t},$$

where all indices are relative to $\mathscr{F}$.

REMARK 1.2. a) The $A^\alpha{}_\beta$ need not be components of a tensor field. For example, if $\Gamma_{\alpha\beta}{}^\lambda$ are the Christoffel symbols in a frame (X_α), $\nabla_\alpha X_\beta = \Gamma_{\alpha\beta}{}^\lambda X_\lambda$, the formula for the covariant derivative of a tensor field of arbitrary type reads

$$\nabla_\alpha \varphi^I{}_J = X_\alpha \varphi^I{}_J - (\Gamma_{\alpha\bullet}{}^\bullet \# \varphi)^I{}_J. \tag{1.5}$$

Here I and J are multi-indices, and $X_\alpha \varphi^I{}_J$ is just the derivative of the component $\varphi^I{}_J$ viewed as a scalar function.

b) The commutator of covariant derivatives on tensor fields of arbitrary type is given by a $\#$ action of the Riemann tensor:

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\varphi^I{}_J = -(R^\bullet{}_{\bullet\alpha\beta} \# \varphi)^I{}_J. \tag{1.6}$$

c) If the $A^\alpha{}_\beta$ are the components of a tensor field, then $A^\bullet{}_\bullet \#$ can be characterized as the unique type-preserving derivation on the mixed tensor algebra of M which annihilates functions, commutes with contractions, and has $(A^\bullet{}_\bullet \# X)^\alpha = -A^\alpha{}_\beta X^\beta$ on vector fields.

d) Conversely, suppose that B is a type-preserving derivation on the mixed tensor algebra which annihilates functions and commutes with contractions. Then $B = A\#$ for some tensor field $A = (A^\alpha{}_\beta)$. For if f is a function and φ a tensor field, $B(f\varphi) = fB\varphi$. This means that B "lives at points"; i.e., is a zeroth order differential operator with C^∞ coefficients. Thus on vector fields, $(BX)^\alpha = -A^\alpha{}_\beta X^\beta$ for some tensor field $(A^\alpha{}_\beta)$, and this determines B as $A\#$ on tensor fields of all types.

e) Suppose $A = (A^\alpha{}_\beta)$ is a tensor field, (X_α) is a local frame, and (η^β) is the dual coframe. Then on differential forms,

$$A\# = A^\alpha{}_\beta \varepsilon(\eta^\beta)\iota(X_\alpha). \tag{1.7}$$

f) If B is an order-preserving derivation on the Grassmann algebra which annihilates functions, $B = A\#$ for some A which is determined by the effect of B on coordinate coframe forms dx^α.

b. *Conformal deformation of canonical linear differential operators.* The main objects of study in this paper are linear differential operators (LDO) of the form

$$D = \text{polynomial } (\nabla, R\otimes, g\otimes, g^1\otimes, \text{contractions}) \tag{1.8}$$

on tensor fields.

Our concern will be with how such operators deform under *conformal change of metric* $\underline{g} = \Omega^2 g$, $0 < \Omega \in C^\infty(M)$. To see how the "building blocks" ∇ and R deform, we adopt the notation of [8]. Fix a local g-orthonormal frame (X_α) (in which $g_{\alpha\beta} = \pm\delta_{\alpha\beta}$), and the corresponding $\underline{g}$-orthonormal frame (ΩX_α). We underline all Christoffel symbols, covariant derivatives, and curvatures calculated with respect to $\underline{g}$, and all indices relative to the frame (ΩX_α). We set $\omega = \log\Omega$, and abbreviate expressions like $\nabla_\alpha\omega$, $\nabla_\beta\nabla^\lambda\nabla_\alpha\omega$ by ω_α, $\omega_\alpha{}^\lambda{}_\beta$ respectively.

LEMMA 1.3. a) (See, e.g., [8].)

$$\underline{\Gamma_{\alpha\beta}{}^\lambda} = \Omega(\Gamma_{\alpha\beta}{}^\lambda - \delta_\alpha{}^\lambda\omega_\beta + g_{\alpha\beta}\omega^\lambda). \tag{1.9}$$

b) (See, e.g. [25].)

$$\begin{aligned} \underline{R^{\alpha\beta}{}_{\lambda\mu}} &= \Omega^2(R^{\alpha\beta}{}_{\lambda\mu} - u_\lambda{}^\beta\delta^\alpha{}_\mu + u_\mu{}^\beta\delta^\alpha{}_\lambda - u_\mu{}^\alpha\delta^\beta{}_\lambda + u_\lambda{}^\alpha\delta^\beta{}_\mu), \\ u_{\alpha\beta} &= \omega_{\alpha\beta} + \omega_\alpha\omega_\beta - \frac{1}{2}\omega^\lambda\omega_\lambda g_{\alpha\beta}. \end{aligned} \tag{1.10}$$

DEFINITION 1.4. a) A *canonical tensor field space* is a vector subspace of the mixed tensor algebra gotten by specifying a certain contravariant/covariant type (s,t), and (possibly) some list of symmetry/antisymmetry conditions.

b) A *canonical* LDO (CLDO) is an LDO of the form (1.8) carrying one canonical tensor space to another.

c) Suppose D is a CLDO carrying a canonical space $\mathscr{T}_1$ of (s_1,t_1)-tensor fields to a canonical space $\mathscr{T}_2$ of (s_2,t_2)-tensor fields. (s_i is the contravariant degree, or number of upper indices, and t_i is the covariant degree, or number of lower indices.) D is of *level* l, $l = 0,1,2,\ldots$, if under a uniform dilation $g = A^2\underline{g}$ of the metric, $0 < A \in \mathrm{R}$, D deforms according to

$$\underline{D} = A^{l-(t_2-s_2)+(t_1-s_1)}D. \tag{1.11}$$

REMARK 1.5. The objects in the definition are actually "CLDO schemes", or rules which assign a CLDO to each ΨR manifold of a given dimension n. One should really take the real vector space of such polynomial schemes and mod out by "universal identities" like the Leibniz rule

$$\nabla_\lambda(r^\alpha{}_\beta\varphi_\alpha) = r^\alpha{}_\beta\nabla_\lambda\varphi_\alpha + (\nabla_\lambda r^\alpha{}_\beta)\varphi_\alpha$$

and the Bianchi identity

$$\nabla_\alpha r^\alpha{}_\beta = \tfrac{1}{2}\nabla_\beta K.$$

Looking at how ∇ and R deform under uniform dilation of the metric ((1.5) and special cases of (1.9), (1.10)), we see that all terms in such an identity must be of the same level, and thus one has a space of level l CLDO schemes modulo universal identities. To avoid this sort of language, we shall adopt a consistent abuse of terminology in which we speak of "level l CLDO", instead of the more precise "equivalence classes of level l CLDO schemes."

REMARK 1.6. The level acts as a counter for the number of differentiations involved in a term of (1.8), R being viewed as a second derivative of g. In other words, we get the level by assigning a score of 1 for each ∇, 2 for each $R\otimes$, and 0 for each $g\otimes$, $g^1\otimes$, or contraction. For example, the operator

$$(\varphi_{\alpha\beta}) \to (\nabla_\alpha\nabla^\lambda\nabla_\beta\varphi_{\lambda\mu} + (\nabla_\alpha K)\varphi_{\mu\beta} + r^\lambda{}_\beta\nabla_\mu\varphi_{\alpha\lambda})$$

from 2-forms to (0,3)-tensors is of level 3.

(1.11) leads us to expect that under conformal change $g = \Omega^2\underline{g}$, the operator of Definition 1.4 (c) deforms according to

$$\underline{D}\varphi = \Omega^{l-(t_2-s_2)+(t_1-s_1)}(D\varphi + (\text{remainder})_0),$$

where the remainder is some sort of canonical expression which is linear in φ. However, we can also expect that for any $a \in \mathrm{R}$,

$$\underline{D}(\Omega^a\varphi) = \Omega^{a+l-(t_2-s_2)+(t_1-s_1)}(D\varphi + (\text{remainder})_a),$$

where the new remainder is the same kind of expression. The qualitative observation about these remainders which will be essential here is given in Definition 1.7 and Proposition 1.8 below.

DEFINITION 1.7. Let f be a C^∞ function on M. An *f-augmented* CLDO is an LDO $G(f)$ which carries some canonical space $\mathscr{T}_1$ of (s_1,t_1)-tensor fields to another canonical space $\mathscr{T}_2$ of (s_2,t_2)-tensor fields, and which is given by a formal expression

$$G(f) = \text{polynomial } (\nabla, R\otimes, df\otimes, g\otimes, g^1\otimes, \text{contractions}). \tag{1.12}$$

$G(f)$ is of level l if the effect of a uniform dilation $\underline{g} = A^2 g$, $0 < A \in \mathrm{R}$, is to make

$$\underline{G}(f) = A^{l-(t_2-s_2)+(t_1-s_1)}G(f).$$

$G(f)$ is *proper* if each term in some representation (1.12) for $G(f)$ contains a $df\otimes$.

PROPOSITION 1.8. *Let D be a level l* CLDO *carrying a canonical space $\mathscr{T}_1$ of (s_1,t_1)-tensor fields to a canonical space $\mathscr{T}_2$ of (s_2,t_2)-tensor fields. Under conformal change $\underline{g} = \Omega^2 g$, with $\omega = \log\Omega$, D deforms according to*

$$\underline{D}(\Omega^a\varphi) = \Omega^{a+l-(t_2-s_2)+(t_1-s_1)}(D\varphi + D^{+(a)}(\omega)\varphi), \tag{1.13}$$

where for each $a \in \mathrm{R}$, $D^{+(a)}(\omega)$ is a proper level l ω-augmented CLDO.

PROOF. We proceed by induction on the length of the "string"

$$\text{monomial } (\nabla, R\otimes, g\otimes, g^1\otimes, \text{contractions})$$

which gives a term of (1.8). The assertion is certainly true of the simple operators $g\otimes, g^1\otimes$, and contractions, and is true of ∇ and $R\otimes$ by (1.5), (1.9), and (1.10). In the case of ∇,

$$\begin{aligned}\underline{\nabla}_\alpha\varphi^{\underline{I}}{}_{\underline{J}} = \Omega^{1-|I|+|J|}(\nabla_\alpha\varphi^I{}_J &+ (|J|-|I|)\omega_\alpha\varphi^I{}_J + \\ &+ [(\delta_\alpha{}^\bullet\omega_\bullet - g_{\alpha\bullet}\omega^\bullet)\ \#\varphi]^I{}_J),\end{aligned} \tag{1.14}$$

where I and J are multi-indices of lengths $|I|$ and $|J|$; thus

$$\begin{aligned}\underline{\nabla}_\alpha(\Omega^a\varphi^{\underline{I}}{}_{\underline{J}}) = \Omega^{a+1-|I|+|J|}(\nabla_\alpha\varphi^I{}_J &+ (a+|J|-|I|)\omega_\alpha\varphi^I{}_J + \\ &+ [(\delta_\alpha{}^\bullet\omega_\bullet - g_{\alpha\bullet}\omega^\bullet)\ \#\varphi]^I{}_J).\end{aligned} \tag{1.15}$$

Written without the indices, this says that

$$\underline{\nabla}(\Omega^a\varphi) = \Omega^a(\nabla\varphi + \nabla^{+(a)}(\omega)\varphi), \tag{1.16}$$

where $\nabla^{+(a)}(\omega)$ has the desired form.

Suppose now that D is a level l operator satisfying the assertion of the theorem. Then the compositions $g\otimes \circ D$, $g^1\otimes \circ D$, and (any contraction) $\circ D$ are also level l operators satisfying the assertion. By (1.16), $\nabla \circ D$ is a level $l+1$ operator satisfying the assertion, and by (1.10), $R\otimes \circ D$ is a level $l+2$ operator satisfying the assertion.

REMARK 1.9. By keeping track of the number of derivatives at each stage of the proof, we find that the order of $D^{+(a)}(\omega)$ as a differential operator cannot exceed that of D.

An important special case of the above is gotten by considering zeroth order, level l CLDO D carrying (t,s)-tensor fields to scalar functions. Such an object is given, through the natural duality, by an (s,t)-tensor field T:

$$D\varphi = T^{\lambda_1 \dots \lambda_s}{}_{\mu_1 \dots \mu_t} \varphi^{\mu_1 \dots \mu_t}{}_{\lambda_1 \dots \lambda_s}.$$

Unraveling the definitions, one finds that T must be an R-linear combination of expressions

$$C(\underbrace{\nabla \dots \nabla}_{i_1 \text{ times}} R \otimes \dots \otimes \underbrace{\nabla \dots \nabla}_{i_m \text{ times}} R),$$

where $i_u \geqq 0$,

$$(2+i_1) + \dots + (2+i_m) = l,$$

and C is some composition of $g\otimes$, $g^1\otimes$, and contractions which leaves s free (unsummed) upper indices and t free lower indices. In other words, T is a level l *local* (ΨR) *invariant* (see, for example, [4]). Proposition 1.8 and Remark 1.9 imply that under conformal change $g = \Omega^2 \underline{g}$, T deforms according to

(1.17) $$\underline{T}^I{}_J = \Omega^l (T^I{}_J + T^+(\omega)^I{}_J),$$

where $T^+(\omega)$ is a *proper ω-augmented level l local invariant*, i.e., an R-linear combination of expressions

$$C(\underbrace{\nabla \dots \nabla}_{i_1 \text{ times}} R \otimes \dots \otimes \underbrace{\nabla \dots \nabla}_{i_m \text{ times}} R \otimes \underbrace{\nabla \dots \nabla}_{j_1 \text{ times}} \omega \otimes \dots \otimes \underbrace{\nabla \dots \nabla}_{j_p \text{ times}} \omega),$$

where C is as above, $p \geqq 1$, $m \geqq 0$, $i_u \geqq 0$, $j_v \geqq 1$, and

$$(2+i_1) + \dots + (2+i_m) + j_1 + \dots + j_p = l.$$

In this setting, (1.14) and (1.17) yield

$$\underline{\nabla}_\alpha \underline{T}^I{}_J = \Omega^{l+1}\{\nabla_\alpha(T^I{}_J + T^+(\omega)^I{}_J) + \\ + l\omega_\alpha(T^I{}_J + T^+(\omega)^I{}_J) + \\ + [(\delta_\alpha{}^\bullet \omega_\bullet - g_{\alpha\bullet}\omega^\bullet) \# (T + T^+(\omega))]^I{}_J\}. \tag{1.18}$$

c. *The equivalence of finite and infinitesimal conformal covariance.* The "nicest" CLDO D from the conformal deformation point of view are those for which

$$\underline{g} = \Omega^2 g \Rightarrow \underline{D}(\Omega^a\varphi) = \Omega^b D\varphi, \text{ all } \varphi, \tag{1.19}$$

for some $a,b \in \mathrm{R}$. Looking at the behavior of CLDO under uniform dilation (a special case of (1.13)), it is clear that (1.19) is possible only if D is of some level l, and

$$b = a + l - (t_2 - s_2) + (t_1 - s_1),$$

where s_i and t_i are as in Proposition 1.8.

DEFINITION 1.10. Let D be a level l CLDO, defined for ΨR manifolds of a fixed dimension n. Let $s_1, t_1, s_2, t_2, \mathscr{T}_1$, and $\mathscr{T}_2$ be as in Proposition 1.8. D is *conformally covariant of weight* w if (1.19) holds for $a = w - (t_1 - s_1)$ and $b = w + l - (t_2 - s_2)$; that is, if the operator $D^{+(w-(t_1-s_1))}(\omega)$ defined by (1.13) vanishes identically (for all M, g, and ω).

As an apparently weaker condition on D, one might demand only that for some a, $D^{+(a)}(\omega)$ vanishes "to first order in derivatives of ω". $D^{+(a)}(\omega)$ can be decomposed as

$$D^{+(a)}(\omega) = D^{1,a}(\omega) + \ldots + D^{l,a}(\omega), \tag{1.20}$$

where $D^{i,a}(\omega)$ is i-homogeneous in ω: if $u \in \mathrm{R}$, $D^{i,a}(ud\omega) = u^i D^{i,a}(\omega)$.

DEFINITION 1.11. a) Let D be a level l CLDO, defined for ΨR manifolds of a given dimension n. For $a \in \mathrm{R}$, we denote by $D'^{(a)}(\omega)$ the operator $D^{1,a}(\omega)$ defined by (1.20).

b) Let s_1 and t_1 be as in Proposition 1.8. D is *infinitesimally conformally covariant of weight* w if $D'^{(w-(t_1-s_1))}(\omega)$ is identically zero (for all (M,g) and $\omega \in C^\infty(M)$).

The equivalence of conformal covariance and infinitesimal conformal covariance is asserted in Corollary 1.14 below. The following slightly more general fact will be needed in Section 2.b.

PROPOSITION 1.12. *Let D be a level l CLDO, defined for ΨR manifolds of a given dimension n, and let s_i, t_i, and $\mathscr{T}_i$ be as in Proposition 1.8. Let (M,g) be a particular n-dimensional ΨR manifold, and let $0 < \Omega \in C^\infty(M)$. Suppose that*

$$(x,u) \to \Omega_u(x)$$

is a C^∞ *map* $M \times (-\varepsilon, 1+\varepsilon) \to \mathbf{R}^+$ *for some* $\varepsilon > 0$, *and that* $\Omega_0 \equiv 1$, $\Omega_1 = \Omega$. *Let* D_u (*respectively* $D_u^{+(a)}$, $D_u^{\prime(a)}$) *be* D (*respectively* $D^{+(a)}$, $D^{\prime(a)}$) *evaluated in* $(M, \Omega_u^{-2} g)$. *Set* $\omega_u = \log \Omega_u$, *and assume that for some* $a \in \mathbf{R}$,

$$D_{u_0}^{\prime(a)} \left(\frac{d\omega_u}{du}\bigg|_{u=u_0} \right) = 0 \tag{1.21}$$

for all $u_0 \in [0,1]$. *Then*

$$D_1(\Omega^a \varphi) = \Omega^{a+l-(t_2-s_2)+(t_1-s_1)} D_0 \varphi \tag{1.22}$$

for all $\varphi \in \mathscr{T}_1$.

PROOF. Let $b = a + l - (t_2 - s_2) + (t_1 - s_1)$. We want to show that given $\varphi \in \mathscr{T}_1$ and $x \in M$,

$$\frac{d}{du}\bigg|_{u=u_0} [(\Omega_u^{-b} D_u(\Omega_u^a \varphi) - D_0\varphi)(x)] \overset{?}{=} 0 \tag{1.23}$$

for all $u_0 \in [0,1]$. Since everything is C^∞ and the expression in square brackets is zero when $u = 0$, (1.22) will follow. Since $(D_0\varphi)(x)$ is independent of u, the left side of (1.23) is equal to

$$\frac{d}{du}\bigg|_{u=u_0} \left(\Omega_u^{-b} D_u(\Omega_u^a \varphi) - \Omega_{u_0}^{-b} D_{u_0}(\Omega_{u_0}^a \varphi) \right)(x)$$

$$= \frac{d}{dh}\bigg|_{h=0} \left(\Omega_{u_0+h}^{-b} D_{u_0+h}(\Omega_{u_0+h}^a \varphi) - \Omega_{u_0}^{-b} D_{u_0}(\Omega_{u_0}^a \varphi) \right)(x).$$

By (1.13) and (1.20), this last expression equals

$$\frac{d}{dh}\bigg|_{h=0} \left(\Omega_{u_0}^{-b} D_{u_0}^{+(a)} (\log \Omega_{u_0+h}/\Omega_{u_0}) \right)(x)$$

$$= \frac{d}{dh}\bigg|_{h=0} \left(\Omega_{u_0}^{-b} D_{u_0}^{+(a)} (\omega_{u_0+h} - \omega_{u_0}) \right)(x)$$

$$= \Omega_{u_0}^{-b}(x) \left(D_{u_0}^{\prime(a)} \left(\frac{d}{dh}\bigg|_{h=0} \omega_{u_0+h} \right) \right)(x)$$

$$= \Omega_{u_0}^{-b}(x) \left(D_{u_0}^{\prime(a)} \left(\frac{d\omega_u}{du}\bigg|_{u=u_0} \right) \right)(x).$$

Thus the assertion is reduced to (1.21).

COROLLARY 1.13. *Let* D, s_i, t_i, $\mathscr{T}_i$, (M,g), *and* Ω *be as above. Let* D_u

(respectively $D_u'^{(a)}$) be D (respectively $D'^{(a)}$) evaluated in $(M,\Omega^{-2u}g)$. If for some $a \in \mathbf{R}$, we have $D_u'^{(a)}(\omega) = 0$, $u \in [0,1]$, $\omega = \log\Omega$, then (1.22) *holds.*

PROOF. Set $\Omega_u = \Omega^u$ in Proposition 1.12.

COROLLARY 1.14. *A* CLDO *is conformally covariant of weight w if and only if it is infinitesimally conformally covariant of weight w.*

PROOF. Infinitesimal conformal covariance means we have the hypotheses of Corollary 1.13 for all M,g, and ω. The conclusion implies that

$$\underline{g} = \Omega^2 g \Rightarrow \underline{D}(\Omega^{w-(t_1-s_1)}\varphi) = \Omega^{w+l-(t_2-s_2)}D\varphi$$

for all M,g,Ω, and φ, and thus that the identity holds *universally* (i.e., it is a formal identity in the sense of Remark 1.5).

REMARK 1.15. Let D, s_i, t_i, and $\mathscr{T}_i$ be as above. If $\tilde{D}$ is another CLDO acting on $\mathscr{T}_2$, then

$$(\tilde{D} \circ D)'^{(a)} = \tilde{D} \circ D'^{(a)} + \tilde{D}'^{(a+l-(t_2-s_2)+(t_1-s_1))} \circ D. \tag{1.24}$$

Loosely speaking, we have defined a derivation on the algebra of CLDO whose kernel consists of the conformally covariant CLDO. We shall use this to construct explicit conformally covariant operators in Sections 1.d, 2.a, and 2.b.

REMARK 1.16. a) In the special case of a level l local invariant T, $T'^{(a)} \equiv T'$ is independent of a. Corollary 1.14 says that if T' is identically zero, then T is a universal conformal invariant:

$$\underline{g} = \Omega^2 g \Rightarrow \underline{T}^I{}_J = \Omega^l T^I{}_J.$$

b) The operation $T \to T'$ commutes with contractions, the raising and lowering of indices, and, in the case of type (1,1) local invariants, the # action.

We shall adopt the notations

$$T'(\omega) \equiv ((T')^I{}_J) \equiv ((T^I{}_J)'),$$

which suppress the dependence on ω.

REMARK 1.17. By (1.18), for T a level l local invariant,

$$\begin{aligned}(\nabla_\alpha T^I{}_J)' = \nabla_\alpha (T^I{}_J)' + l\omega_\alpha T^I{}_J + \\ + [(\delta_\alpha{}^\bullet \omega_\bullet - g_{\alpha\bullet}\omega^\bullet) \# T]^I{}_J.\end{aligned} \tag{1.25}$$

d. *Known conformally covariant operators.* When dealing with conformal deformation properties, it is convenient to write local invariants in terms of

$$J = \frac{1}{2(n-1)}K, \quad n > 1$$

(1.26) $$V^{\alpha}{}_{\beta} = \frac{1}{n-2}(r^{\alpha}{}_{\beta} - J\delta^{\alpha}{}_{\beta}), \quad n > 2$$

$$C^{\alpha\beta}{}_{\lambda\mu} = R^{\alpha\beta}{}_{\lambda\mu} + V_{\lambda}{}^{\beta}\delta^{\alpha}{}_{\mu} - V_{\mu}{}^{\beta}\delta^{\alpha}{}_{\lambda} + V_{\mu}{}^{\alpha}\delta^{\beta}{}_{\lambda} - V_{\lambda}{}^{\alpha}\delta^{\beta}{}_{\mu}, \quad n > 2$$

instead of the Riemann tensor R, the Ricci tensor r, and the scalar curvature K. C is the *Weyl conformal curvature tensor*; by (1.10),

$$g = \Omega^2 \underline{g} \Rightarrow \underline{C}^{\alpha\beta}{}_{\lambda\mu} = \Omega^2 C^{\alpha\beta}{}_{\lambda\mu};$$

in particular $C' = 0$, and

(1.27) $$(V^{\alpha}{}_{\beta})' = \omega^{\alpha}{}_{\beta}, \quad J' = \omega_{\alpha}{}^{\alpha}.$$

By the definitions and the Bianchi identity $r^{\alpha}{}_{\beta|\alpha} = \frac{1}{2}K_{\beta}$,

(1.28) $$V^{\alpha}{}_{\alpha} = J, \quad V^{\alpha}{}_{\beta|\alpha} = J_{\beta}.$$

Here and below, indices after the bar denote covariant derivatives, and the bar is not written in the case of functions (e.g., $J_{\beta} = J_{|\beta}$).

By (1.1) and (1.2), operators on forms built from d, δ, ∇, and R are CLDO. Conformal change does not affect the exterior derivative, so $\underline{d} = d$ and

$$\underline{d}(\Omega^a \varphi) = (\Omega^a(d + a\varepsilon(d\omega))\varphi.$$

Using this and the fact that δ and d are formal adjoints, we get

$$\underline{\delta}(\Omega^a \varphi) = \Omega^{a+2}(\delta + (n-2k-a)\iota(d\omega))\varphi \text{ on } k\text{-forms.}$$

(By (1.1)–(1.4), operators built from $d, \delta, \nabla, R, \varepsilon(d\omega), i(d\omega)$, and $d\omega$ are ω-augmented CLDO.) Thus

(1.29) $$d'^{(a)} = a\varepsilon, \quad \delta'^{(a)} = (n-2k-a)\iota \text{ on } k\text{-forms,}$$

where we have abbreviated $\varepsilon(d\omega)$ and $\iota(d\omega)$ by ε and ι. By Remark 1.15,

(1.30)
$$(\delta d)'^{(a)} = \delta d'^{(a)} + \delta'^{(a)} d = a\delta\varepsilon + (n-2(k+1)-a)\iota d \text{ on } k\text{-forms,}$$

and in particular,

(1.31) $$(\delta d)'^{(a)} = a\delta\varepsilon + (n-2-a)\iota d \text{ on functions.}$$

But if φ is a function, $\delta\varepsilon\varphi = -\nabla^{\lambda}(\omega_{\lambda}\varphi)$ and $\iota d\varphi = \omega^{\lambda}\varphi_{\lambda}$ by (1.1)–(1.4).

Thus

$$(\delta d)'^{((n-2)/2)} = -\frac{n-2}{2}\omega_\lambda{}^\lambda \text{ on functions.}$$

By (1.27) and Corollary 1.14, we have:

THEOREM 1.18 (see, for example, [19]). *The operator*

$$D_2 = \square + \frac{n-2}{2}J = \square + \frac{n-2}{4(n-1)}K$$

on functions in conformally covariant of weight $(n-2)/2$ *for* $n \neq 1$.

As usual, we set $\square = \delta d + d\delta$ on forms, and this reduces to $\square = \delta d$ on functions. Note that our sign convention gives $\square = -(\partial_1^2 + \ldots + \partial_n^2)$ in Euclidean $\mathbf{R}^n$.

The special case $a = 0$, $k = (n-2)/2$ of (1.30) is the familiar result on the conformal covariance of the Maxwell operator on "vector potentials":

THEOREM 1.19 (see, for example, [5]). *On* $(n-2)/2$*-forms for even* n, *the operator* δd *is conformally covariant of weight* $(n-2)/2$.

In 1981, the present author constructed a general second-order conformally covariant operator on forms of arbitrary order. On k-forms,

$$(1.32) \qquad (d\delta)'^{(a)} = d\delta'^{(a)} + d'^{(a+2)}\delta = (n-2k-a)d\iota + (a+2)\varepsilon\delta.$$

Formulas (1.30) and (1.32) suggest looking at the "weighted Laplacian"

$$\tilde{\square}_{n,k} = \frac{n-2k+2}{2}\delta d + \frac{n-2k-2}{2}d\delta,$$

since the apparently first-order differential operator

$$(\tilde{\square}_{n,k})'^{((n-2k-2)/2)} = \frac{n-2k+2}{2}\,\frac{n-2k-2}{2}\,(\iota d + d\iota + \delta\varepsilon + \varepsilon\delta)$$

is actually zeroth-order. To evaluate $\iota d + d\iota + \delta\varepsilon + \varepsilon\delta$, recall that if X is a vector field,

$$(1.33) \qquad L_X = \iota(X)d + d\iota(X) \text{ on forms,}$$

where L_X is the Lie derivative with respect to X. By the symmetry of ∇ ($\nabla_X Y - \nabla_Y X = [X, Y]$), $L_X - \nabla_X$ is a derivation on the mixed tensor algebra which annihilates functions, commutes with contractions, and agrees with $\nabla_\cdot X^\cdot\,\#$ on vector fields; by Remark 1.2 (d), (1.34) $$L_X - \nabla_X = \nabla_{\bullet} X^{\bullet} \# \text{ on arbitrary tensor fields.}$$

Furthermore, the ΨR condition $\nabla g = 0$ implies that $\nabla g^k = 0$ for all the metrics g^k on forms, and this gives

(1.35) $$\nabla_X^* = -\nabla_X - \nabla_\alpha X^\alpha \text{ on forms.}$$

By (1.34), (1.35), and the identity

(1.36) $$(A^{\bullet}{}_{\bullet}\#)^* = A_{\bullet}{}^{\bullet}\# \text{ on forms}$$

(recall Remark 1.2(e)),

(1.37) $$L_X + L_X^* = -\nabla_\alpha X^\alpha + \nabla_{\bullet} X^{\bullet} \# + \nabla^{\bullet} X_{\bullet} \# \text{ on forms.}$$

By the symmetry of the Hessian, $\omega_{\beta\alpha} = \omega_{\alpha\beta}$, we have

(1.38)
$$\iota(d\omega)d + d\iota(d\omega) + \delta\varepsilon(d\omega) + \varepsilon(d\omega)\delta = L_{d\omega} + L_{d\omega}^* = -\omega_\lambda{}^\lambda + 2\omega^{\bullet}{}_{\bullet}\#,$$

where $L_{d\omega}$ is the Lie derivative with respect to the vector field (ω^α) associated to $d\omega$ by g. Thus

$$(\tilde{\Box}_{n,k})'^{((n-2k-2)/2)} = \frac{n-2k+2}{2}\,\frac{n-2k-2}{2}(-\omega_\lambda{}^\lambda + 2\omega^{\bullet}{}_{\bullet}\#).$$

Now if T is a local invariant of tensor type (1,1), $(T^{\bullet}{}_{\bullet}\#)' = (T')^{\bullet}{}_{\bullet}\#$. Thus $(V\,\#)' = \omega^{\bullet}{}_{\bullet}\#$, and we have proved:

THEOREM 1.20 ([5]). *Let $\beta = (n-2k)/2$. The operator on k-forms*

$$D_{2,k} = (\beta+1)\delta d + (\beta-1)d\delta + (\beta+1)(\beta-1)(J - 2V\,\#)$$

is conformally covariant of weight $(n-2)/2$ for $n \neq 1,2$. $D_{2,k}$ specializes to $((n+2)/2)D_2$ on functions, and to the Maxwell operator δd on $(n-2)/2$-forms.

In 1983, S. Paneitz introduced a general fourth-order conformally covariant operator on functions, with leading term $\Box^2$. By (1.31), on functions,

$$\begin{aligned}(\delta d\delta d)'^{(a)} &= \delta d(\delta d)'^{(a)} + (\delta d)'^{(a+2)}\delta d \\ &= \delta d(a\delta\varepsilon + (n-2-a)\iota d) + ((a+2)\delta\varepsilon + (n-2-(a+2))\iota d)\delta d.\end{aligned}$$

Thus

$$(\delta d\delta d)'^{((n-4)/2)} = \frac{n-4}{2}[\delta d(L_{d\omega} + L_{d\omega}^*) + (L_{d\omega} + L_{d\omega}^*)\delta d] + $$
$$+ 2\delta(L_{d\omega} + L_{d\omega}^*)d$$

by (1.33), the fact that ι and δ annihilate functions, and the identities $dd = \delta\delta = 0$. This, plus the experience gained in defining $D_{2,k}$ above, suggests "correcting" $\delta d\delta d$ by adding

$$E_2 = \frac{n-4}{2}(\delta dZ + Z\delta d) + 2\delta Zd,$$

where $Z = J - 2V\#$, since $-Z' = L_{d\omega} + L^*_{d\omega}$. But

$$E_2'^{((n-4)/2)} = \frac{n-4}{2}[\delta dZ' + (\delta d)'^{(n/2)}Z + Z(\delta d)'^{((n-4)/2)} + Z'\delta d] + $$
$$+ 2[\delta Zd'^{((n-4)/2)} + \delta Z'd + \delta'^{(n/2)}Zd],$$

so that

$$(\Box^2 + E_2)'^{((n-4)/2)}$$
$$= \frac{n-4}{2}[(\delta d)'^{(n/2)}Z + Z(\delta d)'^{((n-4)/2)}] + 2[\delta Zd'^{((n-4)/2)} + \delta'^{(n/2)}Zd]$$
$$= -\frac{n-4}{2}\frac{n}{2}[Z'Z + ZZ'] + (n-4)[(\iota Zd - \iota dZ) + (\iota Zd - \iota dZ)^*],$$

using the fact that Z is pointwise self-adjoint. Now if φ is a function,

$$\iota Jd\varphi - \iota d(J\varphi) = -\iota\varepsilon(dJ)\varphi = -\omega^\lambda J_\lambda \varphi,$$
$$\iota V\#d\varphi - \iota dV\#\varphi = \iota V\#d\varphi = \omega^\lambda V^\alpha{}_\lambda \varphi_\alpha.$$

So

$$\iota Zd - \iota dZ = -\omega^\lambda J_\lambda - 2\nabla_{(\omega^\lambda V^\alpha{}_\lambda)}$$

and

$$(\iota Zd - \iota dZ) + (\iota Zd - \iota dZ)^* = -2\omega^\lambda J_\lambda + 2\nabla_\alpha(\omega^\lambda V^\alpha{}_\lambda)$$
$$= 2\omega^\lambda{}_\alpha V^\alpha{}_\lambda = (V^\lambda{}_\alpha V^\alpha{}_\lambda)'$$

by (1.35), (1.28), and (1.27). We conclude that

$$(\Box^2 + E_2)'^{((n-4)/2)} = -\frac{n-4}{2}\frac{n}{2}(Z^2)' + (n-4)(V^\alpha{}_\beta V^\beta{}_\alpha)'.$$

Because of Corollary 1.14, we have proved:

THEOREM 1.21 [20]. *The operator*

$$D_4 = \Box^2 + \frac{n-4}{2}(\Box Z + Z\Box) + 2\delta Zd + \frac{n-4}{2}\left(\frac{n}{2}Z^2 - 2V^\alpha{}_\beta V^\beta{}_\alpha\right)$$

on functions, where $Z = J - 2V\#$, *is conformally covariant of weight* $(n-4)/2$ *for* $n \neq 1,2$.

2. New high-order conformally covariant operators.

a. *A fourth-order operator on forms.* $n = 4$ *as a critical dimension.* We continue with the notational conventions of Section 1.d. By (1.30) and (1.32), on k-forms,

$$(2.1)\quad \begin{aligned} (\delta d\delta d)'^{(a)} &= \delta d(a\delta\varepsilon + [n-2(k+1)-a]\iota d) + \\ &\quad + ((a+2)\delta\varepsilon + [n-2(k+1)-(a+2)]\iota d)\delta d, \\ (d\delta d\delta)'^{(a)} &= d\delta([n-2k-a]d\iota + (a+2)\varepsilon\delta) + \\ &\quad + ([n-2k-(a+2)]d\iota + (a+4)\varepsilon\delta)d\delta. \end{aligned}$$

Choosing $a = \beta - 2$ (as before, $\beta = (n-2k)/2$), and looking at the operator

$$E_4 = (\beta+2)\delta d\delta d + (\beta-2)d\delta d\delta,$$

we get

$$\begin{aligned} E_4'^{(\beta-2)} &= (\beta+2)\{(\beta-2)\delta d\delta\varepsilon + \beta\delta d\iota d + \beta\delta\varepsilon\delta d + (\beta-2)\iota d\delta d\} + \\ &\quad + (\beta-2)\{(\beta+2)d\delta d\iota + \beta d\delta\varepsilon\delta + \beta d\iota d\delta + (\beta+2)\varepsilon\delta d\delta\} \\ &= (\beta+2)(\beta-2)(\square(L_{d\omega} + L_{d\omega}^*) + (L_{d\omega} + L_{d\omega}^*)\square) + \\ &\quad + 2(\beta+2)\delta(L_{d\omega} + L_{d\omega}^*)d - 2(\beta-2)d(L_{d\omega} + L_{d\omega}^*)\delta, \end{aligned}$$

since $\varepsilon d = -d\varepsilon$ and $\delta\iota = -\iota\delta$. This suggests "correcting" by

$$E_2 = (\beta+2)(\beta-2)(\square Z + Z\square) + 2(\beta+2)\delta Z d - 2(\beta-2)dZ\delta.$$

The "corrected" operator satisfies

$$(2.2)\quad \begin{aligned} (E_4 + E_2)'^{(\beta-2)} &= (\beta+2)(\beta-2)(\square'^{(\beta)}Z + Z\square'^{(\beta-2)}) + \\ &\quad + 2(\beta+2)(\delta Z d'^{(\beta-2)} + \delta'^{(\beta)}Zd) - \\ &\quad - 2(\beta-2)(dZ\delta'^{(\beta-2)} + d'^{(\beta+2)}Z\delta) \\ &= (\beta+2)(\beta-2)\beta(\mathscr{L}Z + Z\mathscr{L}) + \\ &\quad + 2(\beta+2)(\beta-2)[-\iota dZ + Zd\iota + \iota Zd - dZ\iota - \\ &\qquad - Z\delta\varepsilon + \varepsilon\delta Z + \delta Z\varepsilon - \varepsilon Z\delta], \end{aligned}$$

where $\mathscr{L} = L_{d\omega} + L_{d\omega}^*$. To evaluate the expression in square brackets, and for future purposes, it is convenient to have the following algebraic lemmas.

LEMMA 2.1. *Let $\mathscr{A} = (\mathscr{A}^n)$ be a* **Z***-graded algebra. Suppose that D is a derivation of degree $|D|$ on $\mathscr{A}$* (i.e., *carrying $\mathscr{A}^n \to \mathscr{A}^{n+|D|}$*), *that E is a derivation of degree $|E|$, that A is an antiderivation of degree $|A|$, and that B is an antiderivation of degree $|B|$. Then:*

a) *$[D,E] = DE - ED$ is a derivation of degree $|D| + |E|$.*
b) *If $|A|$ and $|B|$ are even, $[A,B]$ is a derivation of degree $|A| + |B|$.*
c) *If $|A|$ and $|B|$ are odd, $(A,B) = AB + BA$ is a derivation of degree $|A| + |B|$.*
d) *If $|D|$ is even, $[A,D]$ is an antiderivation of degree $|A| + |D|$.*

PROOF. The proof is just straight computation. For example, to prove (d), suppose that $\varphi \in \mathscr{A}^n$ and $\psi \in \mathscr{A}^m$. Then

$$\begin{aligned} AD(\varphi\psi) &= A(\varphi D\psi + (D\varphi)\psi) \\ &= (-1)^n \varphi AD\psi + (A\varphi)D\psi + (-1)^{n+|D|}(D\varphi)A\psi + (AD\varphi)\psi, \\ DA(\varphi\psi) &= D((-1)^n \varphi A\psi + (A\varphi)\psi) \\ &= (-1)^n \varphi DA\psi + (-1)^n (D\varphi)A\psi + (A\varphi)D\psi + (DA\varphi)\psi. \end{aligned}$$

Thus if $|D|$ is even,

$$[A,D](\varphi\psi) = (-1)^n \varphi [A,D]\psi + ([A,D]\varphi)\psi.$$

LEMMA 2.2. *If A is a symmetric type* (1,1) *tensor field, then on forms,*

$$\begin{aligned} &-\iota dA\# + A\#d\iota + \iota A\#d - dA\#\iota - A\#\delta\varepsilon + \varepsilon\delta A\# + \delta A\#\varepsilon - \varepsilon A\#\delta \\ &\quad = -\omega^\lambda A^\mu{}_{\lambda|\mu} - \omega^\lambda{}_\mu A^\mu{}_\lambda - \omega^\lambda(2A^\bullet{}_{\bullet|\lambda} - A^\bullet{}_{\lambda|\bullet} - A_{\bullet\lambda|}{}^\bullet)\# + \\ &\qquad + (\omega^\bullet{}_\mu A^\mu{}_\bullet + \omega_{\bullet\mu} A^{\mu\bullet})\#. \end{aligned}$$

PROOF. First look at the operator

$$S = -\iota dA\# + A\#d\iota + \iota A\#d - dA\#\iota = ([A\#,d],\iota);$$

ultimately, we want to evaluate $S + S^*$. Since $A\#$ is a derivation of degree 0 on the Grassmann algebra, d is an antiderivation of degree 1, and ι is an antiderivation of degree -1, S is by Lemma 2.1 a derivation of degree 0 on forms. On functions,

$$S = \iota A\#d = \nabla_X, \; X^\alpha = \omega^\lambda A^\alpha{}_\lambda.$$

Thus $S - \nabla_X$ is a derivation of degree 0 on the Grassmann algebra annihilating functions. By Remark 1.2(f), $S-\nabla_X = B\#$ for some (1,1) tensor B which can be determined from its effect on closed one-forms. If η is a closed one-form, i.e. $\eta_{\mu|\lambda} = \eta_{\lambda|\mu}$, then

$$\begin{aligned}(S\eta)_\alpha &= ((-\iota dA\# + A\# d\iota)\eta)_\alpha \\ &= -\omega^\lambda(A^\mu{}_{\alpha|\lambda}\eta_\mu - A^\mu{}_\lambda\eta_{\mu|\alpha} - A^\mu{}_{\lambda|\alpha}\eta_\mu) + \omega^\lambda{}_\mu A^\mu{}_\alpha\eta_\lambda.\end{aligned}$$

This identifies B as

$$B^\alpha{}_\beta = -\omega^\lambda(A^\alpha{}_{\beta|\lambda} - A^\alpha{}_{\lambda|\beta}) + \omega^\alpha{}_\mu A^\mu{}_\beta.$$

Thus

$$\begin{aligned}S+S^* &= \nabla_X + \nabla_X^* + B^\bullet{}_\bullet\# + B_\bullet{}^\bullet\# \\ &= -\nabla_\alpha(\omega^\lambda A^\alpha{}_\lambda) - \omega^\lambda(2A^\bullet{}_{\bullet|\lambda} - A^\bullet{}_{\lambda|\bullet} - A_{\bullet\lambda|}{}^\bullet)\# + \\ &\quad + (\omega^\bullet{}_\mu A^\mu{}_\bullet + \omega_{\bullet\mu}A^{\mu\bullet})\#,\end{aligned}$$

by (1.7), (1.35), and the symmetry of A, as desired.

In the case of immediate interest (recall (2.2)), we set $A=-2V$ in Lemma 2.2 to get

(2.3)

$$\begin{aligned}&-2(-\iota dV\# + V\# d\iota + \iota V\# d - dV\#\iota - V\#\delta\varepsilon + \varepsilon\delta V\# + \delta V\#\varepsilon - \varepsilon V\#\delta) \\ &\quad = 2(\omega^\lambda J_\lambda + \omega^\lambda{}_\alpha V^\alpha{}_\lambda) + 2\omega^\lambda(2V^\bullet{}_{\bullet|\lambda} - V^\bullet{}_{\lambda|\bullet} - V_{\bullet\lambda|}{}^\bullet)\# - \\ &\quad\quad -2(\omega^\bullet{}_\mu V^\mu{}_\bullet + \omega_{\bullet\mu}V^{\mu\bullet})\#,\end{aligned}$$

since $V^\alpha{}_{\lambda|\alpha} = J_\lambda$. Furthermore,

$$-\iota dJ + Jd\iota + \iota Jd - dJ\iota = -\iota\varepsilon(dJ) - \varepsilon(dJ)\iota = -\omega^\lambda J_\lambda$$

is a self-adjoint multiplication operator, so

(2.4) $\quad -\iota dJ + Jd\iota + \iota Jd - dJ\iota - J\delta\varepsilon + \varepsilon\delta J + \delta J\varepsilon - \varepsilon J\delta = -2\omega^\lambda J_\lambda.$

Combining (2.3) and (2.4), we get

$$\begin{aligned}&-\iota dZ + Zd\iota + \iota Zd - dZ\iota - Z\delta\varepsilon + \varepsilon\delta Z + \delta Z\varepsilon - \varepsilon Z\delta \\ &\quad = 2\omega^\lambda{}_\alpha V^\alpha{}_\lambda + 2\omega^\lambda(2V^\bullet{}_{\bullet|\lambda} - V^\bullet{}_{\lambda|\bullet} - V_{\bullet\lambda|}{}^\bullet)\# - 2(\omega^\bullet{}_\mu V^\mu{}_\bullet + \omega_{\bullet\mu}V^{\mu\bullet})\# \\ &\quad = (V^\lambda{}_\alpha V^\alpha{}_\lambda)' - 2(V^\bullet{}_\mu V^\mu{}_\bullet)'\# + 2(2V^\bullet{}_{\bullet|\lambda} - V^\bullet{}_{\lambda|\bullet} - V_{\bullet\lambda|}{}^\bullet)\#.\end{aligned}$$

The problem is now to "integrate" the expression

$$F^\alpha{}_\beta = \omega^\lambda(2V^\alpha{}_{\beta|\lambda} - V^\alpha{}_{\lambda|\beta} - V_{\beta\lambda|}{}^\alpha);$$

i.e., to find a tensor field $(G^\alpha{}_\beta)$ with $G' = F$. For this and other purposes (see Section 2.b below), we need the following lemma.

LEMMA 2.3. *Let $T = (T^\alpha{}_\beta)$ be a type (1,1) level l local invariant. Then*

(2.6) $$(\nabla^\lambda\nabla_\lambda T^\alpha{}_\beta)' = \nabla^\lambda\nabla_\lambda (T^\alpha{}_\beta)' + (2l+2-n)\omega^\lambda T^\alpha{}_{\beta|\lambda} + l\omega_\lambda{}^\lambda T^\alpha{}_\beta - 2\omega_\lambda T^\lambda{}_{\beta|}{}^\alpha - 2\omega^\lambda T^\alpha{}_{\lambda|\beta} + 2\omega^\alpha T^\lambda{}_{\beta|\lambda} + 2\omega_\beta T^\alpha{}_{\lambda|}{}^\lambda.$$

Let S be a scalar level l local invariant. Then

(2.7) $$(\nabla_\beta\nabla^\alpha S)' = \nabla_\beta\nabla^\alpha S' + (l+1)(\omega^\alpha S_\beta + \omega_\beta S^\alpha) + l\omega^\alpha{}_\beta S - \omega^\lambda S_\lambda \delta^\alpha{}_\beta.$$

PROOF. This is just direct iterated calculation with (1.25).

By Lemma 2.3,

(2.8) $$(V^\alpha{}_{\beta|\lambda}{}^\lambda)' = \omega^\alpha{}_{\beta\lambda}{}^\lambda + (6-n)\omega^\lambda V^\alpha{}_{\beta|\lambda} + 2\omega_\lambda{}^\lambda V^\alpha{}_\beta - 2\omega_\lambda V^\lambda{}_{\beta|}{}^\alpha - 2\omega^\lambda V^\alpha{}_{\lambda|\beta} + 2\omega^\alpha J_\beta + 2\omega_\beta J^\alpha$$

and

(2.9) $$(J^\alpha{}_\beta)' = \omega_\lambda{}^{\lambda\alpha}{}_\beta + 3(\omega^\alpha J_\beta + \omega_\beta J^\alpha) + 2\omega^\alpha{}_\beta J - \omega^\lambda J_\lambda \delta^\alpha{}_\beta.$$

By the Ricci identity (1.6) and the Bianchi identity $R^\alpha{}_{\lambda\beta\gamma|}{}^\gamma = r_\beta{}^\alpha{}_{|\lambda} - r_{\beta\lambda|}{}^\alpha$,

(2.10)
$$\begin{aligned}\omega^\alpha{}_{\beta\lambda}{}^\lambda - \omega_\lambda{}^{\lambda\alpha}{}_\beta &= 2R^\alpha{}_\mu{}^\lambda{}_\beta \omega^\mu{}_\lambda + (r_{\beta\mu|}{}^\alpha - r_\beta{}^\alpha{}_{|\mu} + r_\mu{}^\alpha{}_{|\beta})\omega^\mu + \\ &\quad + r^{\mu\alpha}\omega_{\mu\beta} + r^\mu{}_\beta \omega_\mu{}^\alpha \\ &= 2C^\alpha{}_\mu{}^\lambda{}_\beta \omega^\mu{}_\lambda - 2V^\lambda{}_\mu \omega^\mu{}_\lambda \delta^\alpha{}_\beta + n(V_{\beta\mu}\omega^{\mu\alpha} + V^{\mu\alpha}\omega_{\beta\mu}) + \\ &\quad + (n-2)(V_{\beta\lambda|}{}^\alpha - V_\beta{}^\alpha{}_{|\lambda} + V_\lambda{}^\alpha{}_{|\beta})\omega^\lambda + \\ &\quad + J^\alpha\omega_\beta + J_\beta\omega^\alpha - J_\lambda\omega^\lambda\delta^\alpha{}_\beta + 2J\omega^\alpha{}_\beta.\end{aligned}$$

Combining (2.8), (2.9), and (2.10), we get

(2.11) $$\begin{aligned}(V^\alpha{}_{\beta|\lambda}{}^\lambda - J^\alpha{}_\beta)' &= (n-4)\omega^\lambda(V_{\beta\lambda|}{}^\alpha + V^\alpha{}_{\lambda|\beta} - 2V^\alpha{}_{\beta|\lambda}) + \\ &\quad + 2C^\alpha{}_\lambda{}^\mu{}_\beta \omega^\lambda{}_\mu - 2V^\mu{}_\lambda \omega^\lambda{}_\mu \delta^\alpha{}_\beta + \\ &\quad + n(V^\alpha{}_\lambda \omega^\lambda{}_\beta + V^\lambda{}_\beta \omega_\lambda{}^\alpha).\end{aligned}$$

(2.11) above can be written more intelligibly in terms of the *Lichnerowicz Laplacian* [17], defined on covariant tensor fields by

(2.12) $$(LT)_{\alpha_1\ldots\alpha_p} = -T_{\alpha_1\ldots\alpha_p|\lambda}{}^\lambda + \sum_{s=1}^p r^\mu{}_{\alpha_s} T_{\alpha_1\ldots\underset{\uparrow}{\mu}\ldots\alpha_p} -$$

α_s place

$$-2\sum_{t<s} R^\mu{}_{\alpha_t}{}^\lambda{}_{\alpha_s} T_{\alpha_1\ldots\underset{\uparrow}{\mu}\ldots\underset{\uparrow}{\lambda}\ldots\alpha_p},$$

α_t place α_s place

and extended to all tensor fields by the requirement that L commute with raising and lowering of indices, and with contractions. Now

$$\begin{aligned}(LV)^\alpha{}_\beta &= -V^\alpha{}_{\beta|\lambda}{}^\lambda + V^\lambda{}_\beta r_\lambda{}^\alpha + V^\alpha{}_\lambda r^\lambda{}_\beta - 2R^\alpha{}_{\lambda\beta}{}^\mu V^\lambda{}_\mu \\ &= -V^\alpha{}_{\beta|\lambda}{}^\lambda + 2nV^\lambda{}_\beta V^\alpha{}_\lambda - 2C^\alpha{}_{\lambda\beta}{}^\mu V^\lambda{}_\mu - 2V^\lambda{}_\mu V^\mu{}_\lambda \delta^\alpha{}_\beta,\end{aligned}$$

and this simplifies (2.11) to

$$\begin{aligned}((LV)^\alpha{}_\beta + J^\alpha{}_\beta)' = (n-4)\omega^\lambda(2V^\alpha{}_{\beta|\lambda} - V_{\beta\lambda|}{}^\alpha - V^\alpha{}_{\lambda|\beta}) - \\ -(V^\mu{}_\lambda V^\lambda{}_\mu \delta^\alpha{}_\beta)' + n(V^\lambda{}_\beta V^\alpha{}_\lambda)'.\end{aligned} \tag{2.13}$$

Combined with (2.5), this says that

$$\begin{aligned}(n-4)(-\iota dZ + Zd\iota + \iota Zd \\ - dZ\iota - Z\delta\varepsilon + \varepsilon\delta Z + \delta Z\varepsilon - \varepsilon Zd) = (n-4)(V^\lambda{}_\mu V^\mu{}_\lambda)' + 2Y'\#,\end{aligned} \tag{2.14}$$

where

$$Y^\alpha{}_\beta = (LV)^\alpha{}_\beta + J^\alpha{}_\beta + V^\mu{}_\lambda V^\lambda{}_\mu \delta^\alpha{}_\beta - 2(n-2)V^\lambda{}_\beta V^\alpha{}_\lambda. \tag{2.15}$$

(2.2) can now be rewritten

$$\begin{aligned}(E_4+E_2)'^{(\beta-2)} = -(\beta+2)(\beta-2)\beta(Z^2)' + \\ + 2(\beta+2)(\beta-2)\Big[(V^\lambda{}_\mu V^\mu{}_\lambda)' + \frac{2}{n-4}Y'\#\Big],\end{aligned}$$

and this gives the following theorem.

THEOREM 2.4. *Suppose $n \neq 1,2,4$. The operator*

$$\begin{aligned}D_{4,k} = (\beta+2)\delta d\delta d + (\beta-2)d\delta d\delta + \\ + (\beta+2)(\beta-2)(\Box Z + Z\Box) + 2(\beta+2)\delta Zd - 2(\beta-2)dZ\delta + \\ + (\beta+2)(\beta-2)\beta Z^2 - 2(\beta+2)(\beta-2)\Big[V^\lambda{}_\mu V^\mu{}_\lambda + \frac{2}{n-4}Y\#\Big],\end{aligned}$$

on k-forms, where $\beta = (n-2k)/2$, $Z = J - 2V\#$, and Y is given by (2.15), *is conformally covariant of weight* $(n-4)/2$.

Note the introduction of $n = 4$ as a *critical dimension* for $D_{4,k}$ $(k \neq 0)$; like 1 and 2 for $D_{2,k}$ and D_4, a dimension in which the operator is undefined. The argument above *does* give a result in dimension 4, however. By (2.14), the (1,1)-tensor field Y is a conformal invariant in dimension 4:

THEOREM 2.5. *For $n = 4$, the tensor field*

$$Y^\alpha{}_\beta = (LV)^\alpha{}_\beta + J^\alpha{}_\beta + V^\mu{}_\lambda V^\lambda{}_\mu \delta^\alpha{}_\beta - 4V^\lambda{}_\beta V^\alpha{}_\lambda$$

is a conformal invariant in the sense that

(2.16) $$g = \Omega^2 \underline{g} \Rightarrow \underline{Y}^{\alpha}{}_{\underline{\beta}} = \Omega^4 Y^{\alpha}{}_{\beta}.$$

Y is traceless, but the norm-squared of Y is a scalar conformal invariant:

(2.17) $$g = \Omega^2 \underline{g} \Rightarrow \underline{Y}^{\alpha}{}_{\underline{\beta}} \underline{Y}^{\beta}{}_{\underline{\alpha}} = \Omega^8 Y^{\alpha}{}_{\beta} Y^{\beta}{}_{\alpha}.$$

PROOF. (2.16) follows from (2.14) and Remark 1.16(a). That Y is traceless follows from the fact that L commutes with contractions:

$$(LV)^{\alpha}{}_{\alpha} = LJ = -J_{\alpha}{}^{\alpha}.$$

(2.17) follows from (2.16).

Thus one can take the view that there is always a *level* 4 conformally covariant operator in dimension $n > 2$, but that the *order* of this operator collapses from 4 (for $n \neq 4$) to 0 (for $n = 4$, the operator being $Y \#$).

While this paper was being written, it was pointed out to the author by C. R. Graham that, in dimension 4, the tensor field Y and its conformal deformation laws (2.16) and (2.17) were discovered by Bach [2] in 1921. The operator $D_{4,k}$ can in some sense be regarded as a generalization of the Bach tensor to arbitrary dimension $n > 2$.

b. *A sixth-order operator on function. An apparent geometric obstruction.* By (2.1), on functions,

(2.18)
$$(\Box^2)'^{(a)} = a\Box\delta\varepsilon + (n-2-a)\Box\iota d + (a+2)\delta\varepsilon\Box + (n-4-a)\iota d\Box.$$

Together with (1.31), this implies that

$$(\Box^3)'^{(a)} = \Box[a\Box\delta\varepsilon + (n-2-a)\Box\iota d + (a+2)\delta\varepsilon\Box + (n-4-a)\iota d\Box] + \\ + [(a+4)\delta\varepsilon + (n-6-a)\iota d]\Box^2.$$

Choosing $a = (n-6)/2$ gives

$$(\Box^3)'^{((n-6)/2)} = \frac{n-6}{2}[\mathscr{L}\Box^2 + \Box^2\mathscr{L}] + \\ + 4[\Box\delta\mathscr{L}d + \delta\mathscr{L}d\Box] + \frac{n-10}{2}\Box\mathscr{L}\Box,$$

where, as before, $\mathscr{L} = L_{d\omega} + L^*_{d\omega}$. This suggests "correcting" by

(2.19)
$$E_4 = \frac{n-6}{2}[Z\Box^2 + \Box^2 Z] + 4[\Box\delta Z d + \delta Z d\Box] + \frac{n-10}{2}\Box Z\Box.$$

(1.29), (1.31). and (2.18) then give

$$
\begin{aligned}
(\Box^3 + E_4)'^{((n-6)/2)}
&= \left(\frac{n-6}{2}\right)^2 (Z\Box\mathscr{L} + \mathscr{L}\Box Z) + \\
&\quad + \frac{n-6}{2}\frac{n-2}{2}(Z\mathscr{L}\Box + \Box\mathscr{L}Z + \mathscr{L}Z\Box + \Box Z\mathscr{L}) + \\
&\quad + 2(n-6)(Z\delta\mathscr{L}d + \delta\mathscr{L}dZ + \mathscr{L}\delta Zd + \delta Zd\mathscr{L}) + \\
&\quad + 2(n-6)\{(-\iota dZ + \iota Zd - Z\delta\varepsilon + \delta Z\varepsilon)\Box + \\
&\quad + \Box(-\iota dZ + \iota Zd - Z\delta\varepsilon + \delta Z\varepsilon)\} + \\
&\quad + 8\delta(\mathscr{L}Z + Z\mathscr{L} - \iota dZ + Zd\iota - dZ\iota - Z\delta\varepsilon + \varepsilon\delta Z - \varepsilon Z\delta)d \\
&= \left(\frac{n-6}{2}\right)^2 (Z\Box\mathscr{L} + \mathscr{L}\Box Z) + \\
&\quad + \frac{n-6}{2}\frac{n-2}{2}(Z\mathscr{L}\Box + \Box\mathscr{L}Z + \mathscr{L}Z\Box + \Box Z\mathscr{L}) + \\
&\quad + 2(n-6)(Z\delta\mathscr{L}d + \delta\mathscr{L}dZ + \mathscr{L}\delta Zd + \delta Zd\mathscr{L}) + \\
&\quad + 2(n-6)(\mathscr{P}\Box + \Box\mathscr{P}) + 8\delta(\mathscr{L}Z + Z\mathscr{L} + \mathscr{P})d,
\end{aligned}
$$

where

$$\mathscr{P} = -\iota dZ + Zd\iota + \iota Zd - dZ\iota - Z\delta\varepsilon + \varepsilon\delta Z + \delta Z\varepsilon - \varepsilon Z\delta.$$

By (2.14), if $n \neq 4$, $\mathscr{P} = -P'$, where

$$-P = V^\lambda{}_\mu V^\mu{}_\lambda + \frac{2}{n-4} Y\#,$$

and Y is as in (2.15). All this suggests "correcting" $\Box^3 + E_4$ by

$$
\begin{aligned}
\text{(2.20)}\qquad E_2 &= \left(\frac{n-6}{2}\right)^2 Z\Box Z + \frac{n-6}{2}\frac{n-2}{2}(Z^2\Box + \Box Z^2) + \\
&\quad + 2(n-6)(Z\delta Zd + \delta ZdZ) + \\
&\quad + 2(n-6)(P\Box + \Box P) + 8\delta(Z^2 + P)d.
\end{aligned}
$$

For the corrected operator,

$$
\begin{aligned}
\text{(2.21)}\qquad & (\Box^3 + E_4 + E_2)'^{((n-6)/2)} \\
&= \frac{n-6}{2}\frac{n-2}{2}\frac{n+2}{2}(Z\mathscr{L}Z + Z^2\mathscr{L} + \mathscr{L}Z^2) + \\
&\quad + (n-6)(n-2)(Z\mathscr{P} + \mathscr{P}Z + \mathscr{L}P + P\mathscr{L}) + \\
&\quad + 4(n-6)(-\iota ZdZ + \iota Z^2 d - Z\delta Z\varepsilon + \delta Z^2\varepsilon) +
\end{aligned}
$$

$$+4(n-6)(-\iota dP+P\iota d+\iota Pd-P\delta\varepsilon+\delta\varepsilon P+\delta P\varepsilon)$$
$$=-\frac{n-6}{2}\frac{n-2}{2}\frac{n+2}{2}(Z^3)'-(n-6)(n-2)(ZP+PZ)'+4(n-6)(\mathscr{B}+\mathscr{C}),$$

where

$$\mathscr{B}=-\iota ZdZ+\iota Z^2d-Z\delta Z\varepsilon+\delta Z^2\varepsilon,$$
$$\mathscr{C}=-\iota dP+P\iota d+\iota Pd-P\delta\varepsilon+\delta\varepsilon P+\delta P\varepsilon.$$

To simplify $\mathscr{B}+\mathscr{C}$, we calculate directly using (1.35) to get

(2.22)
$$\mathscr{B}=2\omega^\mu V^\lambda{}_\mu J_\lambda+2\omega^\mu{}_\lambda V^\lambda{}_\mu J-4\omega^\beta V^\mu{}_{\beta|\lambda}V^\lambda{}_\mu-4\omega^\beta{}_\lambda V^\mu{}_\beta V^\lambda{}_\mu,$$
$$\mathscr{C}=6\omega^\lambda V^\alpha{}_\beta V^\beta{}_{\alpha|\lambda}+V^\alpha{}_\beta V^\beta{}_\alpha\omega_\lambda{}^\lambda+\frac{2}{n-4}(\omega^\mu Y^\lambda{}_{\mu|\lambda}+\omega^\mu{}_\lambda Y^\lambda{}_\mu).$$

To evaluate $Y^\lambda{}_{\mu|\lambda}$, note that the Lichnerowicz Laplacian L commutes with the divergence operator

$$\operatorname{div}:(T_{\alpha_1\dots\alpha_p})\to(\nabla^\lambda T_{\lambda\alpha_2\dots\alpha_p})$$

(see [17]), so that

$$(LV)^\lambda{}_{\mu|\lambda}=(\operatorname{div}LV)_\mu=(L\operatorname{div}V)_\mu$$
$$=(L(\nabla J))_\mu=-J_{\mu\lambda}{}^\lambda+r^\lambda{}_\mu J_\lambda$$

by (1.28). This and the symmetry of the Hessian, $J_{\mu\lambda}=J_{\lambda\mu}$, give

$$Y^\lambda{}_{\mu|\lambda}=-(n-2)V^\lambda{}_\mu J_\lambda+JJ_\mu+2V^\alpha{}_\beta V^\beta{}_{\alpha|\mu}-2(n-2)V^\lambda{}_{\mu|\alpha}V^\alpha{}_\lambda.$$

Returning to (2.22), we have

(2.23)
$$\mathscr{C}=6\omega^\lambda V^\alpha{}_\beta V^\beta{}_{\alpha|\lambda}+V^\alpha{}_\beta V^\beta{}_\alpha\omega_\lambda{}^\lambda+$$
$$+\frac{2}{n-4}\{\omega^\mu[-(n-2)V^\lambda{}_\mu J_\lambda+JJ_\mu+2V^\alpha{}_\beta V^\beta{}_{\alpha|\mu}-$$
$$-2(n-2)V^\lambda{}_{\mu|\alpha}V^\alpha{}_\lambda]+\omega^\mu{}_\lambda Y^\lambda{}_\mu\}.$$

Since by (2.13)

$$(Y^\alpha{}_\beta)'=-\frac{n-4}{2}\{4V^\alpha{}_\lambda\omega^\lambda{}_\beta-2\omega^\lambda(2V^\alpha{}_{\beta|\lambda}-V^\alpha{}_{\lambda|\beta}-V_{\beta\lambda|}{}^\alpha)\},$$

(2.22) and (2.23) imply that

$$\mathscr{B}+\mathscr{C}=\frac{2}{n-4}\{(V^\alpha{}_\beta Y^\beta{}_\alpha)'+(V^\alpha{}_\beta V^\beta{}_\alpha J)'+$$
$$+(n-2)\omega^\lambda[V^\alpha{}_\beta V^\beta{}_{\alpha|\lambda}-2V^\alpha{}_{\lambda|\beta}V^\beta{}_\alpha]-2\omega^\lambda V^\alpha{}_\lambda J_\alpha+\omega^\lambda JJ_\lambda\}$$

$$= \frac{2}{n-4}\{(V^{\alpha}{}_{\beta}Y^{\beta}{}_{\alpha})' + (V^{\alpha}{}_{\beta}V^{\beta}{}_{\alpha}J)'\} +$$

$$+ \frac{2}{(n-4)(n-2)}\omega^{\lambda}r^{\beta}{}_{\alpha}(r^{\alpha}{}_{\beta|\lambda} - 2r^{\alpha}{}_{\lambda|\beta}),$$

or, returning to (2.21), that if

$$E_0 = \frac{n-6}{2}\frac{n-2}{2}\frac{n+2}{2}Z^3 + (n-6)(n-2)(ZP+PZ) -$$

$$-\frac{8(n-6)}{n-4}(V^{\alpha}{}_{\beta}Y^{\beta}{}_{\alpha} + V^{\alpha}{}_{\beta}V^{\beta}{}_{\alpha}J),$$

then

(2.24)

$$(\Box^3 + E_4 + E_2 + E_0)'^{((n-6)/2)} = \frac{8(n-6)}{(n-4)(n-2)}\omega^{\lambda}r^{\beta}{}_{\alpha}(r^{\alpha}{}_{\beta|\lambda} - 2r^{\alpha}{}_{\lambda|\beta}).$$

Now the ω-augmented scalar local invariant

$$N(\omega) = \omega^{\lambda}r^{\beta}{}_{\alpha}(r^{\alpha}{}_{\beta|\lambda} - 2r^{\alpha}{}_{\lambda|\beta})$$

is not realizable as $T'(\omega)$ for any scalar local invariant T (which would necessarily be of level 6). To see this, we define the natural "second derivative" on the space of level l local invariants $(T^I{}_J)$ of a certain tensor type. Recalling that the "first derivative" satisfied

$$(T^I{}_J)'(\omega)(x) = \frac{d}{d\lambda}\bigg|_{\lambda=0}[e^{-\lambda l\omega}(\underline{T}^{\underline{I}}{}_{\underline{J}})_{e^{-2\lambda\omega}g} - (T^I{}_J)_g](x),$$

for $x \in M$, we define this second derivative by

(2.25)

$$(T^I{}_J)''(\omega,\xi)(x)$$

$$= \left(\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right)\bigg|_{(\lambda,\mu)=(0,0)}[e^{-\lambda l\omega}e^{-\mu l\xi}(\underline{T}^{\underline{I}}{}_{\underline{J}})_{e^{-2\lambda\omega}e^{-2\mu\xi}g} - (T^I{}_J)_g](x).$$

In practice the second derivative can be calculated algebraically by extending the first derivative $'$ to ξ-sugmented local invariants $G(\xi)$, as follows. We require that $G(\xi) \to G(\xi)'(\omega)$ be a derivation annihilating $d\xi$, agreeing with $T \to T'(\omega)$ on ordinary local invariants, and satisfying (1.25). These conditions are consistent, and lead to

$$(T^I{}_J)''(\omega,\xi) = (T^I{}_J)'(\xi)'(\omega).$$

By (2.25) and the equality of mixed partials, $(T^I{}_J)''(\omega,\xi)$ must be symmetric in ω and ξ. This gives a necessary condition for an ω-augmented local invariant like $N(\omega)$ to be $T'(\omega)$ for some T. $N(\omega)$ fails this test, as can be shown by taking for M the manifolds $S^p \times S^q$, with either of the metrics $g_{S^p} \pm g_{S^q}$ built from the standard sphere metrics, and taking for ω and ξ products of eigenfunctions of $\Box_{S^p}$ and $\Box_{S^q}$.

Alternatively, one could look at a basis for the level 6 scalar local invariants. Because of various identities, the following is one such basis:

$J_\alpha{}^\alpha{}_\beta{}^\beta$,

$J_\alpha{}^\alpha J$, $J^\alpha{}_\beta V^\beta{}_\alpha$, $V^\alpha{}_{\beta|\lambda\mu} C_\alpha{}^{\lambda\beta\mu}$, $V^\alpha{}_{\beta|}{}^\lambda{}_\lambda V^\beta{}_\alpha$

$J^\alpha J_\alpha$, $V^\alpha{}_{\beta|\lambda} V^\beta{}_{\alpha|}{}^\lambda$, $V^\alpha{}_{\beta|\lambda} V^{\beta\lambda}{}_{|\alpha}$, $C_{\alpha\beta\lambda\mu|\sigma} C^{\alpha\beta\lambda\mu}{}_|{}^\sigma$

expressions 3-homogeneous in R (involving no derivatives).

Direct calculation then shows that no linear combination T of the above has $T'(\omega) = N(\omega)$. We do not give the details of either approach because, of course, the "non-integrability" of $N(\omega)$ does not constitute a proof that there is no general conformally covariant sixth-order operator on functions. Recalling Theorem 1.12 and Corollary 1.13, however, we have the following positive results:

THEOREM 2.6. *For $n \neq 1,2,4$, let*

(2.26)

$$
\begin{aligned}
D_6 = {} & \Box^3 + \frac{n-6}{2}[Z\Box^2 + \Box^2 Z] + \\
& + 4[\Box\delta Z d + \delta Z d\Box] + \frac{n-10}{2}\Box Z\Box + \\
& + \left(\frac{n-6}{2}\right)^2 Z\Box Z + \frac{n-6}{2}\frac{n-2}{2}(Z^2\Box + \Box Z^2) + \\
& + 2(n-6)(Z\delta Z d + \delta Z d Z) + \\
& + 2(n-6)(P\Box + \Box P) + 8\delta(Z^2 + P)d + \\
& + \frac{n-6}{2}\frac{n-2}{2}\frac{n+2}{2}Z^3 + (n-6)(n-2)(ZP + PZ) - \\
& - \frac{8(n-6)}{n-4}(V^\alpha{}_\beta Y^\beta{}_\alpha + V^\alpha{}_\beta V^\beta{}_\alpha J)
\end{aligned}
$$

on functions, where $Z = J - 2V\#$, Y is given by (2.15), *and*

$$-P = V^{\lambda}{}_{\mu} V^{\mu}{}_{\lambda} + \frac{2}{n-4} Y \#.$$

Let (M,g) be a particular n-dimensional ΨR *manifold, and let $0 < \Omega \in C^{\infty}(M)$. Suppose that $(x,u) \to \Omega_u(x)$ is a C^{∞} map $M \times (-\varepsilon, 1+\varepsilon) \to \mathbb{R}^+$ for some $\varepsilon > 0$, and that $\Omega_0 \equiv 1$, $\Omega_1 = \Omega$. Let $\omega_u = \log \Omega_u$, and $\eta_u = d\omega_u/du$. If*

$$[(\eta_u)^{\lambda} r^{\beta}{}_{\alpha}(r^{\alpha}{}_{\beta|\lambda} - 2r^{\alpha}{}_{\lambda|\beta})]\,|_{(M,\Omega_u^{-2}g)} = 0$$

for all $u \in [0,1]$, then

$$(D_6)_{\Omega^{-2}g}(\Omega^{(n-6)/2}\varphi) = \Omega^{(n+6)/2}(D_6)_g \varphi \tag{2.27}$$

for all $\varphi \in C^{\infty}(M)$.

COROLLARY 2.7. *With notation as in Theorem 2.6, if the one-form*

$$F_{\lambda} = r^{\beta}{}_{\alpha}(r^{\alpha}{}_{\beta|\lambda} - 2r^{\alpha}{}_{\lambda|\beta})$$

vanishes in $(M, \Omega_u^{-2}g)$ for all $u \in [0,1]$, then (2.27) *holds.*

In the special dimension $n = 6$, (2.24) shows that D_6 is fully conformally covariant. Thus we have the following theorem, which is actually a special case of Theorem 2.10 below.

THEOREM 2.8. *Suppose $n = 6$. Then the operator*

$$D_6 = \Box^3 + 4[\Box\delta Z d + \delta Z d \Box] - 2\Box Z \Box + 8\delta(Z^2 + P)d$$

on functions is conformally covariant of weight 0.

The calculations above are in some sense computational evidence for the following:

CONJECTURE 2.9. Suppose $n \neq 6$. *There is no general conformally covariant sixth-order operator on functions, with leading term* $\Box^3$, in n-dimensional ΨR manifolds.

More specifically, it is reasonable to conjecture that the local invariant (F_{λ}) is a genuine geometric obstruction to a covariance relation like (2.27) for a sixth-order operator. It is interesting to note that the condition $F_{\lambda} \equiv 0$ is stable at Einstein manifolds: if (M,g) is Einstein, then

$$(F_{\lambda})'(\omega)|_{(M,g)} \equiv 0.$$

In other words, the condition $F_{\lambda} \equiv 0$, which is slightly weaker than the

Ricci parallel condition $r_{\alpha\beta|\lambda}=0$, is true for infinitesimal conformal deformations of an Einstein manifold.

c. *A sixth-order operator on* $(n-6)/2$-*forms*. By (1.30), on $(n-6)/2$-forms for even $n \geqq 6$,

$$\begin{aligned}(\delta d\delta d\delta d)'^{(0)} &= 4\delta d\delta d\iota d + \delta d(2\delta\varepsilon + 2\iota d)\delta d + 4\delta\varepsilon\delta d\delta d \\ &= 4(\delta d\delta\mathscr{L}d + \delta\mathscr{L}d\delta d) - 2\delta d\mathscr{L}\delta d.\end{aligned}$$

This suggests correcting by

$$E_4 = 4(\delta d\delta Zd + \delta Zd\delta d) - 2\delta dZ\delta d.$$

Now

$$\begin{aligned}((\delta d)^3 + E_4)'^{(0)} &= 8\delta(d\iota Z - dZ\iota + Z\varepsilon\delta - \varepsilon Z\delta + 2\varepsilon\delta Z + 2Zd\iota)d \\ &= 8\delta(\mathscr{L}Z + Z\mathscr{L} - \iota dZ + Zd\iota - dZ\iota - Z\delta\varepsilon + \varepsilon\delta Z - \varepsilon Z\delta)d \\ &= 8\delta(\mathscr{L}Z + Z\mathscr{L} + \mathscr{P})d.\end{aligned}$$

This suggests correcting by

$$E_2 = 8\delta(Z^2 + P)d.$$

Since $d'^{(0)} = \delta'^{(4)} = 0$, we are done:

$$((\delta d)^3 + E_4 + E_2)'^{(0)} = 0,$$

and we have:

THEOREM 2.10. *Suppose $n \geqq 6$ is even. Then the operator*

$$D_{6,(n-6)/2} = (\delta d)^3 + 4(\delta d\delta Zd + \delta Zd\delta d) - 2\delta dZ\delta d + 8\delta(Z^2 + P)d$$

is conformally covariant of weight $(n-6)/2$ on $(n-6)/2$-forms.

The operator $D_{6,(n-6)/2}$ is analogous to the Maxwell operator $D_{2,(n-2)/2} = \delta d$, and to the special case

$$D_{4,(n-4)/2} = 4\delta d\delta d + 8\delta Zd$$

of the operator $D_{4,k}$, in that the leading terms are powers of δd, and the "inside multipliers" (the numbers a in the covariance relation $D'^{(a)} = 0$) are zero.

d. *Nonlinear conformally covariant operators*. As is well-known (see [1, 19]) the nonlinear operators

$$\begin{aligned}S_2(\varphi) &= D_2\varphi + \alpha\varphi^{(n+2)/(n-2)}, \\ N_2(\varphi) &= D_2\varphi + \alpha|\varphi|^{4/(n-2)}\varphi\end{aligned}$$

are conformally covariant in the sense that

$$g = \Omega^2 \underline{g} \Rightarrow \underline{S}_2(\Omega^{(n-2)/2}\varphi) = \Omega^{(n+2)/2} S_2(\varphi), \ n \neq 1,2,$$

and similarly for N_2. (N_2 has the advantage of always being well-defined.) The reason is that the nonlinear terms satisfy the same covariance relation as the linear operator D_2:

$$(\Omega^{(n-2)/2}\varphi)^{(n+2)/(n-2)} = \Omega^{(n+2)/2}\varphi^{(n+2)/(n-2)},$$
$$|\Omega^{(n-2)/2}\varphi|^{4/(n-2)}\Omega^{(n-2)/2}\varphi = \Omega^{(n+2)/2}|\varphi|^{4/(n-2)}\varphi.$$

Analogous statements can clearly be made about

$$S_4(\varphi) = D_4\varphi + \alpha\varphi^{(n+4)/(n-4)},$$
$$N_4(\varphi) = D_4\varphi + \alpha|\varphi|^{8/(n-4)}\varphi.$$

THEOREM 2.11. *For $n \neq 1,2,4$,*

$$g = \Omega^2 \underline{g} \Rightarrow \underline{S}_4(\Omega^{(n-4)/2}\varphi) = \Omega^{(n+4)/2} S_4(\varphi),$$
$$\underline{N}_4(\Omega^{(n-4)/2}\varphi) = \Omega^{(n+4)/2} N_4(\varphi).$$

Subject to the curvature constraint of, say, Corollary 2.7, analogous statements can also be made about

$$S_6(\varphi) = D_6\varphi + \alpha\varphi^{(n+6)/(n-6)},$$
$$N_6(\varphi) = D_6\varphi + \alpha|\varphi|^{12/(n-6)}\varphi.$$

THEOREM 2.12. *Suppose that $n \neq 1,2,4,6$, and that $(M.g)$ and Ω satisfy the curvature constraint of Corollary 2.7. Then*

$$g = \Omega^2 \underline{g} \Rightarrow \underline{S}_6(\Omega^{(n-6)/2}\varphi) = \Omega^{(n+6)/2} S_6(\varphi),$$
$$\underline{N}_6(\Omega^{(n-6)/2}\varphi) = \Omega^{(n+6)/2} N_6(\varphi).$$

If φ is a form, the natural definition of $|\varphi|^2$ is as $g^k(\varphi,\varphi)$. But under conformal change $g = \Omega^2 \underline{g}$, g^k deforms according to

$$\underline{g}^k = \Omega^{2k} g^k. \tag{2.28}$$

Thus the correct nonlinearity with which to augment, for example, $D_{2,k}$, is *not* the $(n-2k+2)/(n-2k-2)$ power, but rather the $(n+2)/(n-2)$ power ([5]). More precisely, let

$$N_{2,k}(\varphi) = D_{2,k}\varphi + \alpha|\varphi|^{4/(n-2)}\varphi.$$

Then:

THEOREM 2.13 [5]. *For $n \neq 1,2,$*

$$g = \Omega^2 \underline{g} \Rightarrow \underline{N}_{2,k}(\Omega^{(n-2)/2}\varphi) = \Omega^{(n+4)/2} N_{2,k}(\varphi).$$

For the same reason, the nonlinear covariant operator associated to $D_{4,k}$ is

$$N_{4,k}(\varphi) = D_{4,k}\varphi + \alpha|\varphi|^{8/(n-4)}\varphi.$$

THEOREM 2.14. *For* $n \neq 1,2,4$,

$$g = \Omega^2 \underline{g} \Rightarrow \underline{N}_{4,k}(\Omega^{(n-4)/2}\varphi) = \Omega^{(n+4)/2} N_{4,k}(\varphi).$$

More can be said when the "inside multiplier" for a covariant CLDO D on forms is Ω^0; that is when D satisfies a covariance relation of the form $\underline{D}\varphi = \Omega^b D\varphi$. This is because the exterior derivative d is also covariant with inside multiplier Ω^0. For example, in dimension $n = 2$, let

$$P_2(\varphi) = D_2\varphi + \alpha|d\varphi|^2|\varphi|^{p-1}\varphi = \Box\varphi + \alpha|d\varphi|^2|\varphi|^{p-1}\varphi, \ p \text{ arbitrary},$$

on functions. Then

$$g = \Omega^2 \underline{g} \Rightarrow \underline{P}_2(\varphi) = \Omega^2 P_2(\varphi),$$

by (2.28). For $p = 1$, the operator P_2 is related to the harmonic maps problem, and to the "nonlinear σ-model" of physics. Along the same lines, we have covariance results for

$$P_4(\varphi) = D_4\varphi + \alpha|d\varphi|^4|\varphi|^{p-1}\varphi, \ p \text{ arbitrary}, \ n = 4,$$

and

$$P_6(\varphi) = D_6\varphi + \alpha|d\varphi|^6|\varphi|^{p-1}\varphi, \ p \text{ arbitrary}, \ n = 6.$$

THEOREM 2.15. *If* $n = 4$,

$$g = \Omega^2 \underline{g} \Rightarrow \underline{P}_4(\varphi) = \Omega^4 P_4(\varphi).$$

THEOREM 2.16. *If* $n = 6$,

$$g = \Omega^2 \underline{g} \Rightarrow \underline{P}_6(\varphi) = \Omega^6 P_6(\varphi).$$

No curvature condition is required in Theorem 2.16, because of Theorem 2.8.

Looking at forms, we have inside multiplier Ω^0 for $D_{2,(n-2)/2} = \delta d$ (the Maxwell operator), $D_{4,(n-4)/2}$, and $D_{6,(n-6)/2}$. Some arithmetic with (2.28) shows that nonlinearities of the form

$$|d\varphi|^q|\varphi|^p\varphi, \ \frac{n-2}{2}p + \frac{n}{2}q = 2$$

satisfies the same covariance relation as $D_{2,(n-2)/2}$. Thus:

THEOREM 2.17. *For* $n = 4,6,8,\ldots$, *let*

$$P_{2,(n-2)/2}(\varphi) = D_{2,(n-2)/2}\,\varphi + \Bigg(\sum_{\substack{i\\(finite)}} \alpha_i |d\varphi|^{q_i} |\varphi|^{p_i}\Bigg)\varphi,$$

where

$$\frac{n-2}{2}p_i + \frac{n}{2}q_i = 2.$$

Then

$$g = \Omega^2 \underline{g} \Rightarrow \underline{P}_{2,(n-2)/2}(\varphi) = \Omega^2 P_{2,(n-2)/2}(\varphi).$$

The analogous statements about $D_{4,(n-4)/2}$ and $D_{6,(n-6)/2}$ are:

THEOREM 2.18. *For* $n = 6,8,10,\ldots$, *let*

$$P_{4,(n-4)/2}(\varphi) = D_{4,(n-4)/2}\,\varphi + \Bigg(\sum_{\substack{i\\(finite)}} \alpha_i |d\varphi|^{q_i} |\varphi|^{p_i}\Bigg)\varphi,$$

where

$$\frac{n-4}{2}p_i + \frac{n-2}{2}q_i = 4.$$

Then

$$g = \Omega^2 \underline{g} \Rightarrow \underline{P}_{4,(n-4)/2}(\varphi) = \Omega^4 P_{4,(n-4)/2}(\varphi).$$

THEOREM 2.19. *For* $n = 8,10,12,\ldots$, *let*

$$P_{6,(n-6)/2}(\varphi) = D_{6,(n-6)/2}\,\varphi + \Bigg(\sum_{\substack{i\\(finite)}} \alpha_i |d\varphi|^{q_i} |\varphi|^{p_i}\Bigg)\varphi,$$

where

$$\frac{n-6}{2}p_i + \frac{n-4}{2}q_i = 6.$$

Then

$$g = \Omega^2 \underline{g} \Rightarrow \underline{P}_{6,(n-6)/2}(\varphi) = \Omega^6 P_{6,(n-6)/2}(\varphi).$$

In dimension 4, one can assert the conformal covariance (and gauge-invariance) of systems which incorporate $D_{2,2}$ and D_4 fields into the Yang-

Mills-Higgs-Dirac system, as "extra Higgs fields." One can then have "mixed" nonlinearities on the right-hand sides of the generalized Higgs equations, and make new contributions to the Yang-Mills current. A system of this type (including $D_{2,2}$ fields and some generalized Dirac fields) is introduced in [6, Section 4].

3. Covariance under conformal transformations and representations of the conformal group.

a. *Conformal transformations.* In this section, let (M,g) and (N,g') be ΨR manifolds of the same dimension and signature.

DEFINITION 3.1. A diffeomorphism $h: M \to N$ is a *conformal transformation* if $(h^{-1})^* g = \Omega^2 g'$ for some $\Omega > 0$ in $C^\infty(N)$.

We write $h\cdot$ for the action of a diffeomorphism h on tensor fields as in [5]; on vector fields, $h \cdot X = (dh)X$, while on covariant tensors, $h\cdot$ acts as $(h^{-1})^*$. To adapt the above results on conformal deformation covariance, it is most convenient to view a conformal transformation $h: M \to N$, $h \cdot g = \Omega^2 g'$, as the composition of an isometry and a conformal deformation:

$$(M,g) \xrightarrow{\text{id}} (M,\underline{g} = (\Omega \circ h)^{-2} g) \xrightarrow{h} (N,g'). \tag{3.1}$$

Since CLDO D are defined to be invariants of ΨR structure, they clearly respect isometries:

$$D_{(N,g')} h \cdot \varphi = h \cdot D_{(M,\underline{g})} \varphi. \tag{3.2}$$

If in addition D is conformally covariant, we have for some a and b a law of the form

$$D_{(M,\underline{g})}[(\Omega \circ h)^a \varphi] = (\Omega \circ h)^b D_{(M,g)}.$$

This and (3.2) show that

$$D_{(N,g')}(\Omega^a h \cdot \varphi) = \Omega^b h \cdot D_{(M,g)} \varphi. \tag{3.3}$$

Analogous statements are true of nonlinear conformally covariant operators like those of Section 2.d. In the case where $(M,g) = (N,g')$, (3.3) has group representation theoretic content, as we shall see immediately below.

DEFINITION 3.2. A conformal transformation *on* (M,g) is a conformal transformation $h: (M,g) \to (M,g)$.

REMARK 3.3. If h_1 and h_2 are conformal transformations on (M,g), $h_i \cdot g = \Omega_i^2 g$, then $h_1 \circ h_2$ is conformal with

$$\Omega_{h_1 \circ h_2} = (h_1 \cdot \Omega_2)\Omega_1, \tag{3.4}$$

and h_1^{-1} is conformal with $\Omega_{h^{-1}} = (\Omega \circ h)^{-1}$. Thus the conformal transformations form a group $\mathscr{C}(M,g)$, which (at least for M connected) is a Lie group of dimension at most $(n+1)(n+2)/2$. (See [16, v.I, Note 9]. This reference works in the Riemannian case, but the results obtained are far more general, as noted in [16, v.II, Note 13].) Of course, there is no reason to expect conformal transformations to exist generically, but in some important special cases, $\mathscr{C}(M,g)$ has the maximal dimension. For example, consider the manifold $M = S^p \times S^q$, $p + q = n$, with the metric $g = g_{S^q} - g_{S^p}$. The group $0(p,q)$ acts conformally on (M,g) by

$$(g,\xi) \to \frac{g\xi}{\sqrt{(g\xi)_0^2 + \ldots + (g\xi)_p^2}},$$

where $\xi = (\xi_0, \ldots, \xi_{n+1})$; $\xi_0, \ldots, \xi_p$ are homogeneous coordinates on S^p; and $\xi_{p+1}, \ldots, \xi_{n+1}$ are homogeneous coordinates on S^q. This example is important as the double cover of the *conformal compactification* of the *standard signature* (p,q) *flat space* $\mathsf{R}^{(p,q)}$; that is, of R^n with the metric

$$-(dx^1)^2 - \ldots - (dx^p)^2 + (dx^{p+1})^2 + \ldots + (dx^n)^2$$

(see [23, 19]).

For the present purposes, we shall define a *tensor representation* of a subgroup H of $\mathscr{C}(M,g)$ to be a homomorphism $u: H \to \operatorname{Aut}\mathscr{T}$ for some canonical tensor space $\mathscr{T}$ on M. (We do not require continuity or put any particular topological vector space structure on $\mathscr{T}$.) By (3.4), the maps

$$u_a : \mathscr{C}(M,g) \to \operatorname{Aut}\mathscr{T}$$
$$h \to \Omega^a h \cdot$$

are tensor representations for each $a \in \mathsf{R}$. (3.3) implies that a conformally covariant operator *intertwines* two of these representations. Even a conditionally conformally covariant operator like the D_6 of Section 2.6 is an intertwining operator so long as (M,g) satisfies the curvature condition $r^\beta{}_\alpha(r^\alpha{}_{\beta|\lambda} - 2r^\alpha{}_{\lambda|\beta}) = 0$. Looking at (3.1) in the presently relevant form,

$$(M,g) \xrightarrow{\text{id}} (M,\underline{g} = (\Omega \circ h)^{-2} g) \xrightarrow{h} (M,g),$$

we see that $(M,\underline{g})$ must also satisfy the curvature condition, being isometric to (M,g). If h can be connected to the identity in $\mathscr{C}(M,g)$ by a one-parameter family (not necessarily a group) of conformal transformations h_t, then each $(M, (\Omega_{h_t} \circ h_t)^{-2}g)$ satisfies the curvature condition by the above argument, and Corollary 2.7 implies that

$$(3.5) \qquad D_6(\Omega^{(n-6)/2}h \cdot \varphi) = \Omega^{(n+6)/2}h \cdot D_6\varphi, \quad \varphi \in C^\infty(M).$$

Since (3.5) is also true when h is an isometry, it holds for the Lie group generated by the connected component $\mathscr{C}_0(M,g)$ of the identity in $\mathscr{C}(M,g)$, and the isometry group $\mathscr{I}(M,g)$. We have proved:

THEOREM 3.4. *Suppose M is connected, and let* $G = \mathscr{C}(M,g)$.

a) D_2 *intertwines* $u_{(n-2)/2}(G)$ *and* $u_{(n+2)/2}(G)$ *on functions,* $n \neq 1$:

$$D_2 u_{(n-2)/2}(h)\varphi = u_{(n+2)/2}(h)D_2\varphi, \quad h \in G.$$

D_4 *intertwines* $u_{(n-4)/2}(G)$ *and* $u_{(n+4)/2}(G)$ *on functions,* $n \neq 1,2$. *If*

$$r^\beta{}_\alpha(r^\alpha{}_{\beta|\lambda} - 2r^\alpha{}_{\lambda|\beta}) = 0$$

in (M,g) *and* $n \neq 1,2,4$, *then* D_6 *intertwines* $u_{(n-6)/2}(H)$ *and* $u_{(n+6)/2}(H)$, *where H is the group generated by* $\mathscr{C}_0(M,g)$ *and* $\mathscr{I}(M,g)$.

b) $D_{2,k}$ *intertwines* $u_{(n-2k-2)/2}(G)$ *and* $u_{(n-2k+2)/2}(G)$ *on k-forms,* $n \neq 1,2$. $D_{4,k}$ *intertwines* $u_{(n-2k-4)/2}(G)$ *and* $u_{(n-2k+4)/2}(G)$ *on k-forms,* $n \neq 1,2,4$.

c) $D_{6,(n-6)/2}$ *intertwines* $u_0(G)$ *and* $u_6(G)$ *on* $(n-6)/2$*-forms,* $n = 6,8,10,\ldots$.

Both $r^\beta{}_\alpha(r^\alpha{}_{\beta|\lambda} - 2r^\alpha{}_{\lambda|\beta}) = 0$ and the existence of a conformal group of nonzero dimension are symmetry conditions of a sort on (M,g). It may be that if (M,g) has an interesting conformal group, then *automatically* some conformally related $(M,\Omega^{-2}g)$ satisfies the curvature condition. This would mean that D_6 is a point of departure for the theories of conformal *deformation* and conformal *transformation*, in the sense that the latter admits more covariant differential operators.

b. *Invariant inner products on the representation spaces.* Suppose D is an intertwining operator on a canonical tensor space $\mathscr{T}$ as above, $Du_a(h) = u_b(h)D$. One then has a decomposition of u_a into representations on $\mathscr{N}(D)$ and $\mathscr{T}/\mathscr{N}(D)$, the latter being equivalent to the representation u_b on $D\mathscr{T}$. It is remarkable that for the operators of Theorem 3.4, u_a-invariant complex inner products on these representation spaces can be produced in great generality; that is, by geometric formulas without passing to examples. In the case of $\mathscr{N}(D)$, the natural setting for these inner products is apparently that of *Lorentz* manifolds (M,g).

The idea for the inner product on $\mathcal{T}/\mathcal{N}(D)$ is very straightforward. Suppose D is an operator on k-forms with

$$Du_a(h) = u_b(h)D, \text{ all } h \in H,$$

where H is some subgroup of $G = \mathscr{C}(M,g)$. Let $\langle \cdot,\cdot \rangle$ be the g^k inner product, and note that if $h \cdot g = \Omega^2 g$, then $h \cdot g^k = \Omega^{-2k} g^k$. For all (complex-valued) φ, $\psi \in \mathcal{T}$,

$$\begin{aligned} \langle u_a(h)\varphi, Du_a(h)\bar{\psi}\rangle &= \langle u_a(h)\varphi, u_b(h)D\bar{\psi}\rangle \\ &= \Omega^{a+b}\langle h \cdot \varphi, h \cdot D\bar{\psi}\rangle \\ &= \Omega^{a+b+2k} h \cdot \langle \varphi, D\bar{\psi}\rangle. \end{aligned} \tag{3.6}$$

Now if E is a normalized orientation on (M,g), we have

$$h \cdot E = \pm \Omega^n E, \tag{3.7}$$

the sign depending on whether h is orientation-preserving or orientation-reversing. (3.6) and (3.7) are all we need to prove the following.

THEOREM 3.5. *Suppose (M,g) is a compact oriented* ΨR *manifold, with normalized orientation E. Suppose D is a formally self-adjoint* CLDO *on k-forms, with*

$$Du_a(h) = u_b(h)D, \text{ all } h \in H,$$

where H is some subgroup of the group of orientation-preserving conformal transformations on (M,g). Then if

$$a + b + 2k = n, \tag{3.8}$$

the formula

$$(\varphi,\psi) = \int_M \langle \varphi, D\bar{\psi}\rangle E \tag{3.9}$$

defines a $u_a(H)$-invariant sesquilinear form on $\mathcal{T}$, the space of smooth complex-valued k-forms on M. The radical of $(\cdot,\cdot)$ is exactly $\mathcal{N}(D)$, and thus $(\cdot,\cdot)$ determines a nondegenerate sesquilinear form on $\mathcal{T}/\mathcal{N}(D)$.

PROOF. It is clear that (φ,ψ) is linear in φ and conjugate-linear in ψ. By the formal self-adjointness of D, we have $(\varphi,\psi) = \overline{(\psi,\varphi)}$. If $\psi \notin \mathcal{N}(D)$, $\eta = D\bar{\psi} \neq 0$, and we can always find a $\varphi \in \mathcal{T}$ for which

$$\int_M \langle \varphi,\eta\rangle E \neq 0,$$

even when g is indefinite.

It remains to show the $u_a(H)$-invariance of $(\cdot,\cdot)$. By (3.6) and (3.7),

$$\begin{aligned}\langle u_a(h)\varphi,\, Du_a(h)\overline{\psi}\rangle E &= \Omega^{a+b+2k-n}h\cdot(\langle\varphi, D\overline{\psi}\rangle E)\\ &= h\cdot(\langle\varphi, D\overline{\psi}\rangle E)\end{aligned}$$

for each $h \in H$, since H consists of orientation-preserving transformations. Thus

$$\begin{aligned}(u_a(h)\varphi,\, u_a(h)\psi) &= \int_M h\cdot(\langle\varphi, D\overline{\psi}\rangle E)\\ &= \int_M \langle\varphi, D\overline{\psi}\rangle E\\ &= (\varphi,\psi),\end{aligned}$$

since h is a diffeomorphism.

Of course, all the operators of Theorem 3.4 are formally self-adjoint, and have inside and outside multipliers satisfying (3.8). Thus we have:

COROLLARY 3.6. *Suppose (M,g) is a compact oriented* ΨR *manifold, and let H be the group of orientation-preserving conformal transformations on (M,g). Then* (3.9) *gives a nondegenerate sesquilinear complex inner product on (k-forms)/$\mathcal{N}(D)$ which is $u_a(H)$-invariant in the following cases*:

(a) $D = D_2$, $k = 0$, $a = (n-2)/2$, $n \neq 1$;
(b) $D = D_4$, $k = 0$, $a = (n-4/2$, $n \neq 1,2$;
(c) $D = D_{2,k}$, $a = (n-2k-2)/2$, $n \neq 1,2$;
(d) $D = D_{4,k}$, $a = (n-2k-4)/2$, $n \neq 1,2,4$;
(e) $D = D_{6,(n-6)/2}$, $k = (n-6)/2$, $a = 0$, $n = 6,8,10,\ldots$.

If $r^{\beta}{}_{\alpha}(r^{\alpha}{}_{\beta|\lambda} - 2r^{\alpha}{}_{\lambda|\beta}) = 0$ in (M,g) and M is connected, the same is true with H replaced by the group generated by the orientation-preserving isometries and the identity component of $\mathscr{C}(M,g)$, in the case $D = D_6$, $k = 0$, $a = (n-6)/2$, $n \neq 1,2,4$.

The assumption that M is compact in the above is just a convenience; it could obviously be replaced by some decay assumption on the fields φ, ψ.

The idea for the inner products on the $\mathcal{N}(D)$ comes from a geometric formula of G. Zuckerman [26] for the well-known [12] inner product on Maxwell potentials in 4-dimensional Minkowski space (or, more precisely, this inner product realized on Maxwell potentials in $S^1 \times S^3$, the double cover of the conformal compactification of Minkowski space). We now restrict to the case where (M,g) is oriented and *Lorentzian*, i.e. g has signature $(n-1,1)$. Let φ and ψ be k-forms, and consider the formal expression

$$\mathscr{A}(\varphi,\psi) = \frac{1}{2i}\int_S (\varphi \wedge * d\overline{\psi} - \overline{\psi} \wedge * d\varphi). \tag{3.10}$$

Here S is a closed, connected, spacelike hypersurface, and $*$ is the Hodge operator in (M,g):

$$g^{n-k}(*\varphi,\eta)=g^n(\varphi\wedge\eta,E) \tag{3.11}$$

for φ a k-form, η an $(n-k)$-form, and E a choice of normalized orientation. (3.10) makes sense because the integrand is an $(n-1)$-form. (Strictly speaking, we should pull back the integrand to S via the inclusion $\iota: S\to M$.) We claim, with certain assumptions on (M,g) and on the subgroup H of $\mathscr{C}(M,g)$ considered, that $\mathscr{A}$ is independent of S, and gives a u_0-invariant complex inner product on Maxwell potentials ($(n-2)/2$-forms in $\mathscr{N}(\delta d)$), and a $u_{(n-2)/2}$-invariant complex inner product on the "wave functions" $\mathscr{N}(D_2)$. On Maxwell potentials, $\mathscr{A}$ will also be gauge-invariant. On $\mathscr{N}(D_{2,k})$, the correct inner product will be

$$\mathscr{B}(\varphi,\psi)=\frac{n-2k+2}{2}\mathscr{A}(\varphi,\psi)-\frac{n-2k-2}{2}\mathscr{A}(*\varphi,*\psi). \tag{3.12}$$

This specializes to the wave ($\mathscr{A}(*\varphi,*\psi)=0$) and Maxwell ($(n-2k-2)/2=0$) cases immediately above.

We first record some identities. By (3.11), on a Lorentz manifold,

$$\delta=(-1)^{n(k+1)}*d* \quad \text{on } k\text{-forms}, \tag{3.13}$$

$$**=(-1)^{k(n-k)+1} \quad \text{on } k\text{-forms}, \tag{3.14}$$

$$\varphi\wedge*\psi=\langle\varphi,\psi\rangle E \quad \text{for } \varphi,\psi \ k\text{-forms}. \tag{3.15}$$

Here $\langle\cdot,\cdot\rangle=g^k$. If φ and ψ are k-forms,

$$d(\varphi\wedge*d\psi)=-\varphi\wedge*\delta d\psi+\langle d\varphi,d\psi\rangle E, \tag{3.16}$$

by (3.13)–(3.15). If h is an orientation-preserving conformal transformation, $h\cdot g=\Omega^2 g$, then

$$*h\cdot\varphi=\Omega^{2k-n}h\cdot*\varphi,\quad \varphi \text{ a } k\text{-form}. \tag{3.17}$$

Thus if $v=h^{-1}\cdot(\log\Omega)$,

$$\begin{aligned} u_a(h)\varphi\wedge*du_a(h)\psi&=\Omega^a h\cdot\varphi\wedge*d(\Omega^a h\cdot\psi)\\ &=\Omega^{2a+2(k+1)-n}h\cdot(\varphi\wedge*(d\psi+a\varepsilon(dv)\psi)), \end{aligned} \tag{3.18}$$

where φ and ψ are k-forms.

THEOREM 3.7. *Let (M,g,E) be Lorentzian, connected, compact, oriented ($g^n(E,E)=-1$), and of even dimension $n\geqq 4$. Suppose that all closed, connected, spacelike hypersurfaces in M represent the same class in homology. Then for $(n-2)/2$-forms $\varphi,\psi\in\mathscr{N}(\delta d)$, the expression* (3.10) *is* *independent of S, and $u_0(H)$-invariant, where H is the group of orientation-preserving conformal transformations on (M,g). That is, $\mathscr{A}$ is a $u_0(H)$-invariant sesquilinear complex inner product on $\mathscr{N}(\delta d)$.*

PROOF. We first show that $\mathscr{A}(\varphi,\psi)$ is independent of S. If S_1 and S_2 are closed, connected, spacelike hypersurfaces, (3.16) gives

$$\left(\int_{S_2} - \int_{S_1}\right)(\varphi \wedge * d\bar{\psi} - \bar{\psi} \wedge * d\varphi) = \int_V (-\varphi \wedge * \delta d\bar{\psi} + \bar{\psi} \wedge * \delta d\varphi),$$

where V is such that $\partial V = S_2 - S_1$. If $\varphi,\psi \in \mathscr{N}(\delta d)$, this is zero.

For the invariance, let h be an orientation-preserving conformal transformation, $h \cdot g = \Omega^2 g$, $v = h^{-1} \cdot (\log \Omega)$. (3.18) gives

$$\begin{aligned} 2i\mathscr{A}(u_0(h)\varphi, u_0(h)\psi) &= \textstyle\int_S (u_0(h)\varphi \wedge * du_0(h)\bar{\psi} - u_0(h)\bar{\psi} \wedge * du_0(h)\varphi) \\ &= \textstyle\int_S h \cdot (\varphi \wedge * d\bar{\psi} - \bar{\psi} \wedge * d\varphi) \\ &= \textstyle\int_{h^{-1}(S)} (\varphi \wedge * d\bar{\psi} - \bar{\psi} \wedge * d\varphi) \\ &= 2i\mathscr{A}(\varphi,\psi), \end{aligned}$$

since the conformality of h implies that $h^{-1}(S)$ is also spacelike.

REMARK 3.8. If either of the Maxwell potentials φ or ψ is exact, say $\varphi = df$ for some $(n-4)/2$-form f, then

$$\begin{aligned} \varphi \wedge * d\bar{\psi} - \bar{\psi} \wedge * d\varphi = df \wedge * d\bar{\psi} &= d(f \wedge * d\bar{\psi}) \pm f \wedge * \delta d\bar{\psi} \\ &= d(f \wedge * d\bar{\psi}), \end{aligned}$$

which integrates to zero over S; thus $\mathscr{A}(\varphi,\psi) = 0$. This says that $\mathscr{A}$ is *gauge-invariant*. As a result, $\mathscr{A}$ determines a complex inner product $\tilde{\mathscr{A}}$ on $\mathscr{N}(\delta d)/\mathscr{R}(d)$, the vector space of gauge equivalence classes. Since $h \cdot df = d(h \cdot f)$, this space also carries the representation u_0, for which $\tilde{\mathscr{A}}$ is invariant. The question of when $\tilde{\mathscr{A}}$ is nondegenerate appears complicated, and we shall not take it up here, except to note that in the important special case treated in [21], $\tilde{\mathscr{A}}$ is nondegenerate.

Exactly the same formula (3.10) serves to provide a $u_{(n-2)/2}(H)$-invariant complex inner product on $\mathscr{N}(D_2)$.

THEOREM 3.9. *Let (M,g,E) and H be as in Theorem 3.7, except that n is allowed to take any value but 1. Then*

$$\mathscr{A}(\varphi,\psi) = \frac{1}{2i}\int_S (\varphi * d\bar{\psi} - \bar{\psi} * d\varphi)$$

gives a well-defined $u_{(n-2)/2}(H)$-invariant sesquilinear complex inner product on $\mathscr{N}(D_2)$.

PROOF. To imitate the proof of Theorem 3.7, we must first show that

$$\xi = d(\varphi \wedge * d\bar{\psi} - \bar{\psi} \wedge * d\varphi) \stackrel{?}{=} 0.$$

But by (3.16),

$$\xi = \frac{n-2}{4(n-1)} K(\varphi * \bar{\psi} - \bar{\psi} * \varphi) = 0.$$

This shows that $\mathscr{A}$ is independent of S.

To get invariance, note that by (3.18),

$$\begin{aligned} &2i\mathscr{A}(u_{(n-2)/2}(h)\varphi, u_{(n-2)/2}(h)\psi) \\ &= \int_S h \cdot [\varphi * (d\bar{\psi} + \tfrac{1}{2}(n-2)\varepsilon(dv)\bar{\psi}) - \bar{\psi} *(d\varphi + \tfrac{1}{2}(n-2)\varepsilon(dv)\varphi] \\ &= \int_S h \cdot (\varphi * d\bar{\psi} - \bar{\psi} * d\varphi) \\ &= \int_{h^{-1}(S)} (\varphi * d\bar{\psi} - \bar{\psi} * d\varphi) \\ &= 2i\mathscr{A}(\varphi, \psi), \end{aligned}$$

since $\varphi * \varepsilon(dv)\bar{\psi} = \bar{\psi} * \varepsilon(dv)\varphi = \varphi\bar{\psi} * dv$.

REMARK 3.10. We cannot expect $\mathscr{A}$ even to be positive semidefinite on $\mathscr{N}(D_2)$, since $\mathscr{A}(\bar{\varphi},\bar{\varphi}) = -\mathscr{A}(\varphi,\varphi)$. To get unitary representations, it seems necessary to get an invariant decomposition of $\mathscr{N}(D_2)$ into *positive* and *negative frequency* subspaces. Consider for the moment the special case of the above in which $M = I \times S$, where I is either S^1 or an interval in $\mathbf{R}$, and S is a compact, oriented Riemannian manifold. Take as the metric on M, $g = -dt^2 + g_S$, where t is the parameter on I, and as the orientation $E = dt \wedge E_S$. The scalar curvature of M is just that of S:

$$K(t,x) = K_S(x),$$

so that

$$D_2 = \frac{\partial^2}{\partial t^2} + \Delta_S + \frac{n-2}{4(n-1)} K_S.$$

(Recall our sign convention which makes $\Delta_S = \Box_S = -\nabla^j\nabla_j$.) Suppose that all the eigenvalues of $B^2 = \Delta_S + (n-2)K_S/4(n-1)$ are positive (in general they need only be bounded below):

$$0 < \lambda_0 < \lambda_1 < \dots.$$

(Multiplicity will not be important.) Let $\mathscr{E}_j$ be the λ_j eigenspace of B^2. C^∞ solutions of $D_2\varphi = 0$ have the form

$$\varphi(t,x) = \sum_{j=0}^{\infty} (e^{i\sqrt{\lambda_j}t}\varphi_j^+(x) + e^{-i\sqrt{\lambda_j}t}\varphi_j^-(x)),$$

where $\varphi_j^\pm \in \mathscr{E}_j$; with some decay condition on the $\|\varphi_j^\pm\|_{L^2(S)}$. This gives the desired positive/negative frequency decomposition.

Now for *any* φ, $d\varphi = \varphi_t dt + d_S\varphi$, so that

$$\int_S (\varphi * d\bar{\varphi} - \bar{\varphi} * d\varphi) = \int_S (-\varphi\bar{\varphi}_t + \bar{\varphi}\varphi_t)E_S, \tag{3.19}$$

since dt pulls back to 0 under $\iota: S \to M$, and $*dt = -E_S$. Here "$\int_S$" is the integral over any of the fixed-time hypersurfaces $t = t_0$. If φ is a positive frequency solution of $D_2\varphi = 0$,

$$\varphi = \Sigma e^{i\sqrt{\lambda_j}t}\varphi_j(x),$$

then (3.19) reduces to

$$2i\mathscr{A}(\varphi,\varphi) = \Sigma 2i\sqrt{\lambda_j}\,\|\varphi_j\|^2_{L^2(S)}.$$

This explains the factor of $1/2i$ in the definition of $\mathscr{A}$, and shows that $\mathscr{A}$ is positive definite on the positive frequency subspace $\mathscr{N}^+(D_2)$. (Similarly, $\mathscr{A}$ is negative definite on the negative frequency subspace.)

To get unitary representations, we would need to know that $\mathscr{N}^+(D_2)$ is $u_{(n-2)/2}(H)$-invariant; however, we are unable to prove this assertion even in the limited generality of this Remark. The special case $I = S^1$, $S = S^{n-1}$ has been worked out in detail however ([19, 21]). For convenience, we restrict to even n. The eigenvalues of B on S^{n-1} are $\lambda_0 = (n-2)/2$, $\lambda_1 = n/2$, $\lambda_2 = (n+2)/2, \ldots, \lambda_j = (n-2+2j)/2, \ldots$, so that all solutions of $D_2\varphi = 0$ are 2π-periodic in t. $\mathscr{E}_j$ consists of jth order spherical harmonics; i.e. restrictions to S^{n-1} of j-homogeneous harmonic polynomials in R^n. The orientation-preserving conformal group H is given by the action of $O(2,n)$ described above in Section 3.a, restricted to $SO(2,n)$. If H_0 is the identity component $SO_0(2,n)$, then $u_{(n-2)/2}(H_0)$ preserves the positive frequency subspace, essentially because an element of the maximal compact $K_0 = SO(2) \times SO(n)$ carries a solution of the type

$$e^{i(n-2+2j)t/2}\varphi(x), \quad \varphi \in \mathscr{E}_j, \tag{3.20}$$

to another of the same type; and an element of the Lie algebra $\mathfrak{h}$ of H carries (3.20) to a solution of the form

$$e^{i(n-4+2j)t/2}\varphi_-(x) + e^{i(n+2j)t/2}\varphi_+(x), \quad \varphi_\pm \in \mathscr{E}_{j\pm 1}.$$

REMARK 3.11. The situation for the representation u_0 on gauge equivalence classes of Maxwell potentials is strongly analogous to that for wave functions. Assume that $M = I \times S$ as above, and that $H^{(n-2)/2}(S) = 0$ (i.e., there are no harmonic $(n-2)/2$-forms on S). Then we can get a unique representative φ of each gauge equivalence class with

$$\iota(\partial/\partial t)\varphi = \delta\varphi = 0;$$

i.e., in the *Coulomb gauge*. Indeed, given an $(n-2)/2$-form φ, let

$$\psi(t,x) = \int_{t_0}^{t} (\iota(\partial/\partial\tau)\varphi)(\tau,x)d\tau$$

for some $t_0 \in I$; then $\varphi' = \varphi - d\psi$ is gauge equivalent to φ, and $\iota(\partial/\partial t)\varphi' = 0$. Thus at each fixed time, φ' is an $(n-2)/2$-form on S; and it may be Hodge decomposed as

$$\varphi' = \varphi'_\delta + \varphi'_d,$$

where $\varphi'_\delta \in \mathscr{R}(\delta_S)$ and $\varphi'_d \in \mathscr{R}(d_S)$. φ'_δ is gauge equivalent to φ' and thus to φ, has $\iota(\partial/\partial t)\varphi'_\delta = \delta\varphi'_\delta = 0$, and is clearly unique with these properties. Gauge equivalence classes of solutions of $\delta d\varphi = 0$ may now be identified with series

$$\sum_{j=0}^{\infty} (e^{i\sqrt{\lambda_j}t}\varphi_j^+(x) + e^{-i\sqrt{\lambda_j}t}\varphi_j^-(x)),$$

where $0 < \lambda_0 < \lambda_1 < \ldots$ are the eigenvalues of $\delta_S d_S$ on coclosed $(n-2)/2$-forms (λ_j corresponding to eigenspace $\mathscr{E}_j$), and $\varphi_j^\pm \in \mathscr{E}_j$.

For *any* φ in the Coulomb gauge,

$$\begin{aligned} 2i\mathscr{A}(\varphi,\varphi) &= \int_S \varphi \wedge *d\bar{\varphi} - \bar{\varphi} \wedge *d\varphi \\ &= \int_S (-\varphi \wedge *_S \partial\bar{\varphi}/\partial t + \bar{\varphi} \wedge *_S \partial\varphi/\partial t) \\ &= \int_S (-\langle \varphi, \partial\bar{\varphi}/\partial t\rangle_S + \langle \bar{\varphi}, \partial\varphi/\partial t\rangle_S)E_S. \end{aligned}$$

If φ is a Coulomb gauge, positive frequency Maxwell potential,

$$\varphi = \sum e^{i\sqrt{\lambda_j}t}\varphi_j(x), \quad \varphi_j \in \mathscr{E}_j,$$

then

$$\mathscr{A}(\varphi,\varphi) = \sum \sqrt{\lambda_j}\, \|\varphi_j\|^2_{L^2(\Lambda^{(n-2)/2}(S))}.$$

Thus $\mathscr{A}$ is positive definite on the space of positive frequency gauge equivalence classes of Maxwell potentials.

If $S = S^{n-1}$ with $n \geqq 4$ even, $\lambda_j = ((n+2j)/2)^2$, so all Coulomb gauge solutions are 2π-periodic. H is given by the action of SO(2,n) above; an element of K_0 carries a solution

$$e^{i(n+2j)t/2}\varphi(x), \quad \varphi \in \mathscr{E}_j, \tag{3.21}$$

to another of the same type; an element of $\mathfrak{h}$ carries (3.21) to a solution

$$e^{i(n+2j-2)t/2}\varphi_-(x) + e^{i(n+2j+2)t/2}\varphi_+(x), \quad \varphi_\pm \in \mathscr{E}_{j\pm 1}.$$

Thus $u_0(H)$ preserves positive frequency, and we have a unitary representation.

We now return to the general case.

THEOREM 3.12. *Let (M,g,E) and H be as in Theorem 3.7, except that the only dimension restriction is $n \neq 1,2$. For k-forms φ,ψ in M, let $\mathscr{A}(\varphi,\psi)$ be given by* (3.10). *Then*

$$\mathscr{B}(\varphi,\psi) = \frac{n-2k+2}{2}\mathscr{A}(\varphi,\psi) - \frac{n-2k-2}{2}\mathscr{A}(*\varphi, *\psi)$$

is a well-defined, $u_{(n-2k-2)/2}(H)$-invariant sesquilinear complex inner product on $\mathscr{N}(D_{2,k})$.

PROOF. Again we imitate the proof of Theorem 3.7. Let $\beta = (n-2k)/2$. To show that $\mathscr{B}$ is well-defined, we use (3.16) and (3.13)–(3.15) to calculate

$$\begin{aligned} d\{\varphi \wedge *d\bar{\psi} - \bar{\psi} \wedge *d\varphi\} &= (-\langle \varphi,\delta d\bar{\psi}\rangle + \langle \bar{\psi},\delta d\varphi\rangle)E, \\ d\{ *\varphi \wedge *d *\bar{\psi} - \bar{\psi} \wedge *d *\varphi\} &= (-\langle *\varphi,\delta d *\bar{\psi}\rangle + \langle *\bar{\psi},\delta d *\varphi\rangle)E \\ &= (-\langle *\varphi, *d\delta\bar{\psi}\rangle + \langle *\bar{\psi}, *d\delta\varphi\rangle)E \\ &= (\langle \varphi,d\delta\bar{\psi}\rangle - \langle \bar{\psi},d\delta\varphi\rangle)E, \end{aligned}$$

since $*$ is an *anti*-isometry for Lorentz manifolds. By the definition of $D_{2,k}$,

$$\begin{aligned} d\{(\beta+1)(\varphi \wedge *d\bar{\psi} - \bar{\psi} \wedge *d\varphi) - (\beta-1)(*\varphi \wedge *d *\bar{\psi} - *\bar{\psi} \wedge *d *\varphi)\} \\ = (\beta+1)(\beta-1)\{\langle \varphi,Z\bar{\psi}\rangle - \langle \bar{\psi},Z\varphi\rangle\}E, \end{aligned}$$

where $Z = J - 2V\#$. This is zero because Z is pointwise self-adjoint; and thus $\mathscr{B}(\varphi,\psi)$ is well-defined (independent of S).

As for the invariance result, for $h \in H$, (3.18) gives

$$\begin{aligned} &u_{\beta-1}(h)\varphi \wedge *du_{\beta-1}(h)\bar{\psi} - u_{\beta-1}(h)\bar{\psi} \wedge *du_{\beta-1}(h)\varphi \\ &\quad = h\cdot\{\varphi \wedge *(d\bar{\psi} + (\beta-1)\varepsilon(dv)\bar{\psi}) - \bar{\psi} \wedge *(d\varphi + (\beta-1)\varepsilon(dv)\varphi)\}, \end{aligned}$$

while by (3.17) and (3.18),

$$\begin{aligned} &*u_{\beta-1}(h)\varphi \wedge *d *u_{\beta-1}(h)\bar{\psi} - *u_{\beta-1}(h)\bar{\psi} \wedge *d *u_{\beta-1}(h)\varphi \\ &\quad = u_{-(\beta+1)}(h) * \varphi \wedge *du_{-(\beta+1)}(h) * \bar{\psi} \end{aligned}$$

$$\begin{aligned}&-u_{-(\beta+1)}(h) *\bar{\psi} \wedge * du_{-(\beta+1)}(h) *\varphi \\ &= h\cdot\{ *\varphi \wedge *(d *\bar{\psi} - (\beta+1)\varepsilon(dv) *\bar{\psi}) \\ &\quad - *\bar{\psi} \wedge *(d *\varphi - (\beta+1)\varepsilon(dv) *\varphi)\}.\end{aligned}$$

Thus

$$\begin{aligned}&2i\mathscr{B}(u_{\beta-1}(h)\varphi, u_{\beta-1}(h)\psi) - 2i\mathscr{B}(\varphi,\psi) \\ &= (\beta+1)(\beta-1)\textstyle\int_{h^{-1}(S)}\{\varphi \wedge *\varepsilon(dv)\bar{\psi} - \bar{\psi} \wedge *\varepsilon(dv)\varphi + \\ &\quad + *\varphi \wedge *\varepsilon(dv) *\bar{\psi} - *\bar{\psi} \wedge *\varepsilon(dv) *\varphi\} \\ &= (\beta+1)(\beta-1)\textstyle\int_{h^{-1}(S)}\{(-1)^k[\varphi \wedge \iota(dv) *\bar{\psi} - \bar{\psi} \wedge \iota(dv) *\varphi] + \\ &\quad + (-1)^{n(k+1)+1}[*\varphi \wedge \iota(dv)\bar{\psi} - *\bar{\psi} \wedge \iota(dv)\varphi]\} \\ &= (\beta+1)(\beta-1)\textstyle\int_{h^{-1}(S)}\iota(dv)(\varphi \wedge *\bar{\psi} - \bar{\psi} \wedge *\varphi) = 0,\end{aligned}$$

since $\iota(dv)$ is an anti-derivation,

$$\iota(dv) = (-1)^{n(k+1)+1} *\varepsilon(dv)* \quad \text{on } k\text{-forms},$$

and $\varphi \wedge *\bar{\psi} = \langle\varphi,\bar{\psi}\rangle E$. This shows that $\mathscr{B}$ is $u_{\beta-1}(H)$-invariant.

As it turns out, $\mathscr{B}$ tends not to be positive definite, even after restriction to a positive frequency subspace. This question, and similar questions involving higher-order intertwining operators, are treated in detail in [7] for the special case $M = S^1 \times S^{n-1}$.

We shall not continue to assemble a catalogue of conformally invariant inner products on the null spaces of D_4, $D_{4,k}$, D_6, and $D_{6,(n-6)/2}$. Rather, to indicate the general direction in which things are headed, we shall just write down the inner product on $\mathscr{N}(D_4)$. Writing D_4 in the form

$$\begin{aligned}D_4 &= \square^2 + \delta T d + Q, \\ T &= (n-2)J - 4V\#, \qquad (3.22)\\ Q &= \frac{n-4}{2}(\square J + \tfrac{1}{2}nJ^2 - 2V^{\alpha}{}_{\beta}V^{\beta}{}_{\alpha}),\end{aligned}$$

the correct inner product is

$$\mathscr{A}_4(\varphi,\psi) = \frac{1}{2i}\int_S\{\varphi \wedge *(d\delta d + Td)\bar{\psi} - \bar{\psi} \wedge *(d\delta d + Td)\varphi + $$
$$+ \delta d\varphi \wedge *d\bar{\psi} - \delta d\bar{\psi} \wedge *d\varphi\}.$$

This also tends to be indefinite, even after restriction to a positive frequency subspace.

4. Remarks and further results.

a. *Global invariants of conformal structure. A fourth-order Yamabe problem.* The conformal covariance relations satisfied by D_2, $D_{2,k}$, D_4, $D_{4,k}$, and $D_{6,(n-6)/2}$ lead to the following.

THEOREM 4.1. *Suppose (M,g) is compact and Riemannian. All critical values of the functionals*

$$\frac{(D_2\varphi,\varphi)_{L^2(M)}}{\|\varphi\|^2_{L^{2n/(n-2)}(M)}}, \qquad n>2 \tag{4.1}$$

$$\frac{(D_{2,k}\varphi,\varphi)_{L^2(\Lambda^k(M))}}{\|\varphi\|^2_{L^{2n/(n-2)}(\Lambda^k(M))}}, \qquad n>2 \tag{4.2}$$

$$\frac{(D_4\varphi,\varphi)_{L^2(M)}}{\|\varphi\|^2_{L^{2n/(n-4)}(M)}}, \qquad n>4 \tag{4.3}$$

$$\frac{(D_{4,k}\varphi,\varphi)_{L^2(\Lambda^k(M))}}{\|\varphi\|^2_{L^{2n/(n-4)}(\Lambda^k(M))}}, \qquad n>4 \tag{4.4}$$

$$\frac{(D_{6,(n-6)/2}\varphi,\varphi)_{L^2(\Lambda^{(n-6)/2}(M))}}{\|\varphi\|^2_{L^{2n/(n-6)}(\Lambda^{(n-6)/2}(M))}}, \qquad n=8,10,12,\ldots \tag{4.5}$$

are conformal invariants of (M,g); i.e., *the same list of critical values is produced by $(M,\Omega^{-2}g)$ for $0<\Omega\in C^\infty(M)$.*

PROOF. For (4.1), this is a key tool in [1]; it follows from the covariance relation

$$\underline{D}_2(\Omega^{(n-2)/2}\varphi) = \Omega^{(n+2)/2}D_2\varphi. \tag{4.6}$$

For (4.2), the assertion follows from

$$\underline{D}_{2,k}(\Omega^{(n-2k-2)/2}\varphi) = \Omega^{(n-2k+2)/2}D_{2,k}\varphi \tag{4.7}$$

and (2.28). (The L^p norms are gotten by integrating powers of the g^k norm.) For (4.3)–(4.5) we use

$$\underline{D}_4(\Omega^{(n-4)/2}\varphi) = \Omega^{(n+4)/2}D_4\varphi, \tag{4.8}$$

$$\underline{D}_{4,k}(\Omega^{(n-2k-4)/2}\varphi) = \Omega^{(n-2k+4)/2}D_{4,k}\varphi, \tag{4.9}$$

$$\underline{D}_{6,(n-6)/2}\varphi = \Omega^6 D_{6,(n-6)/2}\varphi, \tag{4.10}$$

and (2.28).

REMARK 4.2. a) We cannot expect $D_{2,k}$ to be bounded below, or (4.2) to have an infimum, unless $k < (n-2)/2$. This condition guarantees that the δd and $d\delta$ coefficients are both positive. Similarly, to bound $D_{4,k}$ below, we need $k < (n-4)/2$.

b) The nonlinear operators of Theorems 2.11–2.14 are those appearing in the Lagrange multiplier problem for minimizing (4.1)–(4.4). For example, to minimize (4.2), we should solve the nonlinear eigenvalue problem

$$D_{2,k}\varphi + \alpha|\varphi|^{4/(n-2)}\varphi = 0, \quad \alpha \text{ constant.}$$

c) There are continuous imbeddings of the Sobolev spaces involved in (4.1)–(4.5):

$$L_1^2 \to L^{2n/(n-2)}, \quad L_2^2 \to L^{2n/(n-4)}, \quad L_3^2 \to L^{2n/(n-6)}.$$

These exponents put us exactly at the borderline cases of the Sobolev Imbedding Theorem, where the imbeddings are not compact. This is, of course, well-known in the D_2 case (see, e.g., [1]), where the list of critical values has been called the "conformal spectrum" of (M,g) [18]. We can regard the new conformally covariant operators as providing us with new conformal spectral data. It is natural to ask to what extent the various conformal spectra determine the conformal class of (M,g).

d) The curvature condition for the covariance of D_6 seems to make the critical points of

$$\frac{(D_6\varphi,\varphi)_{L^2(M)}}{\|\varphi\|^2_{L^{2n/(n-6)}(M)}}$$

meaningless from the above point of view.

The nonlinear eigenvalue problem corresponding to the Paneitz operator D_4 is analogous to that for D_2, the Yamabe problem. (4.6) with $\varphi \equiv 1$ gives

$$(4.11) \quad \left(\underline{\Delta} + \frac{n-2}{4(n-1)}\underline{K}\right)u = \frac{n-2}{4(n-1)}Ku^{\frac{n+2}{n-2}}, \quad u = \Omega^{\frac{n-2}{2}}, \quad n \neq 1,2.$$

(For Riemannian manifolds, we write $\square = \Delta$.) Thus to prescribe K by conformal deformation of $\underline{g}$, we look for a positive solution u to (4.11). Similarly, (4.8) applied to $\varphi \equiv 1$ gives, in the notation of (3.22),

$$(4.12) \quad (\underline{\Delta}^2 + \underline{\delta T d} + \underline{Q})v = Qv^{\frac{n+4}{n-4}}, \quad v = \Omega^{\frac{n-4}{2}}, \quad n \neq 1,2,4,$$

a Yamabe prescription problem for the level 4 local scalar invariant

$$Q = \frac{n-4}{2}\left(\Delta J + \frac{n}{2}J^2 - 2V^\alpha{}_\beta V^\beta{}_\alpha\right).$$

Among other things, this gives a global obstruction to the possibility of prescribing an Einstein metric by conformal deformation (Corollary 4.5 below).

THEOREM 4.3. *Suppose (M,g,E) is compact, oriented, and Riemannian, $n \neq 1,2,4$, and that $g = \Omega^2 \underline{g}$, $0 < \Omega \in C^\infty(M)$. Then*

$$\int_M \Omega^{\frac{n-4}{2}} \underline{QE} = \int_M \Omega^{-\frac{n-4}{2}} QE.$$

PROOF. Since $(\underline{\Delta}^2 + \underline{\delta T d})\Omega^{(n-4)/2}$ is an exact divergence in the metric $\underline{g}$, it contributes nothing when we integrate both sides of (4.12) against $\underline{E} = \Omega^{-n}E$.

COROLLARY 4.4. *Under the assumptions of Theorem* 4.3: a) *If Q is a positive (respectively negative) constant, $\underline{Q}$ must be positive (respectively negative) somewhere*; b) *If $Q \equiv 0$, then either $\underline{Q} \equiv 0$ or $\underline{Q}$ changes sign on M.*

COROLLARY 4.5. *Under the assumption of Theorem* 4.3, *if g is an Einstein metric of nonzero scalar curvature, then $\underline{Q}$ is positive somewhere. If (M,g) is Ricci flat, then either $\underline{Q} \equiv 0$ or $\underline{Q}$ changes sign on M.*

PROOF. If g is Einstein, $V^\alpha{}_\beta = (J/n)\delta^\alpha{}_\beta$ and $\Delta J = 0$, so

$$Q = \frac{(n-4)^2(n+4)}{4n} J^2.$$

b. *Tensor densities and differential operators depending only on conformal structure.* The vector bundle E of (p,q)-tensors (or (p,q)-tensors with certain symmetry/antisymmetry conditions) over a manifold M can be characterized by the coordinate transformation rule for its C^∞ sections: a (p,q)-tensor field T is a rule which assigns to each coordinate chart $(U;x^1,\dots,x^n)$ a list of functions $T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\dots\beta_q}$), in such a way that the $(U;x^1,\dots,x^n)$ and $(\tilde U, y^1,\dots,y^n)$ lists are related by

$$T^{\lambda_1\dots\lambda_p}{}_{\mu_1\dots\mu_q} = (\partial_{\alpha_1}y^{\lambda_1})\dots(\partial_{\alpha_p}y^{\lambda_p})(\tilde\partial_{\mu_1}x^{\beta_1})\dots(\tilde\partial_{\mu_q}x^{\beta_q})T^{\alpha_1\dots\alpha_p}{}_{\beta_1\dots\beta_q}$$

on $U \cap \tilde U$, where $\tilde\partial_\mu = \partial/\partial y^\mu$. A more general vector bundle can be obtained by weighting this transformation law by a power of the Jacobian determinant

$$\mathcal{J} = \det(\partial x^{\beta}/\partial y^{\mu}).$$

We shall call the bundle defined by

$$\tilde{T}^{\lambda_1 \ldots \lambda_p}{}_{\mu_1 \ldots \mu_q} = \mathcal{J}^{\frac{w-(q-p)}{n}} (\partial_{\alpha_1} y^{\lambda_1}) \ldots (\partial_{\alpha_p} y^{\lambda_p})(\tilde{\partial}_{\mu_1} x^{\beta_1}) \ldots (\tilde{\partial}_{\mu_q} x^{\beta_q}), \quad w \in \mathrm{R},$$

the bundle of *tensor densities of weight* w, and denote it by E^w if the original bundle was E. (In particular, $E = E^{q-p}$.)

If M is given a ΨR metric g, one can calculate $\det(g_{\alpha\beta})$ in each coordinate chart; this scalar quantity has weight $2n$:

$$\det(\tilde{g}_{\lambda\mu}) = \mathcal{J}^2 \det(g_{\alpha\beta}).$$

Thus if T is a (p,q)-tensor density of weight w, the functions

$$(\det g_{\alpha\beta})^{-\frac{w-(q-p)}{2n}} T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q}$$

transform as the components of an ordinary tensor field $[T]$, which we shall call the *core tensor field* for T (relative to g). Conformally related ΨR metrics $g = \Omega^2 \underline{g}$ have

$$\det(g_{\alpha\beta}) = \Omega^{2n} \det(\underline{g}_{\alpha\beta});$$

thus g and $\underline{g}$ determine core tensor fields related by

$$\underline{[T]} = \Omega^{w-(q-p)}[T].$$

The conformally covariant CLDO of Sections 1,2, when viewed as operators on ordinary tensor fields, depend on the ΨR metric g, and not just on its conformal class (e.g., $\underline{D}_2 \neq D_2$ for $g = \Omega^2 \underline{g}$). But the conformal covariance laws say exactly that these operators, when viewed as operators between tensor densities of certain weights, depend only on the conformal class of g. If E and F are canonical tensor bundles and $D: C^{\infty}(E) \to C^{\infty}(F)$ is a CLDO, we define associated differential operators

$$D^{w,w'}: C^{\infty}(E^w) \to C^{\infty}(F^{w'})$$

by

$$[D^{w,w'}T] = D[T].$$

A priori, $D^{w,w'}$ depends on g because $[\cdot]$ does. But if D is a conformally covariant level l CLDO of weight w, then $D^{w,w+l}$ depends only on the conformal class of g. Indeed, if $g = \Omega^2 \underline{g}$, then (supposing that E is a bundle of (p,q)-tensors, and F is a bundle of (r,s)-tensors), the conformal covariance law

$$\underline{D}(\Omega^{w-(q-p)}\varphi) = \Omega^{w+l-(s-r)}D\varphi$$

gives

$$\begin{aligned}[\underline{D}^{w,w+l}T] &= \Omega^{-(w+l-(s-r))}[\underline{D}^{w,w+l}\underline{T}] \\ &= \Omega^{-(w+l-(s-r))}\underline{D}[\underline{T}] \\ &= \Omega^{-(w+l-(s-r))}\underline{D}(\Omega^{w-(q-p)}[T]) \\ &= D[T] \\ &= [D^{w,w+l}T],\end{aligned}$$

as desired.

In this sense, conformally covariant differential operators can be called "differential operators canonically associated to a conformal structure."

It would be desirable to relate, say, the $(D_{2,k})^{(n-2)/2,(n+2)/2}$ to the Laplacian of some elliptic complex in the case of compact Riemannian M, in order to relate conformal geometric information to topological information. This prospect is especially appealing because the $D_{2,k}$ appear to carry more geometric information than the ordinary (de Rham complex) Laplacians. As indicated indirectly in [6, Section 5], the correct route seems to be through some sort of Dirac complex.

c. $\mathcal{N}(D_4)$ *as a partial gauge for the Maxwell equations.* Paneitz' original reason for calculating D_4 involved gauge-fixing for the Maxwell equations on

$$(M,g) = (S^1 \times S^3, -g_{S^1} + g_{S^3} = -dt^2 + g_{S^3}).$$

As shown above (Remark 3.11) one can fix the Coulomb gauge

$$\iota(\partial/\partial t)A = \delta A = 0;$$

but these conditions are not conformally invariant. Paneitz showed that the conformal transform of a Coulomb gauge potential in (M,g) can be reduced to the Coulomb gauge by a solution of $D_4 u = 0$:

$$\iota(\partial/\partial t)(h\cdot A + du) = \delta(h\cdot A + du) = 0, \text{ some } u \in \mathcal{N}(D_4).$$

He also showed that D_4 is in a certain sense unique with this property.

According to calculations of the author, all this remains true in the case of Maxwell potentials ($(n-2)/2$-forms) in $S^1 \times S^{n-1}$ for even $n \geqq 4$, if we replace D_4 by $D_{4,(n-4)/2}$ [7].

d. *Conformal structure and* CR *structure.* Conformally invariant tensor fields and conformally covariant differential operators have recently become important in the problem of deciding when two strictly pseudoconvex domains in $\mathbf{C}^n$ are biholomorphically equivalent. Given a strictly pseudoconvex domain U, Fefferman [11] defines a Lorentz metric g on $F = S^1 \times \partial U$, in such a way that biholomorphisms $H: U_1 \to U_2$ induce conformal transformations

$$h:(F_1,g_1) \to (F_2,g_2).$$

Thus invariants of the conformal structure of (F,g) turn out to be biholomorphic invariants of U.

The natural "intrinsic" structure generalizing the property of being the boundary of a pseudoconvex domain is that of a CR manifold. If N is an intrinsic CR manifold which is strictly pseudoconvex in the sense that its Levi form is positive definite, a version of Fefferman's construction can still be done; this yields a Lorentz metric on an S^1 bundle F over N. (See [15] and references therein.) Invariants of conformal structure like $\|C\|^2$, D_2, and D_4 on F then project to invariants of CR structure on N. In the case of D_2, this process is known to yield a scalar curvature correction to the Kohn–Spencer Laplacian Δ_b. (Again, see [15]. Definitions of the appropriate curvature tensors for a CR structure are worked out in [24].)

One might also work directly with a CR structure and the curvatures of [24], and perform calculations analogous to those of Sections 1.d and 2, to obtain differential operator invariants of CR structure analogous to D_4, D_6, $D_{2,k}$, $D_{4,k}$, $D_{6,(n-6)/2}$.

e. *Heat kernels and conformal invariants.* Let (M,g) be a compact Riemannian manifold. Joint work with B. Ørsted indicates that conformally invariant tensor fields occur at certain points in the Minakshisundaram–Pleijel expansion for the fundamental solution of the heat equation

$$(\partial/\partial t + D)u(x,t) = 0, \quad (x,t) \in M \times \mathbf{R},$$

based on a conformally covariant operator D. Recall that for the heat equation based on the ordinary Laplacian Δ on functions, the heat kernel $K(x,y,t)$ has a certain asymptotic expansion, which on the diagonal reduces to

$$K(x,x,t) \sim (4\pi t)^{-n/2}(a_0 + a_1 t + a_2 t^2 + \ldots), \quad t \downarrow 0,$$

where a_i is a level $2i$ scalar local invariant. Analogous statements can be made when Δ is replaced by a CLDO D, *provided* D has a consistent level l. For example, in the scalar case, if we replace Δ by $\Delta + \alpha K$, $\alpha \in \mathbf{R}$, the a_i are still level $2i$ local invariants, but this is not true if we replace Δ by $\Delta + K^2$.

The flavor of our general results can be illustrated by two examples. A typical "pointwise" result has as a special case that if $n = 6$ and $D = D_2$, then

$$\underline{g} = \Omega^2 g \Rightarrow \underline{a}_2 = \Omega^4 a_2.$$

This special case can be calculated directly: a_2 is a multiple of $\|C\|^2$. A typical "global" result has a special case that for $n = 4$, $D = D_2$,

$$\underline{g} = \Omega^2 g \Rightarrow \int_M \underline{a}_2 \underline{E} = \int_M a_2 E.$$

This special case can also be calculated directly: a_2 is a linear combination of $\|C\|^2$ and the Pfaffian (Euler characteristic density).

ADDED IN PROOF. V. Wünsch reports that he has solved Conjecture 2.9 in the negative, by constructing a general conformally covariant sixth-order operator on functions (for $n > 4$) which appears to differ from our D_6 by a second-order operator.

REFERENCES

1. T. Aubin, *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. 55 (1976), 269–296.
2. R. Bach, *Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs*, Math. Z. 9 (1921), 110–135.
3. H. Bateman, *The transformation of the electrodynamical equations*, Proc. London Math. Soc. 8 (1910), 223–264.
4. M. Beals, C. Fefferman, and R. Grossman, *Strictly pseudoconvex domains in* $\mathbf{C}^n$, Bull. Amer. Math. Soc. 8 (1983), 125–322.
5. T. Branson, *Conformally covariant equations on differential forms*, Comm. Partial Differential Equations 7 (1982), 393–431.
6. T. Branson, *Intertwining differential operators for spinor-form representations of the conformal group*, Adv. in Math. 54 (1984), 1–21.
7. T. Branson, *Group representations arising from Lorentz conformal geometry*, J. Funct. Anal., to appear.
8. Y. Choquet-Bruhat and D. Christodoulou, *Existence of global solutions of the Yang-Mills, Higgs, and spinor field equations in* $3+1$ *dimensions*, Ann. Sci. École Norm. Sup. (4) 14 (1981), 481–500.
9. E. Cunningham, *The principle of relativity in electrodynamics and an extension thereof*, Proc. London Math. Soc. 8 (1910), 77–98.
10. P. A. M. Dirac, *Wave equations in conformal space*, Ann. of Math. 37 (1936), 429–442.
11. C. Fefferman, *Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains*, Ann. of Math. 103 (1976), 395–416.
12. L. Gross, *Norm invariance of mass-zero equations under the conformal group*, J. Math. Phys. 5 (1964), 687–695.
13. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces* (Pure and Appl. Math. 80), Academic Press, New York, 1978.

14. H. P. Jakobsen and M. Vergne, *Wave and Dirac operators, and representations of the conformal group*, J. Funct. Anal. 24 (1977), 52–106.
15. D. Jerison and J. M. Lee, *A subelliptic, nonlinear eigenvalue problem and scalar curvature on* CR *manifolds*, in *Microlocal analysis*, Contemp. Math. 27 (1984), 57–63.
16. S. Kobayashi and K. Nomizu, *Foundations of differential geometry* I, II, Interscience Publishers, New York, 1963 and 1969.
17. A. Lichnerowicz, *Propagateurs et commutateurs en relativité générale*, Inst. Hautes Études Sci. Publ. Math. 10 (1961), 56 pp.
18. J. Morava, *Some conformal invariants of Minkowski space*, preprint, Johns Hopkins University, 1983.
19. B. Ørsted, *Conformally invariant differential equations and projective geometry*, J. Funct. Anal. 44 (1981), 1–23.
20. S. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint, Massachusetts Institute of Technology, 1983.
21. S. Paneitz and I. E. Segal, *Analysis in space-time bundles*, I. *General considerations and the scalar bundle*, J. Funct. Anal. 47 (1982), 78–142, II. *The spinor and form bundles*, J. Funct. Anal. 49 (1982), 335–414.
22. I. E. Segal, *A class of operator algebras which are determined by groups*, Duke Math. J. 18 (1951), 221–265.
23. I. E. Segal, *Mathematical cosmology and extragalactic astronomy*, Academic Press (Pure and Appl. Math. 68), New York, 1976.
24. S. M. Webster, *Pseudohermitian structures on a real hypersurface*, J. Differential Geom. 13 (1978), 25–41.
25. H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka J. Math. 12 (1960), 21–37.
26. G. Zuckerman, *Quantum physics and semisimple symmetric spaces*, in *Lie Group Representations* III (Lecture Notes in Math. no. 1077), Springer-Verlag, Berlin - Heidelberg - New York, 1984.

PRESENT ADDRESS:
MATHEMATICAL INSTITUTE
UNIVERSITY OF COPENHAGEN
UNIVERSITETSPARKEN 5
2100 COPENHAGEN, DENMARK

MATEMATISK INSTITUT
ODENSE UNIVERSITET
5230 ODENSE M, DENMARK

SCHOOL OF MATHEMATICS
INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ. 08540, U.S.A.