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MEASURE DETERMINING CLASSES OF BALLS IN BANACH SPACES

ULLA DINGER

0. Introduction.

Consider two Borel probability measures on $C([0,1])$ – the Banach space of all real-valued continuous functions on $[0,1]$ with the supremum norm. This paper is related to the following question. Are the measures equal, when they agree on all open balls with centres in a closed supporting linear subspace of $C([0,1])$ and radius one? Since any separable Banach space is isometrically isomorphic to a closed subspace of $C([0,1])$ (see [2, p. 185]) we will work in the setting of general separable Banach spaces.

Based on R. O. Davies' construction (in [7]), R. B. Darst shows in [6] that there exists a compact metric space K and two singular Borel probability measures on K which agree on all closed balls (hence, by approximation, on open balls). This implies the existence of two singular Borel probability measures μ and ν on $C(K)$ satisfying the condition

$$(C_0) \qquad \begin{cases} \text{for every } a \in C(K) \text{ there exists a } \delta > 0 \text{ such that} \\ \mu(B_r(a)) = \nu(B_r(a)), \ \ 0 < r < \delta. \end{cases}$$

Here $B_r(a)$ denotes the open ball with centre a and radius r.

However, in [4] C. Borell shows that two Gaussian probability measures on a separable Banach space coincide whenever the condition (C_0) holds.

Throughout this paper, unless otherwise stated, E will denote an arbitrary real separable Banach space. E' denotes the topological dual of E, $\mathscr{B}(E)$ the class of Borel sets in E and $P(E)$ the class of Borel probability measures on E. For $\mu \in P(E)$, $0 < r < +\infty$, $M \subsetneq E$, let

$$\begin{aligned} \mathscr{Q}^r_M(\mu) &= \{\nu \in P(E);\ \nu(B_r(a)) = \mu(B_r(a)), a \in M\}, \\ \mathscr{Q}^{\leqq r}_M(\mu) &= \{\nu \in P(E);\ \nu(B_s(a)) = \mu(B_s(a)), a \in M,\ \ 0 < s \leqq r\}, \\ \mathscr{Q}^{\infty}_M(\mu) &= \{\nu \in P(E);\ \nu(B_s(a)) = \mu(B_s(a)), a \in M,\ \ s > 0\}. \end{aligned}$$

In the case $M = E$, we omit the index M.

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The question whether $\mathcal{Q}^\infty(\mu) = \{\mu\}$ is true or not has been treated by J. Hoffmann-Jørgensen in [8]. He gives an affirmative answer for a large class of Banach spaces including L_p, $1 < p \leqq \infty$, $C(K)$, and c_0. The answer is also in the affirmative for L_1, as was proved by J.P.R. Christensen in [5]. Furthermore, Christensen obtains far stronger results, when E is a Hilbert space. In particular he gets $\mathcal{Q}^r(\mu) = \{\mu\}$.

In this paper a.o. we show that $\mathcal{Q}^\infty(\mu) = \{\mu\}$, if μ has a finite Laplace transform. If

$$E = C_0([0,1]) \equiv \{x \in C([0,1]); x(0) = 0\},$$

and μ is concentrated on the set $\{x \in C_0([0,1]);\ x \text{ is absolutely continuous}, x' \in L_2([0,1])\}$ we show that $\mathcal{Q}^r(\mu) = \{\mu\}$. Actually, we will prove generalizations of this, using Fourier methods.

The Fourier transform of $\mu \in P(E)$ will be denoted $\hat{\mu}$, that is

$$\hat{\mu}(\xi) = \int_E \exp(i\langle x,\xi\rangle)d\mu(x), \quad \xi \in E'.$$

For $\mu \in P(E)$, its Laplace transform, $\tilde{\mu}: E' \to]0,+\infty]$, is defined by

$$\tilde{\mu}(\xi) = \int_E \exp(\langle x,\xi\rangle)d\mu(x), \quad \xi \in E'.$$

Below it is basic that two Borel probability measures in a separable Banach space with the same Fourier transform must be equal.

1. The space c_0.

We start with a result about the Banach space

$$c_0 \equiv \{x = \{x_n\}_1^\infty; x_n \in \mathsf{R},\ x_n \to 0 \text{ as } n \to \infty\}$$

equipped with the supremum norm $\|x\| = \max_n |x_n|$.

THEOREM 1.1. *Let $E = c_0$ and let*

$$M = \{x \in c_0; x_n \neq 0 \text{ only finitely many } n\}.$$

Then $\mathcal{Q}_M^r(\mu) = \{\mu\}$.

PROOF. Suppose $\nu \in \mathcal{Q}_M^r(\mu)$.

We want to show that $\hat{\nu}(\xi) = \hat{\mu}(\xi)$ for each $\xi \in c_0'$. Recall that

$$c_0' = l_1 \equiv \{x = \{x_n\}_1^\infty; \sum |x_n| < \infty\}.$$

Define

$$M_n = \{x \in c_0; x_k = 0\ \forall k > n\} \quad (n \in \mathsf{N})$$

and

$$G_n = \{x \in c_0 ; |x_k| < r \ \forall k > n\} \quad (n \in \mathsf{N}).$$

Below we identify R^n and its image in c_0 under the map

$$(x_1, \dots, x_n) \curvearrowright (x_1, \dots, x_n, 0, 0, \dots).$$

By using the Fubini theorem it now follows that

$$\int_{\mathsf{R}^n} \mu(B_r(x)) \exp\left(i \sum_1^n x_k \xi_k \right) dx$$

$$= \int_{\mathsf{R}^n} \left(\int_{c_0} 1_{B_r(x)}(y) d\mu(y) \right) \exp\left(i \sum_1^n x_k \xi_k \right) dx$$

$$= \int_{c_0 \cap G_n} \left(\int_{\mathsf{R}^n} 1_{B_r(y)}(x) \exp\left(i \sum_1^n x_k \xi_k \right) dx \right) d\mu(y)$$

$$= \int_{c_0 \cap G_n} \left(\int_{y_n - r}^{y_n + r} \cdots \int_{y_1 - r}^{y_1 + r} \exp\left(i \sum_1^n x_k \xi_k \right) dx_1 \dots dx_n \right) d\mu(y)$$

$$= 2^n \left(\prod_{k=1}^n \left(\frac{\sin(r\xi_k)}{\xi_k} \right) \right) \int_{c_0} \exp\left(i \sum_1^n y_k \xi_k \right) 1_{G_n}(y) d\mu(y).$$

Thus the measures $d\mu_n \equiv 1_{G_n} d\mu$ and $d\nu_n \equiv 1_{G_n} d\nu$ satisfy $\hat{\mu}_n(\xi) = \hat{\nu}_n(\xi)$ for each $\xi = (\xi_1, \dots, \xi_n, 0, \dots) \in M_n$. Since $M_n \subset M_{n+1}$, and G_n increases to c_0 as n tends to infinity, this implies that $\hat{\mu}(\xi) = \hat{\nu}(\xi)$ for each $\xi \in M$. Finally, noting that M is norm dense in $c_0' = l_1$ we are done.

2. Measures with finite Laplace transform.

THEOREM 2.1. *If there exists a* $\delta : E' \to]0, +\infty[$, *such that*

$$\tilde{\mu}(\alpha\xi) < +\infty, \quad |\alpha| < \delta(\xi),$$

then $\mathcal{Q}^\infty(\mu) = \{\mu\}$.

COROLLARY 2.1. *Let* $\mu, \nu \in P(E)$. *The condition*

$$\mu(B_1 \cap B_2) = \nu(B_1 \cap B_2), \text{ all open balls } B_1, B_2$$

implies that $\mu = \nu$.

PROOF OF COROLLARY 2.1. To begin with, we have $\mu(B_n(0)) > 0$ for n larger than an appropriate N. Thus, for $n > N$ we can define $\mu_n, \nu_n \in P(E)$ by

$$\mu_n(A) = \frac{\mu(B_n(0) \cap A)}{\mu(B_n(0))}, \quad A \in \mathscr{B}(E)$$

and ν_n analogously.

Evidently, $\nu_n \in \mathscr{Q}^\infty(\mu_n)$. Since μ_n has bounded support, it follows from Theorem 2.1. that $\mu_n = \nu_n$. Thus $\mu(B_n(0) \cap A) = \nu(B_n(0) \cap A)$ for each $A \in \mathscr{B}(E)$. By letting n tend to plus infinity, this yields that $\mu = \nu$.

Given $\mu \in P(E)$, let (X_i) be a sequence of independent identically distributed E-valued random variables (i.i.d.) with distributions $P_{X_i} = \mu$, and set $\bar{X}_n = n^{-1}(X_1 + \ldots + X_n)$. Note that

$$P(\bar{X}_n \in A) = P_{X_1 + \ldots + X_n}(nA) = P_{X_1} * \ldots * P_{X_n}(nA) = \mu * \ldots * \mu(nA).$$

For each open convex subset A of E the limit $\lim_{n\to\infty} n^{-1} \log P(\bar{X}_n \in A)$ exists in $[-\infty, 0]$ and is denoted by $l_\mu(A)$.

DEFINITION. The Cramér transform of μ is defined by

$$\lambda_\mu(x) = -\inf\{l_\mu(A); x \in A, \ A \text{ open convex}\},$$
$$\lambda_\mu : E \to [0, +\infty].$$

LEMMA 2.1. *If* $\nu \in \mathscr{Q}^\infty(\mu)$, *then* $\lambda_\nu = \lambda_\mu$.

PROOF. As l_μ increases on its domain of definition

$$\lambda_\mu(x) = -\inf\{l_\mu(B); x \in B, \ B \text{ open ball}\}.$$

Thus the lemma follows, if

$$P(\bar{X}_n \in B) = P(\bar{Y}_n \in B), \text{ all open balls } B, \ n \in \mathbf{N},$$

where the (X_i) $[(Y_i)]$ is i.i.d. with $P_{X_i} = \mu$ $[P_{Y_i} = \nu]$.

For an arbitrary $\tau \in P(E)$ we have

$$\mu * \tau(B) = \int \mu(B - y)d\tau(y) = \int \nu(B - y)d\tau(y) = \nu * \tau(B).$$

Hence

$$\mu * \mu(B) = \nu * \mu(B) = \mu * \nu(B) = \nu * \nu(B).$$

This yields

$$P(\bar{X}_n \in B) = \mu * \ldots * \mu(nB) = \nu * \ldots * \nu(nB) = P(\bar{Y}_n \in B),$$

and the lemma is proved.

DEFINITION. For $g: E' \to [-\infty, +\infty]$,

$$g^*(x) \equiv \sup_{\xi \in E'} (\langle x, \xi \rangle - g(\xi)), \quad g^*: E \to [-\infty, +\infty]$$

is called the Legendre transform of g.

LEMMA 2.2. ([1, Theorem 3.2].) *Let* $\mu \in P(E)$. *Then*

$$(\log \tilde{\mu})^* = \lambda_\mu.$$

This is a generalization of a theorem by Cramér (1938) and Chernoff (1952).

Given $\mu \in P(E)$, $\log \tilde{\mu}$ is a convex function on E' and $-\infty < \log \tilde{\mu} \not\equiv +\infty$. Moreover, it is lower semicontinuous in any topology compatible with the duality between E and E'. Hence

(1) $$(\log \tilde{\mu})^{**} = \log \tilde{\mu} \quad \text{(see [1, p. 608])}.$$

PROOF OF THEOREM 2.1. Assume $\nu \in \mathscr{Q}^\infty(\mu)$.

By (1), Lemmas 2.1, and 2.2 we obtain

$$\log \tilde{\mu} = \lambda_\mu^* = \lambda_\nu^* = \log \tilde{\nu}.$$

In particular,

$$\int \exp(\alpha\xi) d\mu = \int \exp(\alpha\xi) d\nu < +\infty$$

for every fixed $\xi \in E'$ and each real α satisfying $|\alpha| < \delta(\xi)$. By analytic continuation to the strip $\{\alpha + i\beta; |\alpha| < \delta(\xi),\ \beta \in \mathsf{R}\}$ we especially have $\hat{\mu}(\xi) = \hat{\nu}(\xi)$, which proves Theorem 2.1.

3. Measures supported by a Gaussian reproducing kernel Hilbert space.

Before formulating the next theorem, we recall some basic facts about Gaussian measures. In this we follow the terminology developed by C. Borell in [3].

A Borel probability measure γ on R is said to be a centred Gaussian measure on R, if there exists a $b \geqq 0$, such that $\hat{\gamma}(t) = \exp(-bt^2)$ for each $t \in \mathsf{R}$. A measure $\gamma \in P(E)$ is said to be a centred Gaussian measure on E, if the image measure $\xi(\gamma)$ is a centred Gaussian measure on R for every $\xi \in E'$. The class of all centred Gaussian measures on E is denoted by $\mathscr{G}_0(E)$.

Let $\gamma \in \mathscr{G}_0(E)$ be fixed and denote by $E'_2(\gamma)$ the closure of E' in $L_2(\gamma)$. For every $\eta \in E'_2(\gamma)$, the measure $\eta\gamma$ has a barycentre $\Lambda(\eta) \in E$, that is

$$\langle \Lambda(\eta), \xi \rangle = \int \langle x, \xi \rangle \eta(x) d\gamma(x), \quad \xi \in E'.$$

The map $\Lambda: E'_2(\gamma) \to E$ is linear and injective. We denote its range by $\mathscr{H}(\gamma)$. For brevity we write $\tilde{h} = \Lambda^{-1}(h)$, $h \in \mathscr{H}(\gamma)$. $\mathscr{H}(\gamma)$ is a Hilbert space with the scalar product

$$\langle h, k \rangle_\gamma = \int \tilde{h}\tilde{k} d\gamma, \quad h, k \in \mathscr{H}(\gamma).$$

$\mathscr{H}(\gamma)$ is called the reproducing kernel Hilbert space (RKHS) of γ. The closed unit ball $O(\gamma)$ of $\mathscr{H}(\gamma)$ is a compact subset of E. Moreover, $\operatorname{supp}(\gamma) = \overline{\mathscr{H}(\gamma)}$ and for any $h \in \mathscr{H}(\gamma)$ the measures $\gamma(\cdot - h)$ and γ are mutually absolutely continuous.

LEMMA 3.1. (Borell [4, Lemma 2.2, Theorem 2.2.]) *Assume* $\gamma \in \mathscr{G}_0(E)$. *Then*

(1) $$\gamma(B_r(y)) \leqq \gamma(B_r(0)), \quad y \in E, \ r > 0$$

(2) $$\lim_{r \to 0^+} \frac{\gamma(B_r(y))}{\gamma(B_r(0))} = \exp\left(-\tfrac{1}{2} \| y \|_\gamma^2\right), \quad y \in E,$$

where

$$\| y \|_\gamma^2 = \begin{cases} \langle y, y \rangle_\gamma, & y \in \mathscr{H}(\gamma) \\ +\infty, & y \in E \setminus \mathscr{H}(\gamma). \end{cases}$$

THEOREM 3.1. *Suppose* $\mu \in P(E)$. *If there exists a* $\gamma \in \mathscr{G}_0(E)$ *such that*

$$\mu(\mathscr{H}(\gamma)) = 1,$$

then $\mathscr{Q}^{\leqq r}(\mu) = \{\mu\}$.

PROOF. For $\nu \in P(E)$ and $\xi \in E'$ we have

$$\int_E \nu(B_s(z)) \exp(i\langle z, \xi \rangle) d\gamma(z) = \int_E \left(\int_{B_s(y)} \exp(i\langle z, \xi \rangle) d\gamma(z) \right) d\nu(y).$$

Now using Lemma 3.1 we get, by dominated convergence,

$$\lim_{s \to 0^+} \frac{1}{\gamma(B_s(0))} \int_E \nu(B_s(z)) \exp(i\langle z, \xi \rangle) d\gamma(z)$$

$$= \int_E \exp(i\langle y,\xi\rangle)\exp(-\tfrac{1}{2}\|y\|_\gamma^2)d\nu(y).$$

Thus, if $\nu \in \mathscr{Q}^{\leq r}(\mu)$ it follows that

$$\int_E \exp(i\langle y,\xi\rangle)\exp(-\tfrac{1}{2}\|y\|_\gamma^2)d\nu(y)$$

$$= \int_E \exp(i\langle y,\xi\rangle)\exp(-\tfrac{1}{2}\|y\|_\gamma^2)d\mu(y), \ \ \xi \in E'.$$

Since the measures $\exp(-\frac{1}{2}\|y\|_\gamma^2)d\nu$ and $\exp(-\frac{1}{2}\|y\|_\gamma^2)d\mu$ have the same Fourier transform, they are equal. Finally, as $\mu(\mathscr{H}(\gamma)) = 1$ and ν is a probability measure, this implies that $\nu = \mu$. This proves Theorem 3.1.

We notice that if every $\mu \in P(E)$ is concentrated on the RKHS of an appropriate $\gamma \in \mathscr{G}_0(E)$, then E is isomorphic to a Hilbert space (see [12]). Furthermore, we have

THEOREM 3.2. *Suppose E is a Hilbert space and suppose that for any $\mu \in P(E)$, there exists a $\gamma \in \mathscr{G}_0(E)$ such that $\mu(\mathscr{H}(\gamma)) = 1$. Then E is finite-dimensional.*

In view of Theorem 3.2, Theorem 3.1 does not imply any of Christensen's results on Hilbert spaces (see [5]).

Before the proof of Theorem 3.2 we give an application of Theorem 3.1 to real-valued stochastic processes. For future reference, however, we first prove the following simple lemma.

LEMMA 3.2. *If M is dense in E, then $\mathscr{Q}_M^{\leq r}(\mu) = \mathscr{Q}^{\leq r}(\mu)$.*

PROOF. Let $\nu \in \mathscr{Q}_M^{\leq r}(\mu)$. Take $a \in E$ arbitrarily but fixed.

There is an at most countable set $S \subset]0,r[$, such that

$$\nu(\partial B_s(a)) = \mu(\partial B_s(a)) = 0, \ \ s \in]0,r[\setminus S.$$

Here ∂ denotes the boundary operator.

If a_n is any sequence in M converging to a and $s \in]0,r[\setminus S$, then, by dominated convergence,

$$\nu(B_s(a)) = \lim_{n\to\infty} \nu(B_s(a_n)) = \lim_{n\to\infty} \mu(B_s(a_n)) = \mu(B_s(a)).$$

It follows that $\nu(B_s(a)) = \mu(B_s(a))$ for each $s \in]0,r]$, which proves the lemma.

In the following W denotes the Wiener measure in $C_0([0,1])$ (cf. [11]). Then $W \in \mathscr{G}_0(C_0([0,1]))$ and the RKHS of W is

$$\mathscr{H}(W) = \{x(t) = \int_0^t x'(s)ds; x' \in L_2([0,1])\} \quad \text{(see [9, p. 121])}.$$

Consider a real-valued stochastic process $X = (X(t))_{0 \leqq t \leqq 1}$ such that

$$(*) \qquad \begin{cases} (t,\omega) \rightsquigarrow X(t,\omega) \text{ is measurable and} \\ \int_0^1 X^2(t)dt < +\infty \text{ a.s.}. \end{cases}$$

X induces a random vector in $L_2([0,1])$, again denoted by X.

For given $r > 0$ we define

$$f_X(a) = P(\{|\int_0^t X(s)ds - a(t)| < r,\ 0 \leqq t \leqq 1\}),\quad a \in C_0([0.1]).$$

THEOREM 3.3. *The random variable f_X determines the law of X.*

More explicitely, suppose Y is another real-valued stochastic process possessing property $(*)$, *If* $f_X(a) = f_Y(a)$ a.s. $[W]$, *then* $\mu = \nu$, *where* μ *and* ν *denote the distributions of X and Y in* $L_2([0,1])$, *respectively.*

PROOF. Define $u: L_2([0,1]) \to C_0([0,1])$ by

$$(uf)(t) = \int_0^t f(s)ds.$$

Let μ_u $[\nu_u]$ be the distribution of $u \circ X$ $[u \circ Y]$ in $C_0([0,1])$. Then, $f_X = f_Y$ means that

$$\mu_u(B_r(a)) = \nu_u(B_r(a)) \text{ a.s. } [W].$$

We need the following lemma.

LEMMA 3.3. *Let* $E = C_0([0,1])$. *Suppose* $M \in \mathscr{B}(C_0([0,1]))$ *and*

$$W(M) = 1.$$

Then $\mathscr{Q}_M^r(\mu) = \mathscr{Q}^{\leqq r}(\mu)$ *for each* $\mu \in P(E)$.

PROOF. Choose functions $h_n \in \mathscr{H}(W)$ with the property

$$\begin{cases} h_n(i2^{-n}) = (-1)^i, & i = 1,\dots,2^n, \\ |h_n(t)| \leqq 1, & 0 \leqq t \leqq 1, \end{cases}$$

and set

$$\tilde{M} = \cap (M - qh_n; q \in]0,r[\cap \mathbb{Q},\ n \in \mathbb{N}).$$

Since the measure $W(\cdot - h)$ is absolutely continuous with respect to W for each $h \in \mathscr{H}(W)$ and $W(M) = 1$, $W(\tilde{M}) = 1$. Therefore, since

$$\operatorname{supp}(W) = \overline{\mathscr{H}(W)} = C_0([0,1]),$$

$\tilde{M}$ is dense in $C_0([0,1])$. According to Lemma 3.2 it is enough to show that $\mathscr{Q}_M^r(\mu) \subsetneq \mathscr{Q}_{\tilde{M}}^{\leqq r}(\mu)$.

Suppose $\nu \in \mathscr{Q}_M^r(\mu)$. Take $z \in \tilde{M}$ arbitrarily but fixed. As before, there is an at most countable set $S \subset]0,r[$ such that $\nu(\partial B_s(z)) = \mu(\partial B_s(z)) = 0$ for each $s \in]0,r[\setminus S$.

Let $s \in]0,r[\setminus S$ be fixed. Choose a sequence $q_n \in]0,r[\cap \mathbf{Q}$, $n \in \mathbf{N}$, which converges to $r - s$ as n tends to plus infinity. Then

$$\lim_{n \to \infty} 1_{B_r(z+q_n h_n)}(y) = 1_{B_s(z)}(y), \quad y \in C_0([0,1]) \setminus \partial B_s(z).$$

By dominated convergence this yields

$$\nu(B_s(z)) = \lim_{n \to \infty} \nu(B_r(z + q_n h_n)) = \lim_{n \to \infty} \mu(B_r(z + q_n h_n)) = \mu(B_s(z)),$$

as $z + q_n h_n \in M$ for each n. Hence $\nu \in \mathscr{Q}_{\tilde{M}}^{\leqq r}(\mu)$, which proves the lemma.

Proof of Theorem 3.3. By applying Lemma 3.3 we have

$$\mu_u(B_s(z)) = \nu_u(B_s(z)), \quad z \in C_0([0,1]),\ 0 < s \leqq r.$$

Observing that $u(L_2([0,1])) = \mathscr{H}(W)$ we have

$$\mu_u(\mathscr{H}(W)) = \mu(L_2([0,1])) = 1.$$

It follows from Theorem 3.1 that $\mu_u = \nu_u$. But $u(B) \in \mathscr{B}(C_0([0,1]))$ for every Borel subset B in $L_2([0,1])$ ([13, p. 103]) and, consequently, $\mu = \nu$. This proves Theorem 3.3.

Finally, to prove Theorem 3.2 we will discuss some rather general properties for Gaussian measures in Hilbert spaces.

Let H be a real separable Hilbert space with inner product $\langle \cdot,\cdot \rangle$. A bounded linear operator A on H is said to be a Hilbert–Schmidt operator if

$$\sum_n \| A e_n \|^2 < +\infty$$

for some (and then for each) orthonormal basis (e_n) of H.

For each $\gamma \in \mathscr{G}_0(H)$ there exists a unique positive self-adjoint Hilbert–Schmidt operator A_γ, such that

$$\int \langle x,z \rangle^2 d\gamma(z) = \| A_\gamma x \|^2, \quad x \in H = H'.$$

Conversely, for each positive self-adjoint Hilbert–Schmidt operator A there exists a unique $\gamma \in \mathscr{G}_0(H)$ such that $A = A_\gamma$. In fact $A_\gamma = S_\gamma^{1/2}$, where

S_γ is the covariance operator of γ. These results are well known and can be found, for example, in [11]. Moreover, $\mathscr{H}(\gamma) = \text{range}\, A_\gamma$.

Further, if A is an operator as above, it has a spectral representation

$$A(\cdot) = \sum_n \lambda_n \langle \cdot , e_n \rangle e_n,$$

where (e_n) is an orthonormal sequence in H, $\lambda_n > 0$, and $\sum \lambda_n^2 < +\infty$.

It follows that $(\lambda_n e_n)$ is an orthonormal basis for $\mathscr{H}(\gamma)$ if $A = A_\gamma$, and $(\lambda_n e_n)^\sim = \lambda_n^{-1} e_n$. To avoid trivialities we assume $\dim \mathscr{H}(\gamma) = +\infty$.

In [10], J. Kuelbs shows that for any orthonormal basis (a_n) in $\mathscr{H}(\gamma)$

$$x = \sum_1^\infty \tilde{a}_n(x) a_n \quad \text{a.s. } [\gamma].$$

THEOREM 3.4. *Suppose $\mu \in \mathscr{G}_0(H)$, $\dim \mathscr{H}(\mu) = +\infty$. If*

$$A_\mu = \sum_1^\infty \sigma_n \langle \cdot , e_n \rangle e_n$$

is the spectral representation of A_μ, the following are equivalent

(a) *there exists a $\gamma \in \mathscr{G}_0(H)$, such that $\mu(\mathscr{H}(\gamma)) = 1$,*

(b) $\sum_1^\infty \sigma_n < +\infty$,

(c) *$A_\mu^{1/2}$ is a Hilbert–Schmidt operator.*

It is obvious that Theorem 3.4 implies Theorem 3.2.

PROOF. The equivalence (b)$\Leftrightarrow$(c) follows immediately from the definition.

(c)$\Rightarrow$(a): Of course, $A_\mu^{1/2}$ is also positive and self-adjoint. Thus there exists a $\gamma \in \mathscr{G}_0(H)$ with $A_\gamma = A_\mu^{1/2}$. Then

$$A_\gamma = \sum_1^\infty \sigma_n^{1/2} \langle \cdot , e_n \rangle e_n.$$

Define

$$\varphi(y) = \sum_1^\infty \frac{\langle y, e_n \rangle^2}{\sigma_n}, \quad y \in H.$$

It follows that

$$\text{range}\, A_\gamma = \{y ; \varphi(y) < +\infty\}.$$

However,

$$\int \varphi \, d\mu = \sum_1^\infty \sigma_n < +\infty.$$

Hence, $\varphi(y) < +\infty$ a.s. $[\mu]$, which means that $\mu(\mathscr{H}(\gamma)) = 1$.

(a) $\Rightarrow$ (b): Let

$$A_\gamma = \sum_1^\infty \lambda_n \langle \cdot, f_n \rangle f_n$$

be the spectral representation of A_γ.

Set $b_n = \lambda_n f_n$. Then (b_n) is an orthonormal basis in $\mathscr{H}(\gamma)$ and $\tilde{b}_n = \lambda_n^{-1} f_n$ (here the $\sim$ sign refers to γ as the underlying measure). Without loss of generality we may assume that $E = \operatorname{supp}(\gamma)$, so that (f_n) is an orthonormal basis in E.

Since $\|\cdot\|_\gamma$ is a μ-measurable seminorm which is finite a.s. $[\mu]$,

$$C = \int \|x\|_\gamma^2 \, d\mu(x) < +\infty \quad \text{(see [11])}.$$

By using $x = \sum_1' \tilde{a}_n(x) a_n$ a.s. $[\mu]$, where $a_n = \sigma_n e_n$, $\tilde{a}_n = \sigma_n^{-1} e_n$, we get

$$\int \|x\|_\gamma^2 \, d\mu(x) = \sum_1^\infty \int \langle x, b_n \rangle_\gamma^2 \, d\mu(x) = \sum_1^\infty \frac{1}{\lambda_n^2} \int \langle x, f_n \rangle^2 \, d\mu(x) = \sum_{n,k=1}^\infty \frac{\sigma_k^2}{\lambda_n^2} \langle e_k, f_n \rangle^2.$$

This yields

$$\sum_1^\infty \sigma_k = \sum_{n,k=1}^\infty \sigma_k \langle e_k, f_n \rangle^2 \leqq \left(\sum_{n,k=1}^\infty \frac{\sigma_k^2}{\lambda_n^2} \langle e_k, f_n \rangle^2 \right)^{1/2} \left(\sum_{n,k=1}^\infty \lambda_n^2 \langle e_k, f_n \rangle^2 \right)^{1/2}$$
$$\leqq C^{1/2} \left(\sum_1^\infty \lambda_n^2 \right)^{1/2} < +\infty.$$

This completes the proof of Theorem 3.4.

REMARK. Above we have only investigated different ball problems for Borel probability measures. However, Theorems 1.1, 2.1, and 3.1 still hold, if we work with *finite positive Borel measures* (abbr. $\mathscr{M}(E)$) defining $\mathscr{Q}_M^r(\mu)$, $\mathscr{Q}_{\bar{M}}^{\leqq r}(\mu)$, $\mathscr{Q}_{\bar{M}}^\infty(\mu)$ as before but now replacing $P(E)$ by $\mathscr{M}(E)$. This is obvious for Theorems 1.1 and 2.1. In the proof of Theorem 3.1 we get $\nu(\cdot \cap \mathscr{H}(\gamma)) = \mu$. From the inner regularity of ν it follows that $\nu(\mathscr{H}(\gamma)) = \nu(E)$ and we are through.

Interestingly enough, the above definition leads to new non-trivial problems. For example, if $\mu, \nu \in \mathscr{M}(E)$ and $\mu(B) = \nu(B)$ for all balls of radius smaller than a fixed $r < +\infty$, it is not at all obvious that $\mu(E) = \nu(E)$.

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DEPARTMENT OF MATHEMATICS
CHALMERS UNIVERSITY OF TECHNOLOGY AND THE UNIVERSITY OF GÖTEBORG
S-412 96 GÖTEBORG
SWEDEN