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RANDOM POLYTOPES AND THE VOLUME-PRODUCT OF SYMMETRIC CONVEX BODIES

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1. Introduction.

A well known problem in the theory of convex sets is to find a lower bound for the product of volumes $P(K) = V(K)V(K^*)$, where K is an n-dimensional symmetric convex body (i.e., bounded, symmetric, convex set in the n-dimensional Euclidean space $\mathbf{R}^n$, having non-empty interior) and K^* is the dual body of K with respect to its center of symmetry. An upper bound was obtained by Santalo [10]

$$P(K) \leqq \varkappa_n^2 \tag{1}$$

($\varkappa_n$ denotes the volume of the n-dimensional Euclidean unit ball). For an elementary proof of (1) and related results cf. [9].

A widespread conjecture is that

$$P(K) \geqq 4^n/n! \tag{2}$$

Equality in (2) is known to hold for

$$K = B_1 = B(l_1^n) = \left\{x = (x_1, \ldots, x_n);\ \sum_{i=1}^{n} |x_i| \leqq 1\right\}$$

and for its dual

$$B_\infty = B(l_\infty^n) = \left\{x = (x_1, \ldots, x_n);\ \max_{1 \leqq i \leqq n} |x_i| \leqq 1\right\}.$$

A wider family of convex bodies for which equality holds in (2) was discussed in [9] where (2) was also proved for a special class of convex bodies.

A somewhat weaker problem is whether there is a universal constant $c > 0$ such that $[P(K)]^{1/n} \geqq c/n$ for all n-dimensional K ((1) implies that $[P(K)]^{1/n} \leqq 2\pi e/n$).

In [4] it was shown that

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(3) $$[P(K)]^{1/n} \geqq \frac{c}{n \log n}$$

for all K.

In this paper we show how a result of R. Schneider on the limiting expectation of the number of vertices of a random polytope, can be adapted and used to prove (2) for a class of convex bodies – the class of *zonoids* and for bodies constructed using them. We remark that Mahler [8] proved (2) for $n = 2$, this becomes a special case of Theorem 2 here since all 2-dimensional symmetric convex bodies are zonoids.

As for the general problem of confirming (2) – as far as we know it is still open. We remark that a proof of (2) was given in [5] but there seems to be some errors in the proof, in particular, the uniqueness which is claimed there is known to be false (cf. [9]). We use here freely, terminology and notations from the theories of convex bodies (cf. [3] and [6]) and Banach spaces (cf. [7]).

2.

In this section we adapt the situation of [11] to our needs.

In n-dimensional Euclidean space R^n ($n \geqq 2$) let, for $N \geqq n$

$$G_1 = (H_1, -H_1),\ G_2 = (H_2, -H_2), \ldots, G_N = (H_N, -H_N)$$

bę N random pairs of hyperplanes, where the hyperplanes H_i, $-H_i$ in each pair are symmetric with respect to the origin and meet the Euclidean unit ball B^n whose center is the origin. We denote by $\mathscr{H}_i$ the "strip"

$$\mathscr{H}_i = \{x \in \mathsf{R}^n;\ -1 \leqq f_i(x) \leqq 1\}$$

where $f_i \in \mathsf{R}^n$ is such that $\langle f_i, x\rangle = 1$ for all $x \in H_i$ ($\langle \cdot, \cdot \rangle$ denotes the scalar product of R^n).

Let Y_N be the number of vertices of the symmetric (with respect to the origin) polyhedral set

$$\mathscr{H} = \bigcap_{i=1}^{N} \mathscr{H}_i.$$

Y_N is a real random variable, let $E(Y_N)$ be its expectation. Assume now that the pairs G_i are independent, identically distributed and that their distribution is given in the following way: we are given an even probability measure μ on S^{n-1} and define a measure ν on the space

$$G^n = \{G = (H, -H);\ H \text{ a hyperplane in } \mathsf{R}^n \text{ which intersects } B^n\}$$

by identifying G^n with $S^{n-1} \times [0,1]$ via $G \leftrightarrow (u,\tau) \in S^{n-1} \times [0,1]$ where $G = (H_{u,\tau}, -H_{u,\tau})$,

$$H_{u,\tau} = \{x \in \mathrm{R}^n; \langle x,u\rangle = \tau\}$$

and on $S^{n-1} \times [0,1]$, $\nu = (\mu \times \lambda)$ (λ – Lebesgue measure on $[0,1]$).

We assume also that the support of μ is not contained in any $(n-1)$-dimensional subspace of R^n. G_i are distributed by the probability ν.

The proof of the following theorem is an easy modification of the proof of [11] and we omit it.

THEOREM 1.

$$\lim_{N\to\infty} E(Y_N) = 2^{-n} n!\, V(Z_\mu)V(Z_\mu^*),$$

where Z_μ is the convex body whose support function $h(x)$ is given by

$$h(x) = \frac{1}{2}\int_{S^{n-1}} |\langle u,x\rangle|\, d\mu(u).$$

DEFINITION. Once the existence of the limit is proved we define

$$E_\mu = \lim_{N\to\infty} E(Y_N).$$

REMARK. Theorem 1 and [11] show that also $E_\mu = \lim_{N\to\infty} E(X_N)$, where X_N is the number of vertices of the random polyhedral set generated by N random single hyperplanes, rather than pairs of hyperplanes, with the same distribution.

3.

The convex bodies $K = K_\nu$ whose support functions h_K are given by

$$h_K(x) = \int_{S^{n-1}} |\langle u,x\rangle|\, d\nu(u)$$

with a given even nonnegative Borel measure ν on S^{n-1}, are called *zonoids*. Considered as unit balls of n-dimensional Banach spaces, these are exactly the n-dimensional Banach spaces whose duals can be embedded isometrically in L_1-spaces (cf. [2]).

Let $K = K_\nu$ be a zonoid, in order to estimate $P(K) = V(K)V(K^*)$ we may assume that $\|\nu\| = 1/2$, i.e., that $K = Z_\mu$ ($\mu = 2\nu$) in the terminology of Theorem 1. We get:

THEOREM 2. a) *Let μ be as in Theorem* 1, *then $E_\mu \geqq 2^n$;*
b) *Let K be a zonoid of dimension n, then $V(K)V(K^*) \geq 4^n/n!$.*

PROOF. With the preceding remark it is clear that b) follows from Theorem 1 and a). In order to prove a) it is sufficient to assume that μ is atomic with finitely many atoms, i.e.

$$\mu = \sum_{j=1}^{m} \lambda_j \delta_{u_j} + \sum_{j=1}^{m} \lambda_j \delta_{-u_j}$$

where $m \geqq n$, $\lambda_j > 0$ and $u_j \in S^{n-1}$ for all j, $\sum_{j=1}^{m} \lambda_j = 1/2$ and $\{u_j\}_{j=1}^{m}$ spans R^n.

The proof of Theorem 2 is based on three lemmas, the first lemma shows that the polyhedron $\mathscr{H}$ is a polytope with probability which approaches 1. Throughout this proof the letter P denotes probability.

LEMMA 1.

$$P(\exists 1 \leqq j \leqq m \text{ such that } \forall 1 \leqq i \leqq N,\ H_i \not\perp u_j) \xrightarrow[N\to\infty]{} 0.$$

PROOF. The set whose probability is to be computed is

$$A_N = \bigcup_{j=1}^{m} \bigcap_{i=1}^{N} (H_i \not\perp u_j)$$

hence

$$P(A_N) \leqq \sum_{j=1}^{m} P\left[\bigcap_{i=1}^{N} (H_i \not\perp u_j)\right] = \sum_{j=1}^{m} [P(H_1 \not\perp u_j)]^N.$$

Now

$$P(H_1 \not\perp u_j) = 1 - P(H_1 \perp u_j)$$
$$P(H_1 \perp u_j) = P(\exists -1 \leqq \tau \leqq 1 \text{ such that } H_1 = H_{u_j,\tau}) = 2\lambda_j$$

hence

$$P(A_N) \leqq \sum_{j=1}^{m} (1 - 2\lambda_j)^N \xrightarrow[N\to\infty]{} 0.$$

An n-dimensional polyhedral set is said to be *simple* if all its vertices are intersection points of exactly n $(n-1)$-dimensional faces.

LEMMA 2. *For all* $N \geqq n$

$$P(\mathscr{H} \text{ is not simple}) = 0.$$

PROOF. Let B_N be the set whose probability is to be computed. We have with the notation of Theorem 1:

$$B_N \subset \bigcup_{\substack{k=(\varepsilon_1 k_1,\dots,\varepsilon_{n+1}k_{n+1}\\ \varepsilon_j=\pm 1}} (\varepsilon_1 H_{k_1} \cap \dots \cap \varepsilon_{n+1} H_{k_{n+1}} = \{x\} \text{ for some } x)$$

Therefore

$$P(B_N) \leqq 2^{n+1}\binom{N}{n+1} P(H_1 \cap \dots \cap H_{n+1} = \{x\} \text{ for some } x)$$

$$= 2^{n+1}\binom{N}{n+1} \int_{H_1\cap\dots\cap H_n=\{x\}} \cdots \int P(x \in H_{n+1})dv(G_1)\dots dv(G_n).$$

The nature of v guarantees that for fixed x, $P(x \in H_{n+1}) = 0$, hence $P(B_N) = 0$.

The third lemma is a result of I. Barany and L. Lovasz.

LEMMA 3. [1] *Let K be an n-dimensional symmetric, simple convex polytope. Let v be the number of vertices of K, then $v \geqq 2^n$.*

We complete the proof of Theorem 2 as follows: for fixed N, let A_N and B_N be as in the proofs of Lemmas 1 and 2. If $(G_1, \dots, G_N)$ is not in $A_N \cup B_N$. Then $\mathscr{K}$ satisfies the assumptions about K that were made in Lemma 3 and therefore $Y_N \geqq 2^n$. By Lemmas 1 and 2

$$P(A_N \cup B_N) \xrightarrow[N\to\infty]{} 0$$

which completes the proof.

4. Two corollaries.

It is clear that $P(K) = P(K^*)$, hence, if K is such that K^* is a zonoid, then K satisfies (2).

COROLLARY 1. *Let E be a k-dimensional Banach space having a basis $(e_i)_{i=1}^k$ with an unconditional basic constant $\chi((e_i)) = 1$ (i.e., $\|\sum \pm \alpha_i e_i\|_E = \|\sum \alpha_i e_i\|_E$ for all scalars α_i and all choices of signs). Let $E_1, \dots, E_K$ be finite dimensional Banach spaces, each of which is either a subspace of an L_1 space or a quotient space of a $C(K)$ (K-compact) space. Let $F = (E_1 \oplus \dots \oplus E_k)_E$ be the direct sum of (E_i) in the sense of E (for $x = (x_1, \dots, x_k) \in F$, $x_i \in E_i$, $\|x\|_F = \|\sum_{i=1}^k \|x_i\|_{E_i} e_i\|_E$). If $\dim F = n$ then $P(B_F) \geqq 4^n/n!$ (B_F is the unit ball of F).*

PROOF. Use [9, Theorem 15 and Theorem 21] and Theorem 2.

COROLLARY 2. *Let X_N be the number of vertices of a random polyhedral set generated by N independent random hyperplanes in* $\mathbb{R}^n$, *distributed according to the same probability measure as in Theorem* 1. *Then* $\underline{\lim}_{N\to\infty} E(X_N) \geqq 2^n$ *and also*

$$2 \leqq \left[\lim_{N\to\infty} E(X_N) \right]^{1/n} \leqq \pi.$$

PROOF. The first claim follows from Theorem 2 and the remark following the proof of Theorem 1. The right hand side of the second inequality follows from [11] and Stirling's formula.

REMARKS ADDED IN PROOF. a) Recently, we proved a uniqueness in Theorem 2: equality in b) (for a zonoid K) holds if and only if K is a parallelotope.

b) Recently, J. Bourgain and V. Milman proved that $[P(K)]^{1/n} \geqq c/n$ for all n-dimensional K.

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