MATH. SCAND. 57 (1985), 359–374

MEASURABLE SUBGROUPS AND NONMEASURABLE CHARACTERS

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In the first section we prove that certain continuous homomorphic images of locally compact abelian (LCA) groups must be Borel measurable. In particular, we obtain as a corollary that each LCA group G is Borel measurable as a subgroup of its Borel compactification bG. This vastly improves the 1965 result of N. Th. Varopoulos [11] which states that G is necessarily Haar measurable in bG.

The remainder of the paper is devoted to a study of subgroups and characters of LCA groups G that are nonmeasurable with respect to one or another class of regular Borel outer measures on G. Here the term *character* means any homomorphism of G into the circle group $\boldsymbol{T} = \{z \in \mathbf{C} : |z| = 1\}$. All of the subgroups that we produce are obtained as kernels of characters. On p. 133 of Graham and McGehee [1] it is left as an open question whether a measure $\nu \in M(G)$ for which *each* character of G is ν-measurable must be discrete. Our Theorem 2.6 shows that the answer is affirmative for every LCA group G. More dramatically, it follows from our Theorem 2.2 that each nondiscrete metrizable G has a single character that is μ-measurable only for discrete $\mu \in M(G)$. In Theorem 2.4 we give a very simple structural condition that is both necessary and sufficient in order that a nondiscrete separable metrizable LCA group possess a character whose kernel is *universally nonmeasurable* in the sense that if it is μ-measurable for some regular Borel outer measure μ, then μ must be discrete.

Theorem 2.5 gives two simple characterizations of those LCA groups which have a character whose kernel is both dense and not Haar measurable. We conclude the paper with two results that show the intrusive and pervasive properties of certain dense nonmeasurable sets in arbitrary locally compact (LC) groups. These include as special cases certain properties of Lebesgue nonmeasurable subgroups of $\mathbf{R}$ that seem to have gone unnoticed until the recent paper of A. Simoson [8].

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(1) Partially supported by NSF Grant DMS-8320479.
Received June 15, 1984; in revised form September 25, 1984.

1. Borel measurability of subgroups.

Let $f: G_0 \to G_1$ be a continuous group homomorphism from a LCA group into another. Then, for each $\gamma \in \hat{G}_1$ (the dual group of G_1), $\gamma \circ f$ is a continuous character of G_0, that is, $\gamma \circ f \in \hat{G}_0$. The mapping $f^*: \hat{G}_1 \to \hat{G}_0$ defined by $f^*(\gamma) = \gamma \circ f$ is called the adjoint of f. It is a well-known and easy fact that f is one-to-one if and only if $f^*(\hat{G}_1)$ is dense in $\hat{G}_0$.

THEOREM 1.1. *Suppose that G_0 and G_1 are two LCA groups, that $f: G_0 \to G_1$ is a continuous group homomorphism, and that f^* maps $\hat{G}_1$ onto $\hat{G}_0$. Then $f(G_0)$ is Borel measurable in G_1.*

To prove this result, we need the following

LEMMA 1.2. *In addition to the hypotheses of the theorem, suppose that D is an abelian group, that $g: G_0 \to D$ is a group homomorphism, that the kernel of g is open and σ-compact, and that E is an independent subset of D. Then $f[g^{-1}(\langle E \rangle)]$ is Borel measurable in G_1, where $\langle E \rangle$ denotes the subgroup of D generated by E.*

PROOF. Put $H = g^{-1}(0)$. For each x in $g^{-1}(\langle E \rangle)$, there exists a unique integer-valued function m_x on E, with finite support, such that

$$g(x) = \sum_{e \in E} m_x(e)e, \text{ and} \tag{1}$$

$$0 \leqq m_x(e) < \operatorname{ord}(e) \text{ if } \operatorname{ord}(e) < \infty. \tag{2}$$

Given $n \in \mathbf{Z}^+$, define

$$E_n = \{x \in g^{-1}(\langle E \rangle): \sum_{e \in E} |m_x(e)| = n\}. \tag{3}$$

Then $g^{-1}(\langle E \rangle)$ is the union of all E_n for $n \geqq 0$, so it will suffice to prove that each $f(E_n)$ is Borel measurable in G_1. Since $E_0 = g^{-1}(0) = H$ and H is σ-compact, $f(E_0)$ is σ-compact and therefore Borel measurable in G_1.

Fix a natural number n and an element x of E_n. Let $e_1, \ldots, e_r$ be the distinct elements e of E with $m_x(e) \neq 0$. For each $j = 1, 2, \ldots, r$, we define a function γ_j on E and a complex number a_j by setting

$$\gamma_j(e_j) = a_j = \begin{cases} \exp[2\pi i/\operatorname{ord}(e_j)] & \text{if } \operatorname{ord}(e_j) < \infty, \\ \exp[\pi i/(3n)] & \text{otherwise,} \end{cases} \tag{4}$$

and $\gamma_j(e) = 1$ if $e \neq e_j$. Since E is an independent set in D, γ_j extends to a character of D, which we denote by the same symbol γ_j (cf. (A.7) of [3]). Since g is a homomorphism with $H = g^{-1}(0)$ open, $\gamma_j \circ g$ is a continuous character of G_0. Since f^* maps $\hat{G}_1$ onto $\hat{G}_0$ by hypothesis, it follows that there exists a character $u_j \in \hat{G}_1$ such that $f^*(u_j) = u_j \circ f = \gamma_j \circ g$. Notice that u_j annihilates $f(H)$, and that if $y \in g^{-1}(\langle E \rangle)$, then

$$u_j(f(y)) = \gamma_j(g(y)) = \gamma_j(m_y(e_j)e_j) = a_j^{m_y(e_j)} \tag{5}$$

by the definition of γ_j and a_j.

Now put

$$V_x = \{z \in G_1 : |u_j(z) - u_j(f(x))| < |a_j - 1| \forall_j\}. \tag{6}$$

Then V_x is open in G_1 and $f(H+x) \subset V_x \cap f(E_n)$. (Recall that u_j annihilates $f(H)$, that E_n is a union of cosets of H, and that $x \in E_n$.) If $y \in E_n$ satisfies $f(y) \in V_x$, then

$$\begin{aligned} |a_j^{m_y(e_j)} - a_j^{m_x(e_j)}| &= |u_j(f(y)) - u_j(f(x))| \\ &< |a_j - 1| \quad \text{for all } j = 1,2,\ldots,r \end{aligned}$$

by (5) and (6); hence $m_y(e_j) = m_x(e_j)$ for all j by (2), (3), and (4). Moreover, we have

$$n = \sum_{e \in E} |m_y(e)| \geqq \sum_{j=1}^{r} |m_x(e_j)| = n$$

by (3) since $x, y \in E_n$. Hence $m_y(e) = m_x(e)$ for all $e \in E$, so $g(y) = g(x)$, that is, $y \in H + x$. Thus we have proved that $V_x \cap f(E_n) = f(H+x)$.

Finally let $\{H + x_i : i \in I\}$ be the distinct cosets of H which are contained in E_n, let $F_n = \{x_i : i \in I\}$, and write V_i for V_{x_i}. Thus $E_n = H + F_n$ by (3), and $V_i \cap f(E_n) = f(H + x_i)$ for all $i \in I$. Given a compact subset K of H, put

$$B = B_n(K) = f(E_n) \cap f(K + F_n). \tag{7}$$

Notice that f is one-to-one since $f^*(\hat{G}_1) = \hat{G}_0$. Therefore we infer from (7) that

$$\begin{aligned} V_i \cap B &= [V_i \cap f(E_n)] \cap f(K + F_n) \\ &= f(H + x_i) \cap f(K + F_n) = f(K + x_i). \end{aligned} \tag{8}$$

Since $K + x_i$ is compact, its continuous image $f(K + x_i)$ is compact in G_1. Since V_i is open in G_1, it follows from (8) that $f(K + x_i) = V_i \cap B^-$ for all $i \in I$. Therefore we have

$$f(K + F_n) = \bigcup_{i \in I} f(K + x_i) = \left[\bigcup_{i \in I} V_i \right] \cap B^-, \tag{9}$$

whence $f(K + F_n)$ is a Borel set in G_1. Since K was an arbitrary compact subset of the σ-compact set H, we conclude that $f(E_n) = f(H + F_n)$ is a Borel set in G_1. This completes the proof.

PROOF OF THEOREM 1.1. Let G_0, G_1 and f be as in the hypotheses of Theorem 1.1. Choose and fix any σ-compact open subgroup H of G_0. We imbed G_0/H into a divisible abelian group D [3, (A.15)]. Notice that D can be decomposed into a weak direct sum $\bigoplus_{i \in I} G_i$, where each G_i is (isomorphic to) either $\mathbf{Q}$ or $\mathbf{Z}(p^\infty)$ for some prime $p = p_i$ [3, (A.14)]. For each $k \geqq 1$ and $i \in I$, define $x_{k,i} \in G_i \subset D$ by

$$x_{k,i} = \begin{cases} k^{-1} & \text{if } G_i = \mathbf{Q}, \\ p^{-k} \pmod 1 & \text{if } G_i = \mathbf{Z}(p^\infty). \end{cases}$$

Then the set $F_k = \{x_{k,i} : i \in I\}$ is independent, and D is the countable union of all $\langle F_k \rangle$ for $k = 1, 2, \ldots$.

Let $g : G_0 \to G_0/H$ denote the quotient mapping. We shall regard g as a mapping from G_0 into D in an obvious way. By Lemma 1.2, $f(g^{-1}(\langle F_k \rangle))$ is Borel measurable in G_1 for each $k \geqq 1$. Since

$$f(G_0) = f(g^{-1}(D)) = \bigcup_{k=1}^{\infty} f(g^{-1}(\langle F_k \rangle)),$$

it follows that $f(G_0)$ is Borel measurable in G_1, as desired.

Now let G be an arbitrary LCA group with dual $\hat{G} = \Gamma$, and let Γ_d denote the group Γ with the discrete topology. By definition, the Bohr compactification of G, denoted by bG, is the dual group of Γ_d. If f denotes the natural mapping of G into bG, then f^* is nothing but the identity mapping $\Gamma_d \to \Gamma$. As an immediate consequence of Theorem 1.1, we therefore obtain the following

COROLLARY 1.3. *Every* LCA *group is Borel measurable in its Bohr compactification.*

Although the original proof of this corollary in [2] was erroneous, Graham and Ramsey have recently sent us a new version with a pristine proof.

REMARKS. (a) The surjectivity of f^* in the hypotheses of Theorem 1.1 is *not* superfluous. Let G be any nondiscrete LCA group. Then G contains a subgroup S which is nonmeasurable with respect to Haar measure on G, as will be shown in the next section. If S is equipped with the discrete topology, then the identity mapping $f : S \to G$ is continuous (and one-to-one), but $f(S) = S$ is not Borel measurable in G.

(b) The above proofs show that the Borel measurable sets mentioned in 1.1–1.3 are all countable unions of sets each one of which is the intersection of a closed set with an open set.

2. Nonmeasurable subgroups and characters.

Let μ be a nonnegative countably additive Borel measure on a locally compact Hausdorff space X. Recall that such a measure μ is called *regular* if

(i) $\mu(K) < \infty$ for all compact subsets K of X,

(ii) $\mu(A) = \inf\{\mu(U): U \text{ is open and } A \subset U\}$ for all Borel sets A in X, and

(iii) $\mu(U) = \sup\{\mu(K): K \text{ is compact and } K \subset U\}$ for all open sets U in X (cf. (12.39) of Hewitt–Stromberg [4]).

If $\mu(A)$ is defined by the right-hand side of (ii) for each subset A of X, then the resulting set-function μ is an outer measure on X. We denote by $\mathscr{M}_\mu$ the σ-algebra of all μ-measurable subsets of X.

Now let G be a LCA group and H a subgroup thereof. We denote by $I^+(H)$ the family of all nonnegative regular Borel measures μ on G for which every coset of H is a locally μ-null set (cf. [9]). The H-order of $x \in G$ is defined to be the order of $x + H$ as an element of G/H. A subset K of G is said to be independent modulo H (or H-independent) if (a) $K \cap H = \varnothing$, and (b) whenever $x_1,\dots,x_r$ are distinct elements of K, $n_1,\dots,n_r$ are integers, and $n_1x_1 + \dots + n_rx_r \in H$, then $n_jx_j \in H$, for all $j = 1,2,\dots,r$.

The following lemma is a modification of Theorem 1 of [6] and is essentially proved therein. For the convenience of the reader, we shall give a sketchy proof.

Lemma 2.1. *Let H be an F_σ subgroup of a metrizable* LCA *group G and A a subset of G such that $A \in \mathscr{M}_\mu$ and $0 < \mu(A) < \infty$ for some $\mu \in I^+(H)$. Then there exists an H-independent Cantor set K in G and $x_0 \in G$, with $K + x_0 \subset A$, such that all the elements of K have the same H-order.*

Proof. For each natural number r and $y \in G$, put

$$A(r,y) = A \cap \{x \in G : rx \in y + H\}.$$

Since A is μ-measurable and H is an F_σ, $A(r,y)$ is μ-measurable.

Suppose first that $\mu(A(r,y)) > 0$ for some $r \geqq 1$ and some $y \in G$. Let q be the smallest such r, and let y_0 be any corresponding element of G. Notice that $q \geqq 2$ since $\mu \in I^+(H)$. Choose and fix any $x_0 \in A(q,y_0)$. Then $\mu(A(r,rx_0)) = 0$ for $r = 1,2,\dots,q-1$ by our choice of q. Since

$$0 < \mu(A(q,y_0)) < \infty,$$

it follows from the regularity of μ that $A(q,y_0) \setminus \{\bigcup_{r=1}^{q-1} A(r,rx_0)\}$ contains a compact set K_0 with $\mu(K_0) > 0$. Replacing K_0 by the support of the measure $\mu | K_0$, we may assume that each relatively open nonvoid subset of K_0 has positive μ-measure. Notice that μ is a continuous measure since $\mu \in I^+(H)$, so K_0 is a compact perfect set. Moreover, each element of $K_0 - x_0$ has H-order q by construction, and $K_0 - x_0$ has the following property: whenever $V_1, V_2, \ldots, V_m$ are finitely many, pairwise disjoint, relatively open nonvoid subsets of $K_0 - x_0$, then there exist $x_j \in V_j$ such that $\{x_1, \ldots, x_m\}$ is H-independent (cf. Lemma 2 of [6]). It is now easy to construct an H-independent Cantor set $K \subset K_0 - x_0$ (cf. Lemma 5 of [7]).

Finally assume that $\mu(A(r,y)) = 0$ for all natural numbers r and all $y \in G$. Then

$$A \setminus \{x \in G : rx \in H \text{ for some } r \in \mathsf{N}\}$$

contains a compact perfect set K_0 with $\mu(K_0) > 0$. One checks that K_0 contains a totally disconnected perfect Kronecker set K such that $\langle K \rangle \cap H = \{0\}$, where as before $\langle K \rangle$ denotes the subgroup of G generated by K.

THEOREM 2.2. *Let H be a nonopen F_σ subgroup of a metrizable* LCA *group G, and D any subgroup of G with* Card $D < c =$ Card R. *Then there exist a character of G which annihilates $D + H$ and which is nonmeasurable with respect to all nonzero members of $I^+(H)$.*

PROOF. Choose and fix a σ-compact open subgroup G_0 of G. Let $\mathscr{K}$ denote the family of all H-independent Cantor sets K in G_0 such that the elements of K all have the same H-order r_K. We shall first show that Card $\mathscr{K} = c$.

Indeed, $H \cap G_0$ is a σ-compact nonopen subgroup of G_0 by the hypotheses, so it has Haar measure zero (otherwise, $H \cap G_0$ would be open by Steinhaus' Theorem [10]). It follows from Lemma 2.1 (applied to G_0) that G_0 contains an $(H \cap G_0)$-independent (hence H-independent) Cantor set with the desired property. Therefore Card $\mathscr{K} \geqq c$ since each Cantor set K is homeomorphic to $K \times K$ and has cardinality equal to c. On the other hand, G_0 is σ-compact and metrizable, so the family of all Borel subsets of G_0 has cardinality $\leqq c$ (cf. (10.25) of [4]). Thus we have Card $\mathscr{K} = c$.

Now we use the same symbol c to denote the least ordinal number having c predecessors. Observe that this set of predecessors admits a partition into c disjoint subsets each having c elements. Therefore we can index $\mathscr{K}$ as $\{K_a : a < c\}$ in such a way that each member of $\mathscr{K}$ appears with c different indices (i.e., Card $\{a : a < c \text{ and } K_a = K\} = c$ for each $K \in \mathscr{K}$). For each $a < c$, let r_a denote the common H-order of the elements of K_a. We proceed by transfinite induction to select $x_a \in K_a (a < c)$ as follows.

Suppose that $b < c$ and that the elements x_a have been chosen for all $a < b$. Put

$$E_b = D + \langle\{x_a : a < b\}\rangle,$$

so that $E_b = D$ if $b = 0$. Then Card $E_b < c$. We claim that K_b contains an element x_b such that $kx_b \notin E_b + H$ for all natural numbers $k < r_b$. Indeed, notice that Card $K_b = c$ and that Card $(N \times E_b) < c$. So, if our claim were false, there would exist a natural number $k < r_b$ and $y \in E_b$ such that

$$\text{Card}\,\{x \in K_b : kx \in y + H\} = c.$$

So $kx_1 - kx_2 \in H$ for some different elements x_1 and x_2 of K_b, whence $kx_1 \in H$ by the H-independence of K_b. Since $k < r_b$, this contradicts the definition of r_b. Thus we can find an element $x_b \in K_b$ with the desired property, which completes our transfinite induction. Notice that the set $E = \{x_a : a < c\}$ is independent modulo $D + H$ and that each x_a has $(D + H)$-order r_a.

Now define $\boldsymbol{T}(r) = \{z \in \boldsymbol{T} : z^r = 1\}$ for each natural number r and $\boldsymbol{T}(\infty) = \boldsymbol{T}$. Let $\gamma : E \to \boldsymbol{T}$ be any function such that

$$\gamma(\{x_a : K_a = K\}) = \boldsymbol{T}(r_K) \text{ for every } K \in \mathscr{K}.$$

(Recall that each such K appears c times in the list $\{K_a : a < c\}$ and that $r_a = r_K$ if $K_a = K$.) By the last remark and (A.7) of [3], γ extends to a character of G which annihilates $D + H$. It remains to show that γ is nonmeasurable with respect to all nonzero members of $I^+(H)$. Put

$$S = \gamma^{-1}\{e^{it} : 0 \leqq t < \pi\},$$

and assume by way of contradiction that S is μ-measurable for some nonzero $\mu \in I^+(H)$. Choose any compact subset A of G with $\mu(A) > 0$. Then either $\mu(A \cap S) > 0$ or $\mu(A \setminus S) > 0$. It follows from Lemma 2.1 and our definition of $\mathscr{K}$ that there exist $a < c$ and $y \in G$ such that either $K_a + y \subset A \cap S$ or $K_a + y \subset A \setminus S$. But $\gamma(K_a + y) = \boldsymbol{T}(r_a)\gamma(y)$ is a coset of the nontrivial subgroup $\boldsymbol{T}(r_a)$ of $\boldsymbol{T}$, and neither $\gamma(S)$ nor $\gamma(G \setminus S)$ can contain such a coset by the definition of S. This is of course absurd and so the proof is complete.

LEMMA 2.3. *Suppose in addition to the hypotheses of Theorem 2.2 that G contains a σ-compact open subgroup G_0 such that either*

(i) *each nonzero element of $G_0/(G_0 \cap H)$ has infinite order, or*

(ii) *$(H + (p \times G)) \cap G_0 \subset D + H$ for some prime p, where $p \times G = \{px : x \in G\}$.*

Then there exists a character of G whose kernel both contains $D+H$ and is nonmeasurable with respect to all nonzero members of $I^+(H)$.

PROOF. Let $\mathscr{K} = \{K_a : a < c\}$ be as in the proof of Theorem 2.2.

In case (i), it is obvious that $r_a = \infty$ for all $a < c$. Therefore the character γ constructed in the proof of Theorem 2.2 has the property that $\gamma(K_a) = \boldsymbol{T}$ for all $a < c$. If $A \in \mathscr{M}_\mu$ and $0 < \mu(A) < \infty$ for some $\mu \in I^+(H)$, then A contains a translate of some K_a by Lemma 2.1. Hence $\gamma(A) = \boldsymbol{T}$ for all such A. Accordingly neither $\ker\gamma$ nor its complement can contain any set A as above, so $\ker\gamma$ is nonmeasurable with respect to every nonzero member of $I^+(H)$.

In case (ii), we modify the proof of Theorem 2.2 as follows. Let $b < c$ and let E_b be as there. Assume by way of contradiction that for each $x \in K_b$ there exists a natural number $k < p$ such that $kx \in E_b + H + (p \times G)$. Since p is a prime, this means that $K_b \subset E_b + H + (p \times G)$. By a cardinality argument, we can therefore find a $y \in E_b$ and a subset F of K_b such that $\operatorname{Card} F = c$ and $F \subset y + H + (p \times G)$. Choose any $y_1 \in F$. Then $F - y_1 \subset H + (p \times G)$. But $F - y_1 \subset K_b - K_b \subset G_0$, so $F - y_1 \subset D + H$ by (ii). We again use a cardinality argument to obtain $d \in D$ and a set $F_1 \subset F$ such that $\operatorname{Card} F_1 = c$ and $F_1 - y_1 \subset d + H$. Hence $F_1 - F_1 = (F_1 - y_1) - (F_1 - y_1) \subset H$. Since F_1 is an infinite subset of K_b, this contradicts the H-independence of K_b. It follows that the set $E = \{x_a : a < c\}$ in the proof of Theorem 2.2 can be chosen to be independent modulo $D + H + (p \times G)$. So there exists a character γ of G which annihilates $D + H + (p \times G)$ and which satisfies $\gamma(K_a) = \boldsymbol{T}(p)$ for all $a < c$. It is now easy to see that $\ker\gamma$ has the desired property.

A subset of G is called a *universally nonmeasurable* set if it is nonmeasurable with respect to all nonzero members of $I^+(\{0\})$.

THEOREM 2.4. *Let G be a nondiscrete, σ-compact, metrizable* LCA *group, and*

$$G(n) = \{x \in G : nx = 0\} \quad \textit{for } n = 1, 2, \ldots.$$

Then the following two conditions are equivalent:

(a) *G has a character whose kernel is a universally nonmeasurable set.*

(b) *For each prime p, G(p) is either open or discrete.*

PROOF. First assume that there exists a prime p such that $G(p)$ is neither open nor discrete. The mapping $x \to px : G \to G$ induces a continuous one-to-one homomorphism $G/G(p) \to G$ in an obvious way. Since $G(p)$ is nonopen, $G/G(p)$ is nondiscrete and therefore carries a nonzero continuous regular Borel measure $\mu \geqq 0$ supported by a compact set. Being a continuous one-to-one image of $G/G(p)$, the set $p \times G = \{px : x \in G\}$ carries such a measure ν as well.

Now suppose that S is a universally nonmeasurable *subset* of G. Then $p \times G \not\subset S$ (otherwise, S would be ν-measurable), so there exists $x_0 \in G$ such that $px_0 \notin S$. Since $G(p)$ is a nondiscrete LC group, we also have $S \cap [x_0 + G(p)] \neq \varnothing$. Pick any element s from this nonvoid intersection. Then $ps = px_0 \notin S$ by our choices of s and x_0. Accordingly no subsemigroup of G can be universally nonmeasurable. Therefore we have proved that (a) implies (b). (Notice that neither the σ-compactness nor the metrizability of G has been used in the above argument.)

Now suppose conversely that (b) obtains. We shall split the proof of (a) into two cases. In each of these two cases, we shall first make use of the σ-compactness hypothesis to obtain a certain structural condition on G. As soon as this is done, the σ- compactness of G will play no role in our construction of a character with the desired property.

CASE 1. Suppose that $G(p)$ is discrete for every prime p. Then we claim that the torsion part of G, denoted by G_t, is (at most) countable. Indeed, assume this is false. Since G_t is the countable union of all $G(n)$ for $n = 2, 3, \ldots$, it follows that $G(n)$ is uncountable for some n. But $G(n)$ is a closed subgroup of the σ-compact group G, so $G(n)$ is nondiscrete. Since a Haar measure on $G(n)$ is a member of $I^+(H)$ where $H = \{0\}$, it follows from Lemma 2.1 that $G(n)$ contains an independent Cantor set K such that all of the elements of K have the same order q. Let p be any prime divisor of q. Then $(q/p) \times K$ is a perfect subset of $G(p)$, which contradicts our Case 1 hypothesis. Hence G_t must be countable.

We replace the countability of G_t by the weaker hypothesis that G is a nondiscrete metrizable LCA group such that the F_σ subgroup G_t does *not* contain any perfect set. Then $I^+(\{0\}) = I^+(G_t)$, so Lemma 2.3 with $H = G_t$ and $D = \{0\}$ yields a character of G with the desired property.

CASE 2. Finally assume that $G(p)$ is open for some prime p. Since G is σ-compact, it follows that $G/G(p)$ is countable, whence so is the subgroup $p \times G$ of G. In particular, if G_0 is any σ-compact open subgroup of $G(p)$, then $\operatorname{Card}[(p \times G) \cap G_0] < c$. Thus the desired result follows from Lemma 2.3 with $H = \{0\}$ and $D = (p \times G) \cap G_0$.

Now we proceed to study the existence or nonexistence of dense subgroups of LCA groups G which are nonmeasurable with respect to λ, a Haar measure on G. The existence of such subgroups is known in some special cases (see, e.g., p. 227 of [3]). Recall that G is called an I-group if each neighborhood of 0 in G contains an element of infinite order.

THEOREM 2.5. *For a nondiscrete* LCA *group G, the following conditions are equivalent to each other:*

(a) *G has a character whose kernel is both dense in G and not λ-measurable.*

(b) *G contains a dense proper subsemigroup.*

(c) *Either G is an I-group, or $p \times G$ is nonopen for some prime p.*

PROOF. Recall that $p \times G = \{px : x \in G\}$ for all p.

First suppose that G is an I-group. Then G contains a closed metrizable subgroup I which is an I-group by (2.5.5) of [5]. Choose a sequence $(V_n)_1^\infty$ of compact symmetric neighborhoods of $0 \in G$ so that $2V_{n+1} \subset V_n$ for all n and $I \cap P = \{0\}$, where $P = \bigcap_1^\infty V_n$. Then P is a compact subgroup of G and G/P is a metrizable I-group (cf. (8.7) of [3]). Let H denote the torsion part of G/P. Then H is a nonopen F_σ subgroup of G/P and every nonzero element of $(G/P)/H$ has infinite order. Thus Lemma 2.3 yields a character γ of G/P whose kernel both contains H and is nonmeasurable with respect to all nonzero members of $I^+(H)$.

Let $g : G \to G/P$ be the quotient mapping. We claim that $\ker(\gamma \circ g)$ is nonmeasurable with respect to each nonzero member of $I^+(g^{-1}(H))$. Indeed, assume that $\ker(\gamma \circ g)$ is μ-measurable for some nonzero $\mu \in I^+(g^{-1}(H))$. Then there exists a compact subset K of G such that $\mu(K) > 0$ and either

$$K \cap \ker(\gamma \circ g) = \varnothing \quad \text{or} \quad K \subset \ker(\gamma \circ g).$$

Define $\nu \in M(G/P)$ by $\nu(E) = \mu(K \cap g^{-1}(E))$ for Borel sets E in G/P. It is evident that ν is carried by $g(K)$ and either

$$g(K) \cap \ker\gamma = \varnothing \quad \text{or} \quad g(K) \subset \ker\gamma.$$

Thus $\ker\gamma$ is ν-measurable. But $0 \neq \nu \in I^+(H)$ since $\nu(g(K)) = \mu(K) > 0$ and $\mu \in I^+(g^{-1}(H))$. This contradicts our choice of γ and so confirms our claim. Since $g^{-1}(H)$ is a nonopen F_σ-subgroup of G, it follows from Steinhaus' theorem [10] that $\lambda \in I^+(g^{-1}(H))$. Therefore $\ker(\gamma \circ g)$ is nonmeasurable with respect to $\lambda | V$ for every nonvoid open subset V of G; in particular, $\ker(\gamma \circ g)$ must be dense in G. This proves that every I-group has properties (a) and (b).

To complete the proof, we suppose that G is *not* an I-group. Then

$$G_t = \cup \{G(n) : n \in \mathbf{N}\}$$

contains a neighborhood of 0 in G, and so Baire's category theorem shows that the subgroup $G(n)$ is open for some $n \in \mathsf{N}$. Let $q = q(G)$ be the least such n. Notice $q > 1$ since G is nondiscrete. Now suppose that G contains a proper dense subsemigroup S. We claim that S is actually a subgroup. Indeed, $G(q)$ is open; so given $s \in S$, there exist $x \in G(q)$ and $t \in S$ such that $x - s = t$. Then $-qs = qt$, whence $-s = (q-1)s + qt \in S$. This confirms our claim that S is a (dense) subgroup. Let T_t denote the torsion part of the circle group T. Since T_t is divisible, we can easily construct a non-constant character γ of G such that $\gamma = 1$ on S and $\gamma(G) \subset T_t$. Plainly $\ker \gamma$ is a proper dense subgroup of G with countable index. No such subgroup D of G can be λ-measurable. (Indeed, assume that D is λ-measurable. Since D has a countable index in G, D must contain a compact set having positive λ-measure. Then D is open in G by Steinhaus' Theorem and so equal to G.) Thus we have proved that (b) implies (a).

Now assume that there is a prime p such that $p \times G$ is nonopen. Notice that each nonconstant character γ of G which annihilates $p \times G$ must satisfy

$$\gamma(G) = T(p) = \{z \in T : z^p = 1).$$

If $(p \times G)^-$ is open in G, choose any such character γ such that $\gamma((p \times G)^-) = T(p)$. Given $x \in G$, there exists $y \in (p \times G)^-$ such that $\gamma(y) = \gamma(x)$, Then

$$x \in (y + \ker \gamma) \subset (\ker \gamma)^- + \ker \gamma = (\ker \gamma)^-.$$

Thus $\ker \gamma$ is dense in G and its index is p; hence, as before, it is not λ-measurable. If $(p \times G)^-$ is nonopen, Lemma 6 of [6] assures that G contains a compact subgroup Q such that G/Q is metrizable and such that $R = (p \times G)^- + Q$ is nonopen. (Notice that R is closed, being the algebraic sum of a closed set and a compact set.) Thus G/R is a nondiscrete metrizable LCA group and all of its nonzero elements have order p. Therefore Lemma 2.3 applies to G/R with $H = D = \{0\}$, and yields a character γ of G/R such that $\ker \gamma$ is nonmeasurable with respect to all nonzero members of $I^+(\{0\})$. So, just as in the I-group case, we infer that $\ker(\gamma \circ g)$ is dense in G but is not λ-measurable, where $g: G \to G/R$ is the quotient mapping. Our proof that (c) implies (a) is complete.

Finally assume that (c) fails to hold, i.e., that $p \times G$ is open for all primes p. We need to show that there exists no character as in (a). To this end, factor $q = q(G)$ as $q = p_1^{k_1} \dots p_r^{k_r}$, where the p_j are distinct primes and the k_j are natural numbers. Let γ be any character of G with dense kernel. Then

$$(x + G(q)) \cap (\ker \gamma) \neq \varnothing \quad \text{for each } x \in G.$$

Hence there exists $y = y_x \in G(q)$ such that $\gamma(x + y) = 1$. But $qy = 0$, so

$\gamma(qx) = \gamma(q(x+y)) = 1$. Since $x \in G$ is arbitrary, this proves that $\gamma^q = 1$ on G.

Now put $H = G(q) \cap \bigcap_1^r (p_j \times G)$, and notice that H is an open subgroup of G by the present hypotheses. Since γ has finite order, $\gamma(H)$ is a finite subgroup of T. Choose the largest natural number n and $h \in H$ such that $\gamma(h) = \exp[2\pi i/n]$. Let p denote any one of the p_j. By the definition of H, there exists $y \in G$ such that $py = h$. Since H is open and $\ker\gamma$ is dense in G, we can find $z \in H$ such that $y - z \in \ker\gamma$. Then

$$\gamma(y) = \gamma(z) = \exp[2\pi i m/n] \quad \text{for some } m \in \mathsf{N}$$

by the definition of n. Therefore

$$\exp[2\pi i/n] = \gamma(h) = \gamma(py) = \exp[2\pi i p m/n]. \tag{1}$$

Evidently n divides q, and (1) holds for all $p = p_j$ (and some $m = m_p$). It follows immediately that $n = 1$, that is, $\gamma = 1$ on the open subgroup H. Since $\ker\gamma$ is dense in G, we conclude that $\gamma = 1$ on G. In other words, the constant character is the only character of G whose kernel is dense in G, provided (c) fails to hold. Thus (a) implies (c), which completes the proof.

It is a well-known fact that every λ-measurable character of a LC group is continuous (cf. (22.19) of [3]). Since each nondiscrete LCA group has a discontinuous character, it follows immediately that each such group has a λ-nonmeasurable character. The following theorem provides a stronger result.

THEOREM 2.6. *Let H be a nonopen F_σ subgroup of a* LCA *group G, D a subgroup of G with* Card $D < c$, *and $0 \neq \nu \in I^+(H)$. Then there exists a character of G whose kernel both contains $D + H$ and is nonmeasurable with respect to ν.*

PROOF. Choose and fix a compact subset A of G with $\nu(A) > 0$. Replacing ν by $\nu|A$, we may and do assume that $\nu(G) < \infty$.

Since H is an F_σ and $\nu(x+H) = 0$ for all $x \in G$, it follows from Lemma 6 of [6] that G contains a compact subgroup P such that G/P is metrizable and such that $\nu(x+P+H) = 0$ for all $x \in G$. (First choose any σ-compact open subgroup G_1 of G so that $\nu(G \setminus G_1) = 0$. Then choose a compact subgroup P of G_1, with metrizable G/P, so that $\nu(x + P + (H \cap G_1)) = 0$ for all $x \in G$.) Let $g: G \to G/P$ be the quotient mapping, and let $\mu \in M(G/P)$ be the measure defined by $\mu(E) = \nu(g^{-1}(E))$ for all Borel subsets E of G/P. Plainly $g(H) = g(P+H)$ is a nonopen F_σ subgroup of G/P, μ is a member of $I^+(g(H))$, and $\mu(g(A)) = \nu(A+P) > 0$. So the proof of Lemma 2.1 yields a compact subset B of G/P and $y_0 \in G/P$ such that the elements of B all have the same $g(H)$-order, $B + y_0 \subset g(A)$, and $\mu(B + y_0) > 0$.

Choose and fix any σ-compact open subgroup G_0 of G/P which contains $B \cup \{y_0\}$, and put $D_0 = g(D) + \langle\{y_0\}\rangle$. Let $\mathscr{K}$ and γ be as in the proof of Theorem 2.2 with G, H, and D replaced by G/P, $g(H)$, and D_0, respectively. Then $\gamma \circ g$ is a character of G which annihilates $D + H$. Choose any compact subset K of $g^{-1}(B + y_0)$ with $\nu(K) > 0$. (Notice that $\nu(g^{-1}(B + y_0)) = \mu(B + y_0) > 0$.) We claim that $\gamma(g(K))$ contains a subgroup of T of order > 1.

Indeed, $g(K)$ is a compact subset of $B + y_0$ and $\mu(g(K)) = \nu(P + K) > 0$. It follows from the proof of Lemma 2.1 and the definition of $\mathscr{K}$ that there exists $a < c$ such that $K_a + y_0 \subset g(K)$. But $y_0 \in D_0 \subset \ker \gamma$, so

$$T(r_a) = \gamma(K_a + y_0) \subset (\gamma \circ g)(K),$$

which confirms our claim. Thus the argument at the very end of the proof of Theorem 2.2 shows that $\ker(\gamma \circ g)$ is not ν-measurable, as desired.

REMARKS. (a) In all of the above results, there is enough freedom of choice of the function γ on the set E to produce 2^c distinct characters whose kernels are nonmeasurable in one of various senses.

(b) It should be noted that some λ-nonmeasurable characters have λ-measurable kernels; even kernels that reduce to $\{0\}$. For instance, consider $G = \mathrm{R}$. Let B be a Hamel basis for R (over Q) which contains 1. Any one-to-one function $f: B \to B \setminus \{1\}$ extends to a one-to-one endomorphism of R (still called f) which takes no nonzero rational value. Then the formula

$$\gamma(x) = \exp[2\pi i f(x)] \ \forall x \in \mathrm{R}$$

defines a character of R whose kernel is $\{0\}$. Plainly γ is discontinuous, so it is not λ-measurable.

(c) Let G be a LCA group, and let G_τ be the group G with a LC group topology τ which is strictly stronger than the original topology of G. Regard $M(G_\tau)$ as a subalgebra of $M(G)$ in an obvious way [1]. Then we claim that $M(G_\tau)$ consists of exactly those measures $\mu \in M(G)$ for which every character in $(G_\tau)\hat{}$ is μ-measurable. This gives an affirmative answer to the problem stated on p. 133 of [1].

Indeed, choose and fix any σ-compact open subgroup H of G_τ. Then H is a nonopen F_σ subgroup of G. Given $\mu \in M(G)$, write $\mu = \mu' + \nu$, where $\mu' \in M(G_\tau)$ and $\nu \perp M(G_\tau)$. Choose a countable subgroup D of G such that μ' is concentrated on $D + H$. If $\mu \notin M(G_\tau)$, then $|\nu|$ is a nonzero member of $I^+(H)$, so Theorem 2.6 provides a character γ of G whose kernel contains

$D + H$ but is *not* ν-measurable. Since H is open in G_τ, γ is in $(G_\tau)\hat{}$ but its kernel is obviously not μ-measurable. This confirms our claim.

The corollary of our next result reveals a kind of pervasiveness of nonmeasurability that is carried by all Haar nonmeasurable dense subsemigroups.

THEOREM 2.7. *Let G be a* LC *group, λ a left Haar measure on G, and S a subset of G. Suppose that there exist a dense subset D of G and a λ-measurable set A in G such that*

$$(*) \qquad \lambda(A \cap (DS)) < \lambda(A) < \infty.$$

Then S is a locally λ-null set; in particular, S is λ-measurable.

PROOF. By the definition of the outer measure λ and $(*)$, there exists an open set U in G such that $A \cap (DS) \subset U$ and

$$(1) \qquad \lambda(A \cap (DS)) \leqq \lambda(U) < \lambda(A).$$

It follows from the regularity of λ that $A \setminus U$ contains a compact set K with positive λ-measure. Notice that $K \cap (DS) = \varnothing$ and $\lambda(K^{-1}) > 0$.

Now let H be an arbitrary σ-compact open subgroup of G which contains K. We claim that there exists a countable subset C of D^{-1} such that

$$(2) \qquad \lambda((yK) \setminus (CK)) = 0 \quad \text{for all } y \in D^{-1} \cap H.$$

Indeed, write $H = \bigcup_{n=1}^{\infty} H_n$, where the H_n are compact subsets of H with $H_n \subset (H_{n+1})^0$ for all $n \geqq 1$. For each n, choose a countable subset C_n of $D^{-1} \cap H$ such that

$$(3) \qquad \lambda(H_n \cap (C_n K)) = \sup_F \lambda(H_n \cap (FK)),$$

where the supremum is taken over all countable subsets F of $D^{-1} \cap H$. Define C to be the union of all C_n for $n \geqq 1$. It is easy to check that C satisfies (2).

Now notice that $\lambda((yK) \setminus (CK))$ is a continuous function of $y \in G$. In fact, this is an immediate consequence of the fact that $y \to {}_y f$ is continuous from G into $L_1(G)$ for each $f \in L_1(G)$. Since H is open and D (hence D^{-1}) is dense in G, it follows from (2) that $\lambda((xK) \setminus (CK)) = 0$ for all x in H. Let f and g be the characteristic functions of $B = H \setminus (CK)$ and of K^{-1}, respectively. Then we have

$$\begin{aligned}(f * g)(x) &= \int_G f(xy) g(y^{-1}) dy \\ &= \lambda((x^{-1}B) \cap K) = \lambda((xK) \cap B)\end{aligned}$$

$$= \lambda((xK) \cap [H \setminus (CK)]) = 0$$

for all x in G. (Notice that $K \subset H$, so $(xK) \cap H = \emptyset$ if $x \notin H$). Therefore

$$\lambda(K^{-1}) \cdot \lambda(H \setminus (CK)) = \int_G (f * g)(x) dx = 0, \tag{4}$$

which combined with $\lambda(K^{-1}) \neq 0$ implies $\lambda(H \setminus (CK)) = 0$.

Finally notice that $C \subset D^{-1}$ and $K \cap (DS) = \emptyset$. Therefore $H \cap S \subset H \setminus (CK)$, and so $\lambda(H \cap S) = 0$. Since H was an arbitrary σ-compact open subgroup of G which contains K, we conclude that S is a locally λ-null set, as desired.

COROLLARY 2.8. *Let S be a λ-nonmeasurable dense subsemigroup of a* LC *group G. If $A \in \mathcal{M}_\lambda$ has positive finite λ-measure, then*

$$A \cap S \notin \mathcal{M}_\lambda \quad and \quad \lambda(A \cap S) = \lambda(A \setminus S) = \lambda(A).$$

PROOF. Suppose $A \in \mathcal{M}_\lambda$ and $0 < \lambda(A) < \infty$. Put $D = S \cup \{e\}$, where e is the identity of G. Then $DS = S$ since S is a semigroup, so

$$\lambda(A \cap S) = \lambda(A \cap (DS)) \geqq \lambda(A)$$

by Theorem 2.7; hence $\lambda(A \cap S) = \lambda(A)$. If $A \cap S \in \mathcal{M}_\lambda$, then $(A \cap S) \cdot (A \cap S)$ would contain a nonvoid open set (cf. Hewitt–Ross [3; (20.17)]). Since S is a dense subsemigroup of G, it would follow that $S = G$, which contradicts the nonmeasurability of S. Thus we have proved that $A \cap S \notin \mathcal{M}_\lambda$. Finally assume that $\lambda(A \setminus S) < \lambda(A)$. Then there exists an open set U such that $A \setminus S \subset U$ and $\lambda(U) < \lambda(A)$. Then $B = A \setminus U \in \mathcal{M}_\lambda$, $0 < \lambda(B) < \infty$, and $B \cap S = B$, which all together contradicts what we have just proved. Thus $\lambda(A \setminus S) = \lambda(A)$.

REMARK. It is unknown whether or not every compact group has a subgroup that is not Haar measurable. It would suffice to find a subgroup having a countable infinite index.

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