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REPRESENTATION THEOREMS FOR MULTI-VALUED (REGULAR) L^1-AMARTS

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1. Introduction.

Representation and convergence theorems play a crucial role in the study of multi-valued asymptotic martingales (amarts). Representations of multi-valued (regular) martingales, quasi-martingales and uniform amarts have been considered in [7] and [8], while convergence theorems for multi-valued martingales and L^1-amarts have been presented in [6], [5], and [9]. The main purpose of this paper is to apply these above results to prove some representation theorems for multi-valued (regular) L^1-amarts, given in Sections 3 and 4. A brief summary of definitions and notations will be given in the next section.

2. Mesaurability, integrability and conditional expectations of multi-functions.

Throughout this paper we shall use definitions and notations, given in [6] and [9]. Namely, let $(\Omega,\mathscr{A},P)$ be a probability space, $\mathscr{B}$ a sub σ-field of $\mathscr{A}$, E a real separable Banach space and $K_c(E)$ the class of all closed convex bounded nonempty subsets of E. Thus the Hausdorff topology of $K_c(E)$ is defined by the following complete metric

$$h(X,Y) = \max\left\{\sup_{x\in X} d(x,Y),\ \sup_{y\in Y} d(y,X)\right\}. \tag{2.1}$$

A multi-function $X:\Omega \to K_c(E)$ is called (weakly) $\mathscr{B}$-measurable, write $X \in \mu_c(\mathscr{B},E)$ with $\mu_c(\Omega,E) = \mu_c(\mathscr{A},E)$ if $\{\omega; X(\omega) \cap V \neq \varnothing\} \in \mathscr{B}$ for every open subset V of E. Therefore, if $S_X(\mathscr{B})$ denotes the set of $\mathscr{B}$-measurable selections of X, then the Castaing representation theorem (see [3, Theorem III-9]) shows that an $X:\Omega \to K_c(E)$ is $\mathscr{B}$-measurable if and only if X admits a Castaing representation, i.e. a sequence $\{f^i\}_{i=1}^{\infty}$ in $S_X(\mathscr{B})$ such that for each $\omega \in \Omega$ the sequence $\{f^i(\omega)\}_{i=1}^{\infty}$ is $\|\cdot\|$-dense in $X(\omega)$.

An $X \in \mu_c(\mathscr{B},E)$ is called integrably bounded, if the function

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$$\omega \to \|X(\omega)\| = h(X(\omega), \{0\})$$

is integrable. If this occurs, then we write $X \in \mathscr{L}_c^1(\mathscr{B},E)$ with

$$\mathscr{L}_c^1(\Omega,E) = \mathscr{L}_c^1(\mathscr{A},E),$$

where two elements are considered to be identical, if they are equal each other almost everywhere (a.e.). Furthermore, the integral of X over Ω is defined as the set

$$\int_\Omega X\,dP = \left\{\int_\Omega f\,dP\,; f \in S_X\right\},$$

where $S_X = S_X(\mathscr{A})$, and if $A \in \mathscr{A}$ then

$$\int_A X\,dP = \int_\Omega 1_A X\,dP,$$

where 1_A denotes the characteristic function of A.

Further, if we define for two any $X, Y \in \mathscr{L}_c^1(\Omega,E)$ the following distance

$$H[X,Y] = \int_\Omega h(X(\omega), Y(\omega))\,dP, \tag{2.2}$$

then $\langle \mathscr{L}_c^1(\Omega,E), H\rangle$ becomes a complete metric space.

Finally, let $\mathscr{E}(X,\mathscr{B})$ denote the $\mathscr{B}$-conditional expectation of $X \in \mathscr{L}_c^1(\Omega,E)$.

For further informations on measurability, integrability or conditional expectations of multi-functions we refer to ([3, Chapter III]), [1] or [6], respectively.

3. Representation theorems for multi-valued L^1-amarts.

Hereafter, let $\langle \mathscr{A}_n\rangle$ be an increasing sequence of sub σ-fields of $\mathscr{A}$ with $\mathscr{A}_n \uparrow \mathscr{A}$. A sequence $\langle X_n\rangle$ in $\mathscr{L}_c^1(\Omega,E)$ is called to be adapted to $\langle \mathscr{A}_n\rangle$, if each X_n is $\mathscr{A}_n$-measurable. All sequences considered are assumed to be adapted to $\langle \mathscr{A}_n\rangle$ and taken from $\mathscr{L}_c^1(\Omega,E)$. Let (P) be any but fixed property of $\langle X_n\rangle$. Call a sequence $\langle f_n\rangle$ of E-valued functions a P-selection of $\langle X_n\rangle$, write $\langle f_n\rangle \in \mathrm{PS}(\langle X_n\rangle)$, if each f_n is an $\mathscr{A}_n$-measurable selection of X_n and $\langle f_n\rangle$ has the property (P). It is desirable to get general theorems which guarantee the existence of P-selections and represent $\langle X_n\rangle$ in terms of $\mathrm{PS}(\langle X_n\rangle)$.

Call $\langle X_n\rangle$ a martingale (M), if for all $m \geq n \in \mathrm{N}$ $X_n = X_n(m)$, where

$$X_n(m) = \mathscr{E}(X_m, \mathscr{A}_n)$$

denotes the $\mathscr{A}_n$-conditional expectation of X_m. Thus, it seems to be reasonable to call $\langle X_n\rangle$ an L^1-amart (L^1A), if

$$\lim_{n\to\infty} \sup_{m\geqq n} H[X_n(m),X_n] = 0. \tag{3.1}$$

It has been shown in [9, Theorem 3.2] that (3.1) is equivalent to the existence of a (unique) martingale $\langle M_n\rangle$ with

$$\lim_{n\to\infty} H[X_n,M_n] = 0. \tag{3.2}$$

Moreover, if this occurs then $\langle M_n\rangle$ is given by

$$\lim_{m\to\infty} H[X_n(m),M_n] = 0 \quad (n\in \mathrm{N}). \tag{3.3}$$

Consequently, a sequence $\langle f_n\rangle$ of E-valued functions is an L^1-amart if and only if $\langle f_n\rangle$ can be essentially written in a form

$$f_n = q(f_n) + p(f_n) \quad (n\in \mathrm{N}) \tag{3.4}$$

where $\langle q(f_n)\rangle$ is a martingale and $\langle p(f_n)\rangle$ an L^1-potential, i.e.

$$\lim_{n\to\infty} \|p(f_n)\|_1 = 0. \tag{3.5}$$

To formulate the main theorem we shall need the following additional notations:

Let $\langle X_n\rangle$ be an L^1-amart, $\langle p_n\rangle$ a positive L^1-potential.

Define

$$\begin{aligned}
[L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle} &= \{\langle f_n\rangle \in L^1\mathrm{AS}(\langle X_n\rangle);\ \|p(f_n)\| \leqq p_n \text{ a.e. } (n\in \mathrm{N})\},\\
Q(L^1\mathrm{AS}(\langle X_n\rangle)) &= \{Q(\langle f_n\rangle) = \langle q(f_n)\rangle;\ \langle f_n\rangle \in L^1\mathrm{AS}(\langle X_n\rangle)\},\\
\pi^k(L^1\mathrm{AS}(\langle X_n\rangle)) &= \{\pi^k(\langle f_n\rangle) = f_k;\ \langle f_n\rangle \in L^1\mathrm{AS}(\langle X_n\rangle)\}.
\end{aligned}$$

The main purpose of this section is to prove the following general representation theorem for multi-valued L^1-amarts.

THEOREM 3.1. *A sequence $\langle X_n\rangle$ is an L^1-amart if and only if there is a positive L^1-potential $\langle p_n\rangle$ such that the following conditions are satisfied:*

(a) $S_k(k) = \pi^k([L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle})$ $(k\in \mathrm{N})$,
(b) $Q(L^1\mathrm{AS}(\langle X_n\rangle)) = Q([L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle})$,

where $S_n(m) = S_{X_n}(\mathscr{A}_m)$ $(n,m\in \mathrm{N})$.

To prove the necessity of the conditions we shall need the first two of the next lemmas. However the last one will be applied to prove the sufficiency of the conditions. Thus, we begin with

LEMMA 3.2. (see [7, Corollary 2.5]). *If $\langle X_n\rangle$ is a martingale, then there is a sequence $\{\langle f_n^i\rangle\}_{i=1}^{\infty}$ of* M*-selections of $\langle X_n\rangle$ such that for each k, $\{f_k^i\}_{i=1}^{\infty}$ is a Castaing representation of X_k.*

This result with its complete proof will be published in the reference mentioned above. But let us repeat here the main idea of the proof. Indeed, given an $\varepsilon > 0$ and $f'_k \in S_k(k)$, one can find, in view of the proof of Theorem 6.5, [6], a quasi-martingale $\langle f_n\rangle$ such that each $f_n \in S_n(n)$; $f_k = f'_k$ and

$$\sum_{n\in\mathbb{N}} \|f_n(n+1) - f_n\|_1 \leqq \varepsilon. \tag{3.6}$$

Hence, $\langle f_n\rangle$ is an L^1-amart selection of $\langle X_n\rangle$. Therefore, by (3.3), (3.4), and (3.5) (see also [10, Theorem 1.1]), $\langle f_n\rangle$ can be written in the form (3.4) with $\langle q(f_n)\rangle \in \mathrm{MS}(\langle X_n\rangle)$, noting that each $S_n(n)$ is L^1-closed. Consequently, by (3.6)

$$\|f'_k - q(f_k)\|_1 = \|f_k - q(f_k)\|_1 \leqq \varepsilon.$$

This implies that $\pi^k(\mathrm{MS}(\langle X_n\rangle))$ is L^1-dense in $S_k(k)$ for each $k \in \mathbb{N}$. Combining this fact with Theorem III.9 in [3], we get the lemma.

LEMMA 3.3. *Let $\mathscr{B}_1 \subset \mathscr{B}$ be two sub σ-fields of $\mathscr{A}$; $X \in \mathscr{L}_c^1(\mathscr{B}_1, E)$; $Y \in \mathscr{L}_c^1(\mathscr{B}, E)$ and φ a $\mathscr{B}_1$-measurable positive real-valued function. Then for each $f \in S_X(\mathscr{B}_1)$, there is some $g \in S_Y(\mathscr{B})$ such that*

$$\|f - E^{\mathscr{B}_1}(g)\| \leqq h(X, \mathscr{E}(Y, \mathscr{B}_1)) + \varphi \text{ a.e.} \tag{3.7}$$

and hence

$$\|f - E^{\mathscr{B}_1}(g)\|_1 \leqq H[X, \mathscr{E}(Y, \mathscr{B}_1)] + \int_\Omega \varphi\, dP.$$

Consequently, if Y is $\mathscr{B}_1$-measurable ($\mathscr{B}_1 = \mathscr{B}$) then there is some $g \in S_Y(\mathscr{B}_1)$ such that

$$\|f - g\| \leqq h(X, Y) + \varphi \text{ a.e.} \tag{3.8}$$

and hence

$$\|f - g\|_1 \leqq H[X, Y] + \int_\Omega \varphi\, dP.$$

PROOF. Let $\mathscr{B}_1 \subset \mathscr{B}$; X; Y; φ and f be as in the hypotheses of the lemma. It suffices to prove (3.7). First, since $\mathscr{E}(Y,\mathscr{B}_1)$ is $\mathscr{B}_1$-measurable, by Theorem III.9 in [3] there is a sequence $\{g^i\}_{i=1}^{\infty}$ in $S_{\mathscr{E}(Y,\mathscr{B}_1)}(\mathscr{B}_1)$ such that $\{g^i\}_{i=1}^{\infty}$ is a Castaing representation of $\mathscr{E}(Y,\mathscr{B}_1)$, i.e.

$$\mathscr{E}(Y,\mathscr{B}_1)(\omega) = \mathrm{cl}\ (\{g^i(\omega); i \in \mathsf{N}\})\ \ (\omega \in \Omega). \tag{3.9}$$

Now, applying Theorem 5.3 in [6] to $\mathscr{E}(Y,\mathscr{B}_1)$, we have

$$S_{\mathscr{E}(Y,\mathscr{B}_1)}(\mathscr{B}_1) = \mathrm{cl}\{E^{\mathscr{B}_1}(g); g \in S_Y(\mathscr{B})\}.$$

Thus, for each $i \in \mathsf{N}$, one can choose a sequence $\{g^{ij}\}_{j=1}^{\infty}$ in $S_Y(\mathscr{B})$ such that

$$\lim_{j\to\infty} \| E^{\mathscr{B}_1}(g^{ij}) - g^i \|_1 = 0\ \ (i \in \mathsf{N}). \tag{3.10}$$

But as from every L^1-convergent sequence one can extract an almost surely convergent subsequence, so by (3.10), one can suppose without any loss of generality that

$$g^i(\omega) \in \mathrm{cl}\{E^{\mathscr{B}_1}(g^{ij})(\omega); j \in \mathsf{N}\}\ \ (\omega \in \Omega).$$

This with (3.9) implies that the sequence $\{E^{\mathscr{B}_1}(g^{ij})\}_{i,j=1}^{\infty}$ is a Castaing representation of $\mathscr{E}(Y,\mathscr{B}_1)$. Renumbering the sequence $\}g^{ij}\}_{i,j=1}^{\infty}$ and taking the resulting sequence $\{g_n\}_{n=1}^{\infty}$, we define $\tau : \Omega \to \mathsf{N}$ by

$$\begin{aligned}\tau(\omega) = \inf\{n \in \mathsf{N}; \| f(\omega) - E^{\mathscr{B}_1}(g_n)(\omega) \| \\ \leqq d(f(\omega), \mathscr{E}(Y,\mathscr{B}_1)(\omega) + \varphi(\omega)\}\ \ (\omega \in \Omega).\end{aligned}$$

It is easily checked that τ is a well-defined $\mathscr{B}_1$-measurable function. Hence $g(\,.\,) = g_{\tau(.)}(\,.\,)$ is a $\mathscr{B}$-measurable selection of Y and

$$\begin{aligned}&\| f(\omega) - E^{\mathscr{B}_1}(g)(\omega) \| \\ &= \| f(\omega) - E^{\mathscr{B}_1}\left[\sum_{n=1}^{\infty} 1_{\{\tau=n\}} g_n\right](\omega) \| \\ &= \| f(\omega) - \sum_{n=1}^{\infty} 1_{\{\tau=n\}} E^{\mathscr{B}_1}(g_n)(\omega) \| \\ &\leqq \sum_{n=1}^{\infty} 1_{\{\tau=n\}} \| f(\omega) - E^{\mathscr{B}_1}(g_n)(\omega) \|\ \ (\omega \in \Omega).\end{aligned}$$

But by definition of τ,

$$\| f(\omega) - E^{\mathscr{B}_1}(g_n)(\omega) \| \leqq d(f(\omega), \mathscr{E}(Y,\mathscr{B}_1)(\omega)) + \varphi(\omega),\ \ (\omega \in \{\tau = n\}, n \in \mathsf{N}).$$

Then

$$\| f(\omega) - E^{\mathscr{B}_1}(g)(\omega) \| \leqq \sum_{n=1}^{\infty} 1_{\{\tau=n\}}[d(f(\omega),\mathscr{E}(Y,\mathscr{B}_1)(\omega)) + \varphi(\omega)]$$
$$= d(f(\omega),\mathscr{E}(Y,\mathscr{B}_1)(\omega)) + \varphi(\omega)$$
$$\leqq h[X(\omega),\mathscr{E}(^{Y},\mathscr{B}_1)(\omega)] + \varphi(\omega).$$

This proves (3.7) and hence (3.8), etc. Therefore, the proof of the lemma is completed.

LEMMA 3.4. *Let $X,Y \in \mathscr{L}_c^1(\mathscr{B},E)$, then*

$$H[X,Y] = h(S_X(\mathscr{B}),\ S_Y(\mathscr{B})) \tag{3.11}$$

where $h(.,,.)$ denotes also the Hausdorff metric defined on $K_c(L^1(\mathscr{B},E))$.

PROOF. Let $X,Y \in \mathscr{L}_c^1(\mathscr{B},E); f \in S_X(\mathscr{B})$ and $\varepsilon > 0$. By Lemma 3.3, (3.8) there is some $g \in S_Y(\mathscr{B})$ such that, in particular,

$$\| f - g \|_1 \leqq H[X,Y] + \varepsilon.$$

Hence $d(f,S_Y(\mathscr{B})) \leqq H[X,Y] + \varepsilon$.

But $f \in S_X(\mathscr{B})$ and $\varepsilon > 0$ were arbitrarily taken, so that the last inequality implies

$$\sup_{f \in S_X(\mathscr{B})} d(f,S_Y(\mathscr{B})) \leqq H[X,Y].$$

Therefore, by symmetry one has

$$\sup_{g \in S_Y(\mathscr{B})} d(g,S_X(\mathscr{B})) \leqq H[X,Y].$$

This yields

$$h(S_X(\mathscr{B}),\ S_Y(\mathscr{B}) \leqq H[X,Y]. \tag{3.12}$$

To prove the converse inequality, we note first that for each $A \in \mathscr{B}$ we have

$$H[X,Y] = H[1_A X,\ 1_A Y] + H[1_{A^c} X,\ 1_{A^c} Y]$$

and

$$h(S_X(\mathscr{B}),S_Y(\mathscr{B})) = h(S_{1_A X}(\mathscr{B}),S_{1_A Y}(\mathscr{B})) + h(S_{1_{A^c} X}(\mathscr{B}),S_{1_{A^c} Y}(\mathscr{B})).$$

Hence by definitions (2.1) and (2.2) one can suppose without any restrictions that

$$h(X(\omega),\ Y(\omega)) = \sup_{x \in X(\omega)} d(x,Y(\omega)) \ \ (\omega \in \Omega).$$

Thus, by using Theorem 2.2, [6], we get

$$\begin{aligned} H[X,Y] &= \int_\Omega \sup_{x\in X(\omega)} d(x,Y(\omega))dP \\ &= \sup_{f\in S_X(\mathscr{B})} \int_\Omega d(f(\omega),Y(\omega))dP \\ &= \sup_{f\in S_X(\mathscr{B})} \inf_{g\in S_Y(\mathscr{B})} \int_\Omega \|f(\omega)-g(\omega)\|\, dP \\ &= \sup_{f\in S_X(\mathscr{B})} d(f,S_Y(\mathscr{B})) \\ &\leqq h(S_X(\mathscr{B}),S_Y(\mathscr{B})). \end{aligned}$$

Hence by (3.12) we get (3.11) and the lemma, noting that both $S_X(\mathscr{B})$, $S_Y(\mathscr{B})\in K_c(L^1(\mathscr{B},E))$.

PROOF OF THEOREM 3.1. Suppose first that $\langle X_n\rangle$ is an L^1-amart, then there is a (unique) martingale $\langle M_n\rangle$ such that (3.2) is satisfied. Thus if we put

$$p_n(\omega) = h[X_n(\omega), M_n(\omega)] + 1/2^n \quad (\omega\in\Omega,\ n\in \mathsf{N}),$$

then $\langle p_n\rangle$ is a positive L^1-potential. We shall show that $\langle p_n\rangle$ satisfies (a) and (b). To prove (a), we fix $k\in\mathsf{N}$ and $f'_k\in S_k(k)$. Since $\langle M_n\rangle$ is a martingale, by Lemma 3.2 there is a sequence $\{g^i_n\}_{i=1}^\infty$ of M-selections of $\langle M_n\rangle$ such that for each $m\in\mathsf{N}$, $\{g^i_m\}_{i=1}^\infty$ is a Castaing representation of M_m. Let define $\tau:\Omega\to\mathsf{N}$ by

$$\tau(\omega) = \inf\{i\in\mathsf{N}\,; f'_k(\omega)-g^i_k(\omega)\| \leqq d(f'_k(\omega), M_k(\omega)) + 1/2^k\}$$

and

$$g_k(\omega) = \sum_{i\in\mathsf{N}} 1_{\{\tau=i\}} g^i_k(\omega) = g_k^{\tau(\omega)}(\omega) \quad (\omega\in\Omega).$$

It is easily checked that τ is $\mathscr{A}_k$-measurable. Hence g_k is an $\mathscr{A}_k$-measurable selection of M_k and

$$\|f_k(\omega)-g_k(\omega)\| \leqq p_k(\omega) \text{ a.e.} \tag{3.13}$$

Now put

$$g_n = \sum_{i\in\mathsf{N}} 1_{\{\tau=i\}} g^i_n \quad (n\geqq k)$$

and

$$g_m = g_m(k)(=E^{\mathscr{A}_m}(g_k)) \quad (m < k).$$

Obviously, $\langle g_n\rangle$ is an M-selection of $\langle M_n\rangle$. Next, applying (3.8) to each triple M_n, X_n and g_n, there is a sequence $\langle f_n\rangle$ such that $f_n \in S_n(n)$ and

$$\| f_n(\omega) - g_n(\omega)\| \leqq p_n(\omega) \text{ a.e. } (n \in \mathrm{N}).$$

This follows that $\langle f_n\rangle$ is an L^1A-selection of $\langle X_n\rangle$ with

$$\langle q(f_n)\rangle = \langle g_n\rangle \text{ and } \langle p(f_n)\rangle = \langle f_n - g_n\rangle.$$

Hence $f_n \in [L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle}$. But in view of (3.13) one can assume that $f_k = f'_k$, so that

$$f'_k = f_k = \pi^k(\langle f_n\rangle) \in \pi^k([L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle})$$

which proves (a). To prove equality (b), it is sufficient to show the following inclusions

(3.14) $$\mathrm{MS}(\langle M_n\rangle) \subset Q([L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle})$$

and

(3.15) $$Q(L^1\mathrm{AS}(\langle X_n\rangle)) \subset \mathrm{MS}(\langle M_n\rangle).$$

To see (3.14), we fix $\langle g_n\rangle \in \mathrm{MS}(\langle M_n\rangle)$. The above argument shows that there is some $\langle f_n\rangle \in [L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle}$ such that

$$\langle q(f_n)\rangle = \langle g_n\rangle,$$

and hence

$$\langle g_n\rangle = Q(\langle f_n\rangle) = \langle q(f_n)\rangle \in Q([L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle}).$$

This proves (3.14). To establish (3.15) we fix $\langle f_n\rangle \in L^1\mathrm{AS}(\langle X_n\rangle)$. Then by (3.3), (3.4), and (3.5), we get

$$f_n(m) \in X_n(m) \qquad (m \geqq n \in \mathrm{N}),$$

$$\lim_{m\to\infty} H[X_n(m), M_n] = 0 \qquad (n \in \mathrm{N}),$$

$$\lim_{m\to\infty} \| f_n(m) - q(f_n)\|_1 = 0 \quad (n \in \mathrm{N}).$$

Hence $q(f_n) \in S_{M_n}(n)$ $(n \in \mathrm{N})$. This shows that

$$Q(\langle f_n\rangle) = \langle q(f_n)\rangle \in \mathrm{MS}(\langle M_n\rangle).$$

This implies (3.15) which with (3.14) shows equality (b) and the necessity of the conditions.

To prove the sufficiency of the conditions (a–b), for each $k \in \mathrm{N}$ we put

$$\mu^k = \pi^k Q([L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle}). \tag{3.16}$$

(i) By (a), μ^k is non-empty.

(ii) It is easily checked that by (3.16), μ^k is convex bounded.

(iii) We claim that μ^k is $\mathscr{A}_k$-decomposable.

Indeed, let $\langle f_n^1\rangle, \langle f_n^2\rangle \in [L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle}$ and $A \in \mathscr{A}_k$. By definition of $\mathscr{A}_k$-decomposability, we have to show that

$$1_A q(f_k^1) + 1_{A^c} q(f_k^2) \in \mu^k.$$

To see this, let define $f_n = 1_A f_n^1 + 1_{A^c} f_n^2 \quad (n \geqq k)$

and

$$f_m \in S_m(m) \qquad (m < k).$$

Hence, by the uniqueness of decomposition (3.4) for any L^1-amart, we get

$$\begin{aligned} \langle f_m\rangle &\in L^1\mathrm{AS}(\langle X_n\rangle), \\ q(f_n) &= 1_A q(f_n^1) + 1_{A^c} q(f_n^2) \quad (n \geqq k), \\ q(f_m) &= E^{\mathscr{A}_m}(q(f_k)) \qquad (m < k). \end{aligned}$$

But by (b),

$$Q(\langle f_n\rangle) = \langle q(f_n)\rangle \in Q([L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle}),$$

so that

$$\begin{aligned} 1_A f_k^1 + 1_{A^c} f_k^2 &= q(f_k) \\ &= \pi^k(Q(\langle f_n\rangle)) \\ &\in \pi^k(Q[L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle})) = \mu^k. \end{aligned}$$

This proves the $\mathscr{A}_k$-decomposability of μ^k.

Combining these properties (i–iii) of μ^k with Theorem 3.1 and Corollary 1.6, [6], we infer that there is a unique multi-function $M_k \in \mathscr{L}_c^1(\mathscr{A}_k, E)$ such that

$$S_{M_k}(k) = \mathrm{cl}(\mu^k) \quad (k \in \mathrm{N}). \tag{3.17}$$

It is easily seen that by Theorem 5.3, [6], the definition (3.16) and the equality (3.17), we get

$$S_{M_k(k+1)}(k) = \mathrm{cl}\{E^{\mathscr{A}_k}(h), h \in \mu^{k+1}\}$$

$$= \mathrm{cl}(\mu^k)$$
$$= S_{M_k}(k) \quad (k \in \mathsf{N}).$$

Equivalently,

$$M_k = M_k(k+1) \quad (k \in \mathsf{N}).$$

This is equivalent to that $\langle M_n \rangle$ is a martingale. Finally, by the above construction of the martingale $\langle M_n \rangle$ and Lemma 3.4, one can establish the following conclusions

$$\begin{aligned} H[X_k, M_k] &= h[S_k(k), \mathrm{cl}(\mu^k)] \\ &= h[S_k(k), \mu^k] \\ &\leqq \sup_{\langle f_n \rangle \in [L^1 \mathrm{AS}(\langle X_n \rangle)]^{\langle p_n \rangle}} \| f_k - q(f_k) \|_1 \\ &\leqq \int_\Omega \| p_k \| \, dP \to 0, \text{ as } k \uparrow \infty. \end{aligned}$$

Thus $\langle X_n \rangle$ must be an L^1-amart. This completes the proof of the theorem.

Next, combining the condition (a) of the previous theorem with Theorem III.9, [3], one can establish easily the following result.

COROLLARY 3.5. *If $\langle X_n \rangle$ is an L^1-amart, then there is a sequence $\{\langle f^i_n \rangle\}_{i=1}^\infty$ of L^1 A-selections such that for each $k \in \mathsf{N}$, the sequence $\{f^i_k\}_{i=1}^\infty$ is a Castaing representation of X_k.*

Further, in connection with Lemma 3.2, the inspection of the proof of Theorem 3.1 leads to the following representation theorem for multi-valued martingales.

THEOREM 3.6. *$\langle X_n \rangle$ is a martingale if and only if the following conditions hold:*

a) *there is a sequence $\{\langle f^i_n \rangle\}_{i=1}^\infty$ of* M*-selections of $\langle X_n \rangle$ such that for each $k \in \mathsf{N}$, $\{f^i_k\}_{i=1}^\infty$ is a Castaing representation of X_k,*
b) $Q(L^1 \mathrm{AS}(\langle X_n \rangle)) = \mathrm{MS}(\langle X_n \rangle)$.

PROOF. First, suppose that $\langle X_n \rangle$ is a martingale. Then by Lemma 3.2, (a) is satisfied. Further, since $\langle X_n \rangle$ is a martingale, then $\langle X_n \rangle$ is an L^1-amart with

$$M_n = X_n \quad (n \in \mathsf{N})$$

where $\langle M_n\rangle$ is the martingale associated with the L^1-amart $\langle X_n\rangle$, taken in the sense of (3.2). Therefore, the condition (b) given here is equivalent to the following condition

$$Q(L^1\mathrm{AS}(\langle X_n\rangle) = \mathrm{MS}(\langle M_n\rangle).$$

But this is true, by virtue of the proof of the inclusions (3.14) and (3.15).

Conversely, suppose that (a–b) are satisfied, then by using the same arguments given in the proof of the sufficiency of Theorem 3.1, we infer that there is a martingale $\langle M_n\rangle$ such that

$$S_{M_k}(k) = \mathrm{cl}\{\pi^k(\mathrm{MS}(\langle X_n\rangle))\} \quad (n \in \mathrm{N}),$$

which implies

$$M_k \subset X_k \text{ a.e. } (k \in \mathrm{N}).$$

But by virtue of (a), $X_k \subset M_k$ a.e. and hence

$$X_k = M_k \text{ a.e. } (k \in \mathrm{N}).$$

Therefore, since $\langle M_n\rangle$ is a martingale, then so is $\langle X_n\rangle$. This completes the proof of the theorem.

4. Regularity and convergence of multi-valued L^1-amarts.

Call a sequence $\langle X_n\rangle$ a regular L^1-amart (RL^1A), if there is some $X \in \mathscr{L}_c^1(\Omega,E)$ such that

$$\lim_{n\to\infty} H[X_n, \mathscr{E}(X,\mathscr{A}_n)] = 0. \tag{4.1}$$

Consequently, by (3.2), every regular L^1-amart is an L^1-amart. More precisely, (4.1) is equivalent to (3.2), where the martingale $\langle M_n\rangle$ is regular. Note that for the vector-valued L^1-amarts, regularity and L^1-convergence are equivalent. But for the multi-valued case, only H-convergence implies regularity. We shall present at the end an example of a multi-valued regular martingale in $\mathscr{L}_c^1([0,1],l_2)$ which fails to be convergent even in the Pettis distance (cf. [9]). But before giving further relations between regularity and convergence of multi-valued L^1-amarts we prove a general representation theorem for multi-valued regular L^1-amarts. Three basic lemmas, given in the previous section will be frequently used in the proof of the theorem.

THEOREM 4.1. *$\langle X_n\rangle$ is a regular L^1-amart if and only if the following conditions are satisfied:*

(a) $\sup_{n\in\mathbb{N}} \int_\Omega \|X_n\| \, dP < \infty$,

(b) $S_k(k) = \pi^k(\mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_m\rangle})$ $(k \in \mathbb{N})$,

(c) $Q(\mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)) = Q([\mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)]^{\rangle p_n\rangle})$,

for some positive L^1-potential $\langle p_n\rangle$.

Proof. Suppose first that $\langle X_n\rangle$ is a regular L^1-amart, then by definition there is some $X \in \mathscr{L}_c^1(\Omega, E)$ such that (4.1) is satisfied. Thus, if we put $M_n = \mathscr{E}(X, \mathscr{A}_n)$ and

$$p_n = h(X_n, M_n) + 1/2^n \quad (\omega \in \Omega,\ n \in \mathbb{N}),$$

then by (4.1), $\langle p_n\rangle$ is a positive L^1-potential. We shall show that $\langle p_n\rangle$ satisfies the conditions (a–b–c) required in the theorem. Indeed, the condition (a) is easy. To see (b), we fix $k \in \mathbb{N}$ and $f'_k \in S_k(k)$. Applying Lemma 3.3, (3.7) to X_k, X, and f'_k one can find some $g \in S_X$ such that

$$\|f'_k - E^{\mathscr{A}_k}(g)\| \leqq p_k \text{ a.e.} \tag{4.2}$$

and obviously, $\langle E^{\mathscr{A}_n}(g)\rangle \in \mathrm{RMS}(\langle M_n\rangle)$.

Next, applying again Lemma 3.3, (3.8) to each triple M_n, X_n and $E^{\mathscr{A}_n}(g)$, there is thus a dequence $\langle f_n\rangle$ such that each f_n is an $\mathscr{A}_n$-measurable selection of X_n and

$$\|f_n - E^{\mathscr{A}_n}(g)\| \leqq p_n \text{ a.e.}$$

Hence, $\langle f_n\rangle \in [\mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle}$.

But in view of (4.2) one can take $f_k = f'_k$, so

$$f'_k = f_k = \pi^k(\langle f_n\rangle) \in \pi^k([\mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle})$$

which proves (b). To prove (c), let $\langle f_n\rangle \in \mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)$. By (3.3), (3.4), and (3.5) we get

$$Q(\langle f_n\rangle) = \langle q(f_n)\rangle \in \mathrm{RMS}(M_n\rangle).$$

Again, applying Lemma 3.3, (3.8) to each triple M_n, X_n and $q(f_n)$ we infer that there is an element $\langle f'_n\rangle \in [\mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle}$ such that

$$\langle q(f'_n)\rangle = \langle q(f_n)\rangle.$$

Consequently,

$$Q(\langle f_n\rangle) = \langle q(f_n)\rangle = \langle q(f'_n)\rangle = Q(\langle f'_n\rangle) \in Q([\mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle}.$$

This shows (c) and the necessity of the conditions.

Conversely, suppose that the conditions (a–b–c) are satisfied for some positive L^1-potential $\langle p_n\rangle$. Let us define

(4.3) $$\mu = \{g \in L^1(\Omega,E); \langle E^{\mathscr{A}_n}(g)\rangle \in Q([RL^1AS(\langle X_n\rangle)]^{\langle p_n\rangle})\}.$$

(i) By (a), μ is L^1-bounded.

(ii) By (b), μ is nonempty and by (c) with (4.3), μ is convex.

(iii) We now show that μ is L^1-closed. Let $\{g^i\}_{i=1}^{\infty}$ be a sequence in μ which is L^1-convergent to some $g \in L^1(\Omega,E)$, that is

(4.4) $$\lim_{n\to\infty} \|g^n - g\|_1 = 0.$$

Thus, by definition (4.3), there is a sequence $\{\langle f_n^i\rangle\}_{i=1}^{\infty}$ in $[RL^1AS(\langle X_n\rangle)]^{\langle p_n\rangle}$ such that for each $i \in \mathsf{N}$

$$\langle E^{\mathscr{A}_n}(g^i)\rangle \in Q([RL^1AS(\langle X_n\rangle)]^{\langle p_n\rangle})$$

and

(4.5) $$\sup_{i\in N} \|f_n^i - E^{\mathscr{A}_n}(g^i)\|_1 \leqq \|p_n\|_1 \quad (n \in \mathsf{N}).$$

First, obviously $f_n^n \in S_n(n)$ $(n \in \mathsf{N})$. We shall show that $\langle f_n^n\rangle$ is L^1-convergent to g. Combining (4.4), (4.5) with properties of operators $E^{\mathscr{A}_n}(\cdot)$, we get

$$\begin{aligned}\|f_n^n - g\|_1 &\leqq \|f_n^n - E^{\mathscr{A}_n}(g^n)\|_1 + \|E^{\mathscr{A}_n}(g^n) - E^{\mathscr{A}_n}(g)\|_1 + \|E^{\mathscr{A}_n}(g) - g\|_1\\ &\leqq \|p_n\|_1 + \|g^n - g\|_1 + \|E^{\mathscr{A}_n}(g) - g\|_1\end{aligned}$$

which yields

$$\lim \|f_n^n - g\|_1 = 0.$$

Hence $\langle f_n^n\rangle \in RL^1AS(\langle X_n\rangle)$ with $\langle q(f_n^n)\rangle = \langle E^{\mathscr{A}n}(g)\rangle$' Therefore, by (c),

$$\langle E^{\mathscr{A}n}(g)\rangle \in Q([RL^1AS(\langle X_n\rangle)]^{\langle p_n\rangle}).$$

Hence by (4.3), $g \in \mu$, which implies the L^1-closedness of μ.

(iv) Finally, we claim that μ is $\mathscr{A}$-decomposable, i.e. if $g^1, g^2 \in \mu$ and $A \in \mathscr{A}$ then

$$1_A g^1 + 1_{A^c} g^2 \in \mu \quad (A \in \mathscr{A}).$$

But in view of (iii), it suffices to prove the above assertion for all $A \in \bigcup_{k\in\mathsf{N}} \mathscr{A}_k$. To see this, fix $A \in \mathscr{A}_{k_0}$ for some $k_0 \in \mathsf{N}$. Find $\langle f_n^1\rangle$ and $\langle f_n^2\rangle$ in $[RL^1AS(\langle X_n\rangle)]^{\langle p_n\rangle}$ such that

(4.6) $$\|f_n^i - E^{\mathscr{A}n}(g^i)\|_1 \leqq \|p_n\|_1 \quad (i = 1,2).$$

Define

$$f_n = 1_A f_n^1 + 1_{A^c} f_n^2 \quad (n \geqq k_0)$$

and

$$f_m \in S_m(m) \qquad (m < k_0).$$

It is easily checked that $\langle f_n\rangle \in \mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)$ and

$$\lim_{n\to\infty} \| f_n - (1_A g^1 + 1_{A^c} g^2) \|_1 = 0.$$

Hence

$$\langle E^{\mathscr{A}_n}(1_A g^1 + 1_{A^c} g^2)\rangle \in Q(\mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)).$$

Therefore, by (c)

$$\langle E^{\mathscr{A}_n}(1_A g^1 + 1_{A^c} g^2)\rangle \in Q([\mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)]^{\langle p_n\rangle}).$$

This with (4.3) implies

$$1_A g^1 + 1_{A^c} g^2 \in \mu$$

which proves the $\mathscr{A}$-decomposability of μ.

Now applying Theorem 3.1, [6] to μ, there thus exists some $X \in \mathscr{L}_c^1(\Omega,E)$ such that

(4.7) $$S_X = \mu.$$

It follows from (4.7), (4.3), condition (b), and Lemma 3.4 that

$$\lim_{n\to\infty} H[X_n, \mathscr{E}(X,\mathscr{A}_n)] = 0.$$

This proves (4.1) and the sufficiency of the conditions which completes the proof of the theorem.

Combining condition (b) of the theorem and Theorem III.9, [3], one can establish easily the following result.

COROLLARY 4.2. *If $\langle X_n\rangle$ is a regular L^1-amart,* i.e. (4.1) *is satisfied for some $X \in \mathscr{L}_c^1(\Omega,E)$. Then there is a sequence $\{\langle f_n^i\rangle\}_{i=1}^\infty$ in* $\mathrm{R}L^1\mathrm{AS}(\langle X_n\rangle)$ *and a sequence $\{f_\infty^i\}_{i=1}^\infty$ in S_X such that*

(a) *for each $k \in \mathbb{N} \cup \{\infty\}$, $\{f_k^i\}_{i=1}^\infty$ is a Castaing representation of X_k (with $X_\infty = X$) and*

(b) $$\lim_{n\to\infty} \| f_n^i - f_\infty^i \|_1 = 0 \quad (i \in \mathbb{N}).$$

Combining the proof of the theorem with the proof of Theorem 3.6 we get the following representation theorem for multi-valued regular martingales.

THEOREM 4.3. *A sequence $\langle X_n \rangle$ is a regular martingale if and only if the following conditions are satisfied*:

(a)
$$\sup_{n \in \mathbb{N}} \int_{\Omega} \|X_n\| \, dP < \infty,$$

(b) *there is a sequence $\{\langle f_n^i \rangle\}_{i=1}^{\infty}$ of* RMS$(\langle X_n \rangle)$ *such that each $\{f_k^i\}_{i=1}^{\infty}$ is a Castaing representation of X_k, and*

(c)
$$Q(\mathrm{R}L^1\mathrm{AS}(\langle X_n \rangle)) = Q(\mathrm{RMS}(\langle X_n \rangle)).$$

It is known that if a sequence $\langle X_n \rangle$ is H-convergent, then $\langle X_n \rangle$ is a regular L^1-amart. A simple example in ([9, Example 4.3]) shows that there is a regular martingale in $\mathscr{L}_c^1([0,1),l_2)$ which fails to be H-convergent. Now define for two any $X, Y \in \mathscr{L}_c^1(\Omega,E)$ the Pettis distance

$$H_w(X,Y) = \sup_{x^* \in U^*} \int_{\Omega} |\delta^*(x^*,X) - \delta^*(x^*,Y)| \, dP \quad (\text{cf. } [9]).$$

where U^* is the unit ball of the strong dual E^* of E and $\delta^*(.,Z)$ is the support function of $Z \in K_c(E)$. Note that the Pettis distance $H_w(X,Y)$ is not generally a metric but a quasi-metric on $\mathscr{L}_c^1(\Omega,E)$. Further, Property 1.1 [9] shows that every H-convergent sequence is H_w-convergent. Thus, the same example mentioned above gives us an H_w-convergent regular martingale which fails to be H-convergent.

A natural question arises what are relations between H_w-convergence and regularity of multi-valued L^1-amarts. We are unable to answer to this queation for a general real separable Banach space E. We get, however, the following result.

THEOREM 4.4. *Suppose that either the strong dual E^* of E or E has the Radon-Nikodym property, then every H_w-convergent L^1-amart is regular.*

PROOF. Let $\langle X_n \rangle$ be an L^1-amart which is H_w-convergent to some $X \in \mathscr{L}_c^1(\Omega,E)$. Let $\langle M_n \rangle$ be the martingale satisfying (3.2). Then

$$\lim_{n \to \infty} H_w[X_n, X] = 0$$

implies

$$\lim_{n\to\infty} H_w[M_n,X] = 0.$$

By Property 1.1 [9], this yields

$$\mathrm{cl} \int_A M_n dP = \mathrm{cl} \int_A X dP \quad (A \in \mathscr{A}_n,\ n \in \mathsf{N}).$$

Suppose first that E^* has the Radon-Nikodym property, then by Stegall's theorem [12], E^* must be separable. Hence, Theorem 5.4, (4°), [6] implies

$$M_n = \mathscr{E}(X,\mathscr{A}_n) \quad (n \in \mathsf{N}).$$

Now suppose that E has the Radon-Nikodym property. The last equality implies

$$\begin{aligned} \int_A \|M_n\| \, dP &= \sup \sum_{i=1}^{k} \|\mathrm{cl} \int_{A_i} M_n dP\| \\ &= \sup \sum_{i=1}^{k} \|\mathrm{cl} \int_{A_i} X dP\| \\ &\leqq \int_A \|X\| \, dP \quad (A \in \mathscr{A}_n,\ n \in \mathsf{N}), \end{aligned}$$

where the supremum is taken over all finite $\mathscr{A}_n$-measurable partitions $\{A_1,A_2,\ldots,A_k\}$ of A. Therefore, the martingale $\langle M_n \rangle$ is uniformly integrable. Hence, by (3.2), so is $\langle X_n \rangle$. Now, applying either directly Corollary 3.4, [9] to the L^1-amart $\langle X_n \rangle$ or the martingale limit theorem given by Rønnov [11] (see also Chatterji [4]) to each element of $Q(L^1\mathrm{AS}(\langle X_n \rangle))$ we infer that by Theorems 3.1 and 4.1, $\langle X_n \rangle$ must be a regular L^1-amart. This completes the proof of the theorem.

Finally, the following counter example completes the considerations given in [9] and in the paper.

COUNTER-EXAMPLE 4.6. (see [6], Example 3.4). Let $([0,1), \mathscr{A}, P)$ be the Lebesgue measure space on $[0,1)$. Define a multi-function $F: [0,1) \to K_c(l_2)$ by

$$(4.9) \qquad F(\omega) \in l_2; \|x\| \leqq 1, \langle x,e_n \rangle = 0 \text{ if } \omega_n = 0 \ (n \in \mathsf{N})\}$$

where $\omega = \sum_{n=1}^{\infty} 2^{-n}\omega_n$ is the binary expansion of $\omega \in [0,1)$ and $\langle e_n \rangle$ denotes the usual basis for l_2. Thus, by (4.9) Hiai and Umegaki [6] have shown that $F \in \mathscr{L}_c^1([0,1),l_2)$. Now define for each $n \in \mathsf{N}$ the finite σ-field

$$\mathscr{A}_n = \sigma_- \{[2^{-n}(k-1),2^{-n}k); k = 1,2,\ldots,2^n\} \uparrow \mathscr{B}_{[0,1)} = \mathscr{A}.$$

We claim that the regular martingale $\langle \mathscr{E}(F,\mathscr{A}_n)\rangle$ fails to be even H_w-convergent. To see this, let define $M_n, M_F : \cup_k \mathscr{A}_k \to K_c(l_2)$ by

$$M_n(A) = \mathrm{cl} \int_A F_n dP,$$

$$M_F(A) = \mathrm{cl} \int_A F dP \quad (A \in \Sigma_0 = \cup_k \mathscr{A}_k)$$

and

$$Z_n = \{M_n(A); A \in \Sigma_0\},$$
$$Z_F = \{M_F(A); A \in \Sigma_0\},$$

where

$$F_n = \mathscr{E}(F,\mathscr{A}_n) \quad (n \in \mathsf{N}).$$

First, we note that since each $\mathscr{A}_n$ is finite, each F_n is a simple function. Thus by the embedding theorem 3.6, (2°), [6], one can regard each F_n as an $\hat{E}$-valued Bochner integrable function (for definition of $\hat{E}$, see [9]). Consequently, it follows from [13] that each Z_n is relatively compact in $\hat{E}$ hence in $K_c(l_2)$ with respect to the Hausdorff metric $h(.,.)$ given by (2.1). We shall show now that Z_F is however not relatively compact in $\langle K_c(l_2), h\rangle$. Indeed, let us define

$$A_n = \{\omega \in [0,1); \omega_n = 0\},$$
$$f_n = e_n 1_{A_n^c} \quad (n \in \mathsf{N}).$$

It is not hard to check that

(i) $$\delta^*(e_n, \mathrm{cl} \int_{A_n} F dP) = 0 \quad (n \in \mathsf{N}),$$

(ii) $\langle f_n\rangle$ is a sequence of $\mathscr{A}$-measurable selections of F. Consequently,

$$h\left(\mathrm{cl} \int_{A_n} F dP, \mathrm{cl} \int_{A_m} F dP\right) \geqq \left|\delta^*(e_n, \mathrm{cl} \int_{A_n} F dP) - \delta^*(e_n, \mathrm{cl} \int_{A_m} F dP)\right|$$
$$= \left|\delta^*(e_n, \mathrm{cl} \int_{A_m} F dP)\right|$$
$$\geqq \langle e_n, \int_{A_m} f_n dP\rangle = 1/4 \quad (n \neq m).$$

Hence the sequence $\langle M_F(A_n)\rangle$ in $K_c(l_2)$ contains no subsequence h-convergent. This implies that X_F cannot ve relatively compact in $K_c(l_2)$. Thus by properties of $\langle Z_n\rangle$ and Z_F it follows from Property 1.1, [9] that

$\langle F_n \rangle$ cannot be H_w-convergent to F. Further, $\langle F_n \rangle$ cannot be H_w-convergent to any other $G \in \mathscr{L}_c^1([0,1), l_2)$ (see the proof of Theorem 4.4). This completes the proof of the example.

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