MATH. SCAND. 57 (1985), 346–358

HARMONIC ANALYSIS AND NONSTANDARD BROWNIAN MOTION IN THE PLANE

JUHA OIKKONEN

Abstract.

A uniform distribution of directions yields an essentially conformally invariant two dimensional nonstandard Brownian motion which is used to analyse the relations between harmonic analysis and Brownian motion. The construction leads to combinatorial proofs of a theorem of Kakutani relating Dirichlet's problem to Brownian motion and of a result of Lévy on the invariance of Brownian motion and harmonic measure under conformal mappings.

Introduction.

If b_1 and b_2 are independent one dimensional Brownian motions, then $b=(b_1,b_2)$ is a two dimensional Brownian motion. In this way the nonstandard construction by Anderson [1] of a one dimensional Brownian motion leads to a nonstandard two dimensional Brownian motion (see also Keisler [4]). The internal counterpart of this process is a random walk where one takes infinitely often infinitely short steps corresponding to the directions (1,1), $(-1,1)$, $(-1,-1)$, and $(1,-1)$. We are going to construct two dimensional Brownian motion in a similar way by means of an internal random walk, where the directions of the steps correspond to the roots of the equation $z^N=1$ with N infinite.

We assume that the reader is familiar with the elements of nonstandard analysis and the use of Loeb spaces to construct measures; the required details can be found in e.g. Cutland [2] and Keisler [4].

1. The construction.

1.1. Definition. A stochastic process $b\colon \mathsf{R}^+\times\Omega\to\mathsf{R}^2$ is a two dimensional Brownian motion in the space $(\Omega,\mathscr{D},P)$, if

(i) for all $t>0$ the variable $b(t,\cdot)-b(0,\cdot)$ is normally distributed with density function $(2\pi t)^{-1}\exp(-(x_1^2+x_2^2)/2t)$;

Received August 22, 1984; in revised form October 25, 1984.

(ii) if $0 \leqq t_1 < t_2 < \ldots < t_n$, then the variables $b(t_n, \cdot) - b(t_{n-1}, \cdot), \ldots, b(t_2, \cdot) - b(t_1, \cdot)$ are independent;

(iii) for almost all $\omega \in \Omega$, $b(0,\omega) = 0$ and $b(\cdot, \omega)$ is continuous.

1.2. PROPOSITION. *If b is a Brownian motion, then*

$$P(\{\omega \,|\, |b(t,\omega)| < r \text{ for all } t < t_0\}) \to 0$$

as $t_0 \to \infty$.

This is a property shared by Brownian motion and discrete random walks. We next construct the internal random walk to be used below.

1.3. CONSTRUCTION. Let $H, N \in {}^*\mathsf{N} \setminus \mathsf{N}$ and assume $N \geqq H!$ is divisible by four. Denote $\Delta t = 1/H$ and let $T = \{0, \Delta t, \ldots, H\}$. Let S be the set of the roots of the equation $z^N = 1$. Because N is even, $-z \in S$ whenever $z \in S$. Define $\Omega = S^{T \setminus \{0\}}$ = the set of internal sequences $(\omega(\Delta t), \ldots, \omega(H))$. Let $\tilde{\Omega} = (\Omega, \mathscr{D}, P)$ be the Loeb space obtained by giving to every $\omega \in \Omega$ the weight $\Delta \bar{P} = 1/|\Omega| = 1/N^H$. Analogously, $\tilde{T} = (T, \mathscr{F}, M)$ will be the Loeb space obtained by giving to every $t \in T$ the weight Δt. Recall that this space represents the Lebesgue measure via the map st^{-1} (see [2]). The internal random walk B is defined by setting

$$B(0,\omega) = 0 \text{ and } B(t + \Delta t, \omega) = B(t,\omega) + \sqrt{2\Delta t}\, \omega(t + \Delta t).$$

Finally, extend B by linear interpolation to a function

$$B : {}^*[0,H] \times \Omega \to {}^*\mathsf{R}^2.$$

This internal random walk generates a standard process by taking standard parts.

1.4. DEFINITION. We denote $b(t,\omega) = {}^\circ B(t,\omega)$ for $t \in \mathsf{R}^+$ and $\omega \in \Omega$; $B_z(t,\omega) = B(t,\omega) + z$ and $b_z(t,\omega) = b(t,\omega) + z$.

1.5. THEOREM. *The process $b : \mathsf{R}^+ \times \Omega \to \mathsf{R}^2$ is a Brownian motion in $\tilde{\Omega}$, and the internal random walk B is S-continuous for almost all ω.*

PROOF. The proof goes like that for Anderson's construction (see [1]). The continuity and S-continuity parts follow also from Theorem 3.2 in [4]; the required hypermartingale property follows directly from our construction. (See also the proof of Theorem 4.4. below.)

2. Harmonic measures.

We shall consider a bounded domain $G \subsetneq \mathsf{R}^2$. To simplify our discussion we shall make the following assumptions on G.

(i) There is a continuous function $p: G' \setminus G \to \partial G$, where G' is a domain with $G \cup \partial G \subset G'$. Moreover, $p \restriction \partial G = \mathrm{id}$.

(ii) If $x \in {}^*G$ and ${}^\circ x \in \partial G$, then there are $r_x \approx 0$ and $c_x \approx 1/2$, for which $\geqq c_x 2\pi r_x$ of the circumference of the circle with center x and radius r_x is contained in $-{}^*G$. Moreover it is assumed that the function $x \mapsto (r_x, c_x)$ is internal.

2.1. Examples. 1) The function p in (i) can be the perpendicular projection on ∂G or px may be the element of ∂G closest to x.

2) The disc $G = \{z \,|\, |z - z_0| < r\}$ satisfies both of these requirements. More generally, this will follow from the regularity of the boundary of G; it is enough that ∂G is C^2 (actually almost C^1 suffices, see [5]).

We may assume that $0 \in G$.

2.2. Definition. The discrete versions of the interior and boundary of G are defined as follows:

$$IG = \{B(t,\omega) \,|\, t \in T,\ \omega \in \Omega \text{ and } B(s,\omega) \in {}^*G \text{ for } t \geqq s \in T\};$$
$$DG = \{B(t,\omega) \,|\, \omega \in \Omega \text{ and } t \in T \text{ is the smallest with } B(t,\omega) \in -{}^*G\}.$$

Moreover, for $x \in IG$ and $\omega \in \Omega$, $T_x(\omega)$ = the smallest $t \in T$ with $B_x(t,\omega) \in -{}^*G$, when defined. Notice that B_x moves along IG when $x \in IG$. Thus

$$DG = \{B(T_x(\omega), \omega) \,|\, \omega \in \Omega\}.$$

2.3. Lemma. *Assume that* $x \in IG$ *and* $t \in T$ *is finite* $B_x(t,\omega) \in {}^*G$ *and* ${}^\circ B_x(t,\omega_0) \in \partial G$. *Then there exists an internal set* $A \subsetneq \Omega$ *with*

$$\bar{P}(A \,|\, \{\omega \restriction t = \omega_0 \restriction t\}) \approx 1$$

and

$$\omega \in A \Rightarrow B_x(T_x(\omega), \omega) \approx {}^\circ B_x(t,\omega_0).$$

(Here $\omega \restriction t = (\omega(\Delta t), \ldots, \omega(t))$.*)*

Proof. We shall inductively apply condition (ii) imposed on G. Consider B_x as started from point $B_x(t,\omega)$, i.e. from the moment t on. Then the (conditional) probability for B_x not coming out from a given circle is ≈ 0 and we can neglect such ω in the sequel. Denote by A_0 the set of those ω for which B_x comes out from the circle of center $B_x(t,\omega_0)$ and radius $r_{B_x(t,\omega_0)}$ in the complement of ${}^*G$. Let A_{n+1} be the union of A_n and the set of those ω for which B_x has come from the previous circle in a point $B_x(t',\omega) \in {}^*G$ but leaves in the complement of ${}^*G$ the circle with center $B_x(t',\omega)$ and radius $r_{B_x(t',\omega)}$.

It follows that $A_0 \subsetneq A_1 \subsetneq \ldots$ and that for all standard $\varepsilon > 0$ there is an $n_\varepsilon \in \mathsf{N}$ with $\bar{P}(A_n) > 1-\varepsilon$ for $n > n_\varepsilon$. This holds because $\bar{P}(A_{n+1}| -A_n) \approx 1/2$ and $1/2+1/4+\ldots = 1$. (For each n one can find a "smallest c_z" since Ω is hyperfinite.) Finally, by ω_1-saturation we can use an internal prolongation of the sequence $(A_n)_{n\in\mathsf{N}}$; and by overspill, there is some $A = A_k$, $k \in {}^*\mathsf{N} \setminus \mathsf{N}$, for which $A_n \subsetneq A$ for all $n \in \mathsf{N}$. Thus $\bar{P}(A) \approx 1$.

We shall next start to use the internal random walk B to define a harmonic measure for G.

2.4. Definition. The *internal harmonic measure* $\bar{M}_x$ on DG will be defined for every $x \in IG$ by setting for every $y \in DG$

$$\Delta\bar{M}_x(y) = \bar{P}(\{\omega|\, B_x(T_x(\omega),\omega) = y\}).$$

For internal $A \subsetneq DG$ we get

$$\bar{M}_x(A) = \sum_{y\in A} \Delta\bar{M}_x(y);$$

especially,

$$\bar{M}_x(DG) = \bar{P}(\{\omega|\, T_x(\omega) \text{ is defined}\}) \approx 1.$$

The internal space $(DG, {}^*\mathscr{P}(DG), M_x)$ generates a Loeb space $(DG, \mathscr{A}, M_x)$ which will be called the *space of discrete harmonic measure.*

A set $A \subsetneq DG$ will be called an *interval*, if there is an interval

$$I = \{x \in \mathsf{R}^2 |\, a_i \leqq x_i \leqq b_i,\ i = 1,2\}$$

for which $A = DG \cap {}^*I$.

2.5. Lemma. *If $A \subsetneq DG$ is an interval, $x,y \in IG$, ${}^\circ x, {}^\circ y \in G$ and $x \approx y$, then $\bar{M}_x(A) \approx \bar{M}_y(A)$; so the function $x \mapsto \bar{M}_x(A)$ is S-continuous.*

Indeed, the assertion follows from Lemma 2.3.

2.6. Corollary. *If $A \subsetneq DG$ is an interval and $\varepsilon > 0$ is standard, then for each standard $n > 0$ there is a standard $\delta_n > 0$ with $|\bar{M}_x(A) - \bar{M}_y(A)| < \varepsilon$, for all $x,y \in IG$ satisfying*

$$|x-y| < \delta_n,\ \operatorname{dist}(y,\partial G) > 1/n \ \textit{and}\ \operatorname{dist}(x,\partial G) > 1/n.$$

It follows also that for all intervals $A \subsetneq DG$ and all $x,y \in IG$ with $x \approx y$ *and* ${}^\circ x, {}^\circ y \in G$,

$$(*) \qquad M_x(A) = M_y(A).$$

More generally, $A \subsetneq DG$ is *Borel*, if $A = DG \cap {}^*C$ for some Borel set $C \subsetneq \mathbb{R}^2$. Because $\mathscr{A}$ contains all intervals, it contains all Borel sets. Thus $(*)$ can be generalized to yield the following result.

2.7. Lemma. *If $A \subsetneq DG$ is Borel, $x,y \in IG$, ${}^\circ x, {}^\circ y \in G$ and $x \approx y$, then $M_x(A) = M_y(A)$.*

This Lemma makes it possible to define for all $x \in G$, $M_x = M_y$, where y is any element of IG with $x \approx y$. We shall next construct the harmonic measure μ_x corresponding to any $x \in G$.

2.8. Definition. The *harmonic measure* on ∂G corresponding to $x \in G$ is defined by

$$\mu_x(C) = M_x(\mathrm{st}^{-1}(C)),$$

where $\mathrm{st}^{-1}(C) = \{y \in DG \mid {}^\circ y \in C\}$.

If $I = \{z \mid a_i < z_i < b_i\}$ is an interval in $\mathbb{R}^2$, then

$$(**) \qquad \mathrm{st}^{-1}(I \cap G) = \bigcup_{n \in \mathbb{N}^+} {}^*I_n \cap DG,$$

where $I_n = \{z \mid a_i + 1/n < z_i < b_i - 1/n\}$. This implies that μ_x will be defined on all intervals and henceforth on all Borel sets.

2.9. Theorem. *If C is a Borel set of ∂G, then the mapping $x \mapsto \mu_x(C)$ is continuous on G.*

Proof. It is enough to verify the assertion for intervals. In this case it follows from Corollary 2.6 and $(**)$.

Next we shall compare the above constructions based on the internal random walk B with the standard Brownian motion b. For $x \in G$, we shall denote

$$t_x(\omega) = \inf\{t \mid b_x(t,\omega) \in -G\},$$

when defined.

2.10. Theorem. *If $A \subsetneq \partial G$ is Borel, then*

$$P(\{\omega \mid b_x(t_x(\omega),\omega) \in A\}) = \mu_x(A).$$

Proof. Again, it is enough to consider an interval $A = I \cap \partial G$, where I is an interval of $\mathbb{R}^2$. Then for P-almost all ω,

$$b_x(t_x(\omega),\omega) \in I \cap \partial G \Leftrightarrow \exists n \in \mathbb{N}^+ : B_x(T_x(\omega),\omega) \in {}^*I_n \cap DG$$

and

$$t_x(\omega) = {}^\circ T_x(\omega),$$

because $B(\cdot,\omega)$ is S-continuous for almost all ω. Therefore,

$$\begin{aligned} &P(\{\omega \mid b_x(t_x(\omega),\omega) \in I \cap \partial G\}) \\ &\quad = \lim_n {}^\circ\bar{P}(\{\omega \mid B_x(T_x(\omega),\omega) \in {}^*I_n \cap DG) \\ &\quad = \lim_n {}^\circ\bar{M}_x({}^*I_n \cap DG) = \lim_n M_x({}^*I_n \cap DG) = \mu_x(I \cap \partial G). \end{aligned}$$

As a special case we consider the harmonic measure in a disc relative to the center. It will turn out in the next chapter that the following slightly more general situation is of importance. So let G be a disc with radius r and center ${}^\circ x$ with $x \in IG$. Let G' be the disc in ${}^*\mathsf{R}^2$ with center x and radius r. The sets IG and IG' are defined in the obvious way. Because IG' is invariant under rotations with center x and angle a multiple of $2\pi/N$, $M_x^{G'}({}^*I \cap DG')$ will depend only on the length of the arc ${}^*I \cap {}^*\partial G'$, when I is an interval in R^2.

On the other hand, ${}^*\partial G'$ and ${}^*\partial G$ are everywhere infinitesimally close to each other. So Lemma 2.3 implies that for all intervals I,

$$\bar{M}_x^{G'}({}^*I \cap DG') \approx \bar{M}_x^G({}^*I \cap DG) \approx M_x^G({}^*I \cap DG).$$

It follows that also $\mu_x^G(I \cap \partial G)$ depends only on the length of the corresponding arc. We have obtained the following result.

2.11. THEOREM. *If G is a disc, $x \in IG$ and ${}^\circ x$ is the center of G, then $M_x = M_{{}^\circ x}$ and $\mu_{{}^\circ x}$ is uniformly distributed on ∂G.*

Finally, the following result is needed in the next chapter. Here again G is any domain satisfying (i) and (ii).

2.12. LEMMA. *If $x,y \in IG$ and ${}^\circ x, {}^\circ y \in G$, then M_x and M_y have the same null-sets.*

PROOF. By basic properties of Loeb measures, it is enough to show that if $A \subsetneq DG$ is internal and $\bar{M}_x(A) \not\approx 0$, then $\bar{M}_y(A) \not\approx 0$. Assume $\bar{M}_x(A) > \varepsilon > 0$, where ε is standard. By Lemma 2.5, there is a standard $\delta > 0$ with $\bar{M}_z(A) > \varepsilon$ for all $z \in IG$, $|x - z| < \delta$. Since G is connected, there is a finite sequence of discs $D_1, \ldots, D_n \subsetneq G$ with centers $d_1, \ldots, d_n$, such that $d_1 = {}^\circ y$, $d_{k+1} \in \partial D_k$ for all $k = 1, \ldots, n-1$ and ${}^\circ x \in \partial D_n$. It follows that B_y has a noninfinitesimal probability of hitting the disc with center x and radius δ. Hence $\bar{M}_y(A) \not\approx 0$.

3. Dirichlet's problem.

Recall that a continuous function $u: G \to \mathrm{R}$ is *harmonic* in a domain $G \subsetneq R^2$, if it satisfies the following mutually equivalent conditions:

(a) $D_{11}u + D_{22}u = 0$ in G and $u \in C^2(G)$;
(b) if $r > 0$ and $G' = \{z \mid |z - x| \leqq r\} \subsetneq G$, then $u(x) = (1/2\pi r)\int_{\partial G'} u\,dm$, where m is linear Lebesgue measure on $\partial G'$.

Dirichlet's problem is to find for a given continuous function $f: \partial G \to \mathrm{R}$ a continuous function $u: \mathrm{cl}\, G \to \mathrm{R}$ with $u \upharpoonright \partial G = f$ and $u \upharpoonright G$ harmonic.

Below we shall use the internal random walk B to construct explicit solutions to Dirichlet's problem. We assume that G satisfies the requirements stated in the previous chapter. Let $f: \partial G \to \mathrm{R}$ be a bounded function which is μ_x-measurable for all $x \in G$. (Actually, Borel-measurability is enough.) Let $F: DG \to {}^*\mathrm{R}$ be an internal function, which is a lifting of f with respect to some/all the measures M_x, $x \in IG$, ${}^\circ x \in G$. (Recall Lemma 2.12.) Here, ${}^*f$ is assumed to be extended to DG by

$$ {}^*f(y) = {}^*f({}^*py), \quad y \in DG. $$

If f is continuous, F can be simply obtained as ${}^*f({}^*py)$.

3.1. Notation. Given f and F as above, we denote

$$ u(x) = \int_{\partial G} f\, d\mu_x \qquad \text{for } x \in G; $$

$$ U(x) = \sum_{DG} F \Delta \bar{M}_x \quad \text{for} x \in IG. $$

3.2. Lemma. *For $x \in IG$, ${}^\circ U(x) = u({}^\circ x)$.*

The assertion follows from the general properties of liftings in nonstandard integration theory (cf. [2], and from Lemma 2.5.

3.3. Lemma. For $x \in IG$, $U(x) = \bar{E} F(B_x(T_x(\omega), \omega))$, *when $\bar{E}$ denotes expectation with respect to $\bar{P}$.*

Proof. It is enough to collect the terms of the sum

$$ \sum_{\omega} F(B_x(T_x(\omega), \omega)) \Delta \bar{P} = \Delta \bar{P} \sum_{\omega} F(B_x(T_x(\omega), \omega)) $$

according to the elements of DG.

3.4. Theorem. *The function u is harmonic in the domain G.*

Proof. Lemma 3.2 implies that U is S-continuous. This will imply that u

is continuous. We are left with proving that u satisfies property (b). Consider a disc G' with cl $G' \subsetneq G$. We pick $x \in IG$ with ${}^\circ x$ being the center of G'. The sets IG' and DG' are defined in the obvious way; especially, $IG' \cup DG' \subsetneq IG$. The following calculation implies the result:

$$u({}^\circ x) \approx U(x) \approx \sum_{y \in DG'} U(y)\bar{P}(\{\omega \mid B_x(T_x^{G'}(\omega), \omega) = y\})$$

$$= \sum_{y \in DG'} U(y)\Delta \bar{M}_x^{G'}(y)$$

$$\approx \int_{DG'} u({}^\circ y) dM_x^{G'}(y)$$

$$= \int_{DG'} u({}^\circ y) dM_{{}^\circ x}^{G'}(y) = \int_{\partial G'} u \, d\mu_{{}^\circ x}^{G'}.$$

3.5. Theorem. *If f is continuous and F satisfies $F(y) = {}^*f({}^*py)$ for $y \in DG$, then $U(x) \approx f({}^\circ x)$ for $x \in {}^*G$ with ${}^\circ x \in \partial G$.*

Proof. Because f is continuous, there is a standard M with $|F(y)| \leqq M$ in DG. So we are again able to use the important Lemma 2.3.

3.6. Corollary. *If f is continuous, then for all $y \in \partial G$, $u(x) \to f(y)$ as $x \to y$ in G.*

Indeed, this follows from 3.5 and 3.2. We have shown how the internal random walk B and the internal expectation U lead to a solution of Dirichlet's problem. To end this chapter, we show how our Brownian motion b is connected to the harmonic function u.

3.7. Theorem. *If f is Borel-measurable, then*

$$u(x) = Ef(b_x(t_x(\omega), \omega)),$$

where the expectation is taken with respect to P.

Proof. For a simple function f the assertion follows from Theorem 2.10. Clearly this implies the general result.

3.8. Remark. If f were not continuous, f is the limit of u on the boundary of G in the weak sense that ${}^\circ U(x)$ depends only on those values $F(y)$ with $y \in DG$ and $y \approx x$, when ${}^\circ x \in \partial G$.

3.9. Remark. So far the uniform distribution of directions has been very convenient but not necessary. In the next chapter it will be very essential, however.

4. Conformal invariance.

In this chapter R^2 is viewed as the complex plane C. Assume that a function φ taking complex values is defined in some neighbourhood of z. We recall that φ has a *complex derivative* if

$$\frac{\varphi(z+h)-\varphi(z)}{h} \to \varphi'(z)$$

as $h \to 0$ in C. The function φ is *analytic* in a domain G if it has a derivative at every point of G; moreover, φ is *conformal* in G if it is analytic and $\varphi'(z) \neq 0$ everywhere in G.

In the sequel we shall assume φ to be conformal in the domain G. We are going to study how φ transform B and b.

4.1. Notation. We define a new random walk B_φ as follows:

$$\begin{aligned}
&B_\varphi(0,\omega) = \varphi(0);\\
&B_\varphi(t+\Delta t,\omega) = B_\varphi(t,\omega) + |\varphi'(B(t,\omega))|\sqrt{2\Delta t}\,\omega'(t+\Delta t),
\end{aligned}$$

where $\omega'(t+\Delta t) \in S$ is the first element in the positive direction after

$$\varphi'(B(t,\omega))\omega(t+\Delta t) \,/\, |\varphi'(B(t,\omega))|.$$

4.2. Theorem. *There is an internal set A which satisfies $\bar{P}(A) \approx 1$ and*

$$B_\varphi(t,\omega) \approx \varphi(B(t,\omega))$$

for finite $t \in T$ and all $\omega \in A$.

Proof. Taylor's formula gives

$$\begin{aligned}
\varphi(B(t+\Delta t,\omega) = \varphi(B(t),\omega) &+ \sqrt{2\Delta t}\,\varphi'(B(t,\omega))\omega(t+\Delta t)\\
&+ \Delta t \varphi''(B(t,\omega))\omega(t+\Delta t)^2 + \varepsilon(t,\omega).
\end{aligned}$$

Thus

$$|\varphi(B(t,\omega)) - B_\varphi(t,\omega)| \leqq |S_1| + |S_2| + |S_3|,$$

where

$$S_1 = \sum_{s=0}^{t-\Delta t} \sqrt{2\Delta t}\,(\varphi'(B(s,\omega)\omega(s+\Delta t) - |\varphi'(B(s,\omega))|\,\omega'(s+\Delta t)),$$

with ω' as in the definition of B_φ;

$$S_2 = \sum_{s=0}^{t-\Delta t} \Delta t \varphi''(B(s,\omega));$$

and

$$S_3 = \sum_{s=0}^{t-\Delta t} \varepsilon(s,\omega).$$

If t is finite, $|S_1| \approx 0$, because by the definition of B_φ,

$$|S_1| \leqq tH \cdot \sqrt{2\Delta t} \cdot 2\pi \cdot 1/N = t \cdot \sqrt{2H} \cdot 2\pi/N \approx 0.$$

To show $|S_2| \approx 0$ one observes that

$$\Delta t \varphi''(B(s,\omega)) = \sqrt{\Delta t/2}\, \varphi''(B(s,\omega)) \cdot \sqrt{2\Delta t}\, \omega(s+\Delta t)^2.$$

Because $z \mapsto z^2$ preserves uniform distribution on $\{z \mid |z| = 1\}$, S_2 has the form

$$\sqrt{\Delta t/2} \sum_{s=0}^{t-\Delta t} G(s,\omega'')(B(s+\Delta t,\omega'') - B(s,\omega'')),$$

where G is nonanticipating and ω'' varies over Ω with probability measure $\bar{P}$. The density argument used to show that nonstandard approach to stochastic integration gives the standard Itô integrals shows then that $|S_2| \approx 0$, if t is finite. Finally, $|S_3| \approx 0$, because $\varepsilon(s,\omega)$ is of the form $\varepsilon'(s,\omega)\sqrt{\Delta t}^3$ with $\varepsilon'(s,\omega)$ finite and hence

$$|S_3| \leqq \max_s |\varepsilon'(s,\omega)| \sum_{s=0}^{t-\Delta t} \sqrt{\Delta t}^3 \leqq \max_s |\varepsilon'(s,\omega)| \cdot t \cdot \sqrt{\Delta t}.$$

The process b has the following properties:

(i) For almost all $\omega \in \Omega$, the path $b(\cdot,\omega)$ is continuous and unbounded;
(ii) For all $\varepsilon > 0$, the variables $\Delta_1(\varepsilon); \Delta_2(\varepsilon), \ldots$, where

$$\Delta_i(\varepsilon) = b(\tau_i(\varepsilon), \cdot) - b(\tau_{i-1}(\varepsilon), \cdot) \text{ and}$$
$$\tau_i(\varepsilon) = \inf\{t > \tau_{i-1}(\varepsilon) \mid |b(t,\cdot) - b(\tau_{i-1}(\varepsilon),\cdot)| = \varepsilon\}$$

are independent and evenly distributed on the set $\{z \mid |z| = \varepsilon\}$ with respect to Lebesgue measure.

4.3. Definition. A process satisfying (i) and (ii) above is called a *generalized Brownian motion.* (Cf. the outline [3]).

4.4. Theorem. *The process $(t,\omega) \mapsto \varphi(b(t,\omega))$ is a generalized Brownian motion.*

Proof. It is enough to verify the following result:

CLAIM. If u is harmonic in a domain G and continuous on $\mathrm{cl}\, G$ and if $x \in G$, then

$$u(x) = E(u(\varphi \circ b)_x (t_{\varphi,x}(\omega), \omega))),$$

where $(\varphi \circ b)_x$ indicates $\varphi(b)$ started from x and $t_{\varphi,x}$ is the time of the first exit of $(\varphi \circ b)_x$ from G.

Here we assume that G satisfies the assumptions (i) and (ii) made in Chapter 2. The Theorem needs the claim in the case, where G is a disc with center x.

To prove the claim, we consider the discrete version. Given a disc $D \subsetneq {}^*G$ with center y and radius $r \in {}^*\mathrm{R}^+$, we have

$${}^*u(y) = \int_{\partial D} {}^*u \, d\mu,$$

where μ is the uniform probability measure on ∂D. So we obtain the estimate

$$(*) \qquad \left| {}^*u(y) - \frac{1}{N} \sum_{s \in S} {}^*u(y + rs) \right| \leqq \frac{a}{N},$$

for a finite, by approximating the difference between the integrand ${}^*u(z)$ and the summand ${}^*u(y + rs)$ on each arc between the consecutive points $y + rs'$.

Denote by $B_{\varphi,x}$ the version of B_φ started from x. To be exact, $B_{\varphi,x}$ is defined by a choice of $B_\varphi(t_0, \omega_0) \approx x$, $t_0 \in T$, as $B_{\varphi,x}(t,\omega) = B_\varphi(t_0 + t, \omega')$, where $\omega'(t') = \omega_0(t')$ for $t' \leqq t_0$ and $\omega'(t_0 + t') = \omega(t')$.

Next we estimate ${}^*u(B_{\varphi,x}(0,\omega))$ by the average of all ${}^*u(B_{\varphi,x}(\Delta t,\omega')$; then each ${}^*u(B_{\varphi,x}(\Delta t,\omega'))$ by the average of all ${}^*u(B_{\varphi,x}(2\Delta t,\omega'')$ with $\omega'(t) = \omega''(t)$. This process is carried on through all $t \in T$ as long as the points $B_{\varphi,x}(t,\omega)$ lie in ${}^*G$. Collecting these approximations together, we obtain by $(*)$

$${}^*u(x) \approx {}^*u(B_{\varphi,x}(0,\omega')) \approx \bar{E}\, {}^*u(B_{\varphi,x}(T_{\varphi,x}(\omega),\omega)),$$

where $T_{\varphi,x}$ is the time of first exit of $B_{\varphi,x}$ from ${}^*G$. Finally, it follows from the general properties of liftings that the Claim holds.

4.5. REMARK. 1) The nonstandard proof of conformal invariance of Brownian motion has two parts. In the first (4.2) we showed essentially that $\varphi(b)$ is generated by an internal random walk B_φ differing from B only in that the lengths of steps vary (S-continuously). In the other part (proof of 4.4) we showed that every such random walk generates a generalized Brownian motion.

2) The steps of B_φ have been actually defined using the first nonzero term of the Taylor expansion of φ. This can be done also when φ is only supposed to be a nonconstant analytic function. Because the derivative of such a φ can vanish only at finitely many points, those ω leading to such zeros can be neglected in the arguments. So our results and proofs hold also for nonconstant analytic functions.

The following is due to Nevanlinna [6].

4.6. THEOREM. *If $\varphi : \mathrm{cl}\, G \to \mathrm{cl}\, G'$ is a homeomorphism with $\varphi \upharpoonright G$ conformal (or just nonconstant analytic), then for all $x \in G$ and all Borel sets $A \subsetneq \partial G$,*

$$\mu_x^G(A) = \mu_{\varphi(x)}^{G'}(\varphi A),$$

where the indices G and G′ indicate to which domains the harmonic measures correspond.

PROOF. We may assume $0 \in G$. The internal versions $I_\varphi G'$ and $D_\varphi G'$ are the sets of those $B_\varphi(t,\omega)$ with $B(t,\omega) \in IG$ and $B(t,\omega) \in DG$, respectively. Then B_φ generates internal measures $\bar{M}_{\varphi,x}$ and corresponding Loeb measures $M_{\varphi,x}$ on $D_\varphi G'$ in analogy with chapter 2. In the same way we obtain measures $\mu_{\varphi,x}$ on $\partial G'$.

Let $u : \mathrm{cl}\, G' \to \mathsf{R}$ be continuous and harmonic on G'. It follows from the proof of Theorem 4.4 by methods of Chapter 2 that for $x \in I_\varphi G'$,

$$^*u(x) \approx \bar{E}{}^*u(B_{\varphi,x}(T_{\varphi,x}(\omega),\omega)) \approx \int_{D_\varphi G'} {}^*u\, d\bar{M}_{\varphi,x}.$$

Therefore for $x \in G'$,

$$u(x) = \int_{\partial G'} u\, d\mu_{\varphi,x},$$

which means that $\mu_{\varphi,x}$ is the harmonic measure $\mu_x^{G'}$. On the other hand, if $A \subsetneq DG$ is an interval,

$$\bar{P}\{\omega \mid B_x(T_x(\omega),\omega) \in A\} \approx \bar{P}\{\omega \mid B_{\varphi,{}^*\varphi(x)}(\omega),\omega) \in {}^*\varphi A\}.$$

So for all Borel sets $A \subsetneq \partial G$,

$$\mu_x^G(A) = \mu_{\varphi,\varphi(x)}(\varphi A).$$

ACKNOWLEDGEMENT. We wish to thank professor O. Martio for enlightning discussions on the foundations of harmonic analysis.

REFERENCES

1. R. Anderson, *A non-standard representation of Brownian motion and Itô integration*, Israel J. Math. 25 (1976), 15–46.
2. N. Cutland, *Nonstandard measure theory and its applications*, Bull. London Math. Soc. 15 (1983), 529–589.
3. B. Davis, *Brownian motion and analytic functions*, Ann. Probab. 7 (1979), 913–932.
4. H. J. Keisler, *An infinitesimal approach to stochastic analysis*, Mem. Amer. Math. Soc. 297 (1984).
5. S. G. Krantz, and H. R. Parks, *Distance to C^k hypersurfaces*, J. Differential Equations 40 (1981), 116–120.
6. R. Nevanlinna, *Eindeutige analytische Funktionen* (Grundlehren Math. Wiss. 46), Springer-Verlag, Berlin - Göttingen, 1936.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI
HALLITUSKATU 15
00100 HELSINKI 10
FINLAND