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MARKOV RANDOM WALKS ON GROUPS

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Abstract.

Let X be a Markov chain on C and f a continuous function from C to a topological group G. The process

$$S_0 = e, \quad S_{n+1} = S_n f(X_{n+1}), \quad n = 1,2,\ldots$$

is called a Markov random walk on G. We investigate the set of possible values taken by S and, under some regularity conditions on X, we also study the recurrence/transcience properties of S by means of an embedded ordinary random walk on G. We show, for instance, that if G is a transient group then all Markov random walks on G are transient.

Let G be a multiplicatively written locally compact second countable topological group. The object of this paper is to investigate the sequence of successive products $S_1, S_2, \ldots$ of factors of the form $f(X_1), f(X_2), \ldots$ where the X_n's form a Markov chain on some topological space C and f is a continuous function from C to G. In particular, we are interested in the possible values and the recurrence properties of the sequence $(S_n)_{n=0}^{\infty}$.

Muthsam [8] and Wolff [13] studied the problem for the case G a discrete semigroup and G a compact semigroup, respectively. (In their work $C \subseteqq G$ and S_n is simply $X_1 X_2 \ldots X_n$.) We will draw heavily on these two papers.

Niemi and Nummelin [9] studied central limit properties of $(S_n)_{n=0}^{\infty}$, with $G = \mathbf{R}$ and f an arbitrary measurable function. In our investigation, we will make use of their technique of introducing an artificial recurrent atom for the Markov chain.

1.

Let C be a locally compact second countable Hausdorff space and $(X_n)_{n=0}^{\infty}$ a Markov chain on C with transition kernel P, which we will assume to be Feller, cf. [12, p. 34]. P_x will denote the probability on the

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canonical probability space (path space) C^∞ induced by P and the initial condition $X_0 = x$. Let f be a continuous function from C to G. The process

$$S_n = \prod_{k=1}^{n} f(X_k), \quad n = 1,2,\ldots; \quad S_0 = e$$

will be called a *Markov random walk* on G.

EXAMPLE 1. Let X be a Markov chain on the integers Z, with transition probability matrix P. Then

$$S_n = X_1 + X_2 + \ldots + X_n, \quad n = 1,2,\ldots$$

is a Markov random walk on Z. It reduces to ordinary random walk if the rows of P are all identical (i.e. the X_i's are independent). If $C = \{1, -1\}$ with $P(1,1) = P(-1,1) = \frac{1}{2}$, then S is a symmetric simple random walk.

EXAMPLE 2. Let $C = \{0\} \cup \{n^{-1} \mid 1,2,\ldots\}$ endowed with the topology of the real line. Let P be defined as follows: $P(0,1) = 1$

$$P(n^{-1}, (n+1)^{-1}) = \alpha_n, \quad P(n^{-1},1) = 1 - \alpha_n, \quad n = 1,2,\ldots$$

where $0 < \alpha_n < 1$, $n = 1,2,\ldots$. If X is the random walk on C with the transition probability matrix P, then

$$S_n = X_1 + X_2 + \ldots + X_n, \quad n = 1,2,\ldots$$

is a Markov random walk on the real line R.

EXAMPLE 3. Let μ be a probability distribution on the set of idempotent matrices of the form

$$(1+\alpha\beta)^{-1}\begin{pmatrix} 1 & \alpha \\ \beta & \alpha\beta \end{pmatrix}$$

where $\alpha,\beta \in$ R, $\alpha\beta \neq -1$. The behavior of a product

$$X_1 X_2 \ldots X_{n-1} X_n$$

of independent μ-distributed random matrices is conveniently (for most purposes) studied by considering the (1,1) entry of the product:

$$S_n = (1+\alpha_1\beta_1)^{-1}(1+\alpha_1\beta_2)\ldots(1+\alpha_{n-1}\beta_n)(1+\alpha_n\beta_n)^{-1}$$

where the subscript i corresponds to the random matrix X_i. As we shall see below S_n is, under mild conditions on the probability μ, a Markov random walk on the multiplicative group R $\setminus \{0\}$.

In our analysis of the Markov random walks we will make extensive use of the fact that the auxiliary process $Z_n = (X_n, S_n)$, $n = 0,1,2,\ldots$, is a Markov chain on $C \times G$ with transition kernel

$$U((x,s), A \times B) = \int 1_A(z) 1_B(sf(z)) P(x,dz)$$
$$x \in C, \ s \in G, \ A \in \mathscr{C}, \ B \in \mathscr{G}$$

where $\mathscr{C}$ and $\mathscr{G}$ are Borel σ-algebras of C and G, respectively.

Following [13] we call a $(n+1)$-tuple $(x_0, x_1, \ldots, x_n)$ a *chain* of length n from x to y if $x = x_0$, $y = x_n$, and x_{i+1} is in the support of the measure $P(x_i, \cdot)$, $i = 0,1,\ldots,n-1$. Denote the chain by k_x and define the functions p and q by

$$p(k_x) = f(x_1) f(x_2) \ldots f(x_n) \quad \text{and}$$
$$q(k_x) = x_n.$$

Suppose a sequence $(k_x^n)_{n=1}^{\infty}$ of chains has the property that $q(k_x^n)$ converges to u. If w is a limit point of the sequence $p(k_x^n)$, it is necessarily of the form $w = vf(u)$, since $p(k_x^n) = v_n f(q(k_x^n))$ for some $v_n \in G$. Note that the assumptions that G is a group and f a continuous function are used in this argument.

For $x,u \in G$ consider the set of all sequences of chains (k_x^n) starting from x such that $q(k_x^n)$ converges to u (the sequence may be finite if there is a chain from x to u).

DEFINITION. $\Pi(x,u)$ is the set of all limit points of the sequences $p(k_x^n)$ such that $q(k_x^n)$ converges to u.

If $q(k_x^n) \to u$, then we say that x *leads to* u, $x \rightharpoonup u$. If $x \rightharpoonup u$ implies $u \rightharpoonup x$, we say that x is *essential*. Note that $x \rightharpoonup u$ is equivalent to

$$u \in \text{closure}\left(\bigcup_{n=1}^{\infty} \text{supp}(P^n(x, \cdot)) \right),$$

where P^n is the n-step transition kernel for X.

We will make the following general assumption about the Markov chain X (cf. [13]),

(A) C is an irreducible class and all elements of C are essential.

(A) guarantees that all elements of C communicate. I.e. for every $x,u \in G$ there exists a sequence (k_x^n) of chains starting from x such that $q(k_x^n)$ converges to u.

In our Example 1 above $\Pi(0,0)$ is the set of all sums $x_1 + x_2 + \ldots + x_{n-1} + 0$ such that the transition probabilities $P(0,x_1)$, $P(x_1,x_2), \ldots,$

$P(x_{n-1},0)$ are all positive. If, for example, $P(0,nd)$, $P(nd,0)$, $P(0,-md)$, $P(-md,0) > 0$ for some relatively prime positive integers m, n, and d, then $\Pi(0,0) \supseteqq d\mathbb{Z}$.

In Example 2, $\Pi(1,1)$ is a subset of the real interval $[1,\infty)$. Note that $\Pi(1,1)$ is necessarily a semigroup and that it is asymptotically dense. It is also easily seen that $\Pi(0,1) = \Pi(1,1)$ but $\Pi(0,0)$ is empty because all chains with $q(k_0^n)$ close to 0 have

$$p(k_0^n) \geqq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m_n}$$

for some large positive integer m_n.

$\Pi(x,x)$ is either empty or a closed subsemigroup of G, the *invariance semigroup* of the element $x \in C$. To see this, notice that (x,g) can be reached by the Z-chain from (x,e), if and only if $g \in \Pi(x,x)$, cf. [13]. So if $(x,e) \rightharpoonup (x,h)$ ((x,e) leads to (x,h)) as well we get $(x,e) \rightharpoonup (x,g)$ and $(x,g) \rightharpoonup (x,gh)$ whence, by the transitivity of the relation $\rightharpoonup$, we obtain the desired result $(x,e) \rightharpoonup (x,gh)$.

Remark. The intuitively obvious transitivity of $\rightharpoonup$ can be seen as follows, cf. [13]: If g is a bounded measurable function on $C \times G$, then we can define

$$Ug(x,s) = \int g(z, sf(z)) P(x,dz) \quad (x,s) \in C \times G$$

($U((x,s),N)$ is to be interpreted as $U1_N(x,s)$ for a set $N \in \mathscr{C} \times \mathscr{G}$).

Using the assumption that P is Feller, the local compactness of C and the continuity of the multiplication in G we can show that U and all its iterates U^n are Feller. Note that the functions $U^n(\cdot, N)$ are then lower semicontinuous for open sets N. Suppose $(x,s) \rightharpoonup (x',s')$ and $(x',s') \rightharpoonup (x'',s'')$. This means, by definition, that for any neighborhoods N' and N'' of (x',s') and (x'',s''), respectively, there are n' and n'' with $U^{n'}((x,s),N')$, $U^{n''}((x',s'),N'') > 0$. Choose N' to be such that $U^{n''}(\cdot, N'') > 0$ on N' to obtain $U^{n'+n''}((x,s),N'') > 0$.

Suppose that there is an essential element $(c,g) \in C \times G$ for the Z-chain. Clearly, $(c,g) \rightharpoonup (c,g)$ so $e \in \Pi(c,c)$. If $h \in \Pi(c,c)$, then $(c,g) \rightharpoonup (c,gh)$ and, by essentiality, $(c,gh) \rightharpoonup (c,g)$, that is $h^{-1} \in \Pi(c,c)$. All elements communicating with (c,g) are essential (as well as all (c,h), $h \in G$) so $\Pi(u,u)$ is also a group for all those $u \in C$ for which $\Pi(c,u) \neq \varnothing$. This is, for instance, the case for all

$$u \in \bigcup_{n \geqq 1} \text{support}(P^n(c, \cdot)),$$

a set which is dense in C by assumption (A). The set of essential elements thus contains the set

$$\bigcup_{\Pi(c,u)\neq\emptyset} \{u\}\times G,$$

cf. [8], [13].

The groups $\Pi(c,c)$ and $\Pi(u,u)$ are, in fact, conjugate. To see this, consider the transitions $(u,e)\rightharpoonup(c,l)\rightharpoonup(u,e)$ or, equivalently, $(u,e)\rightharpoonup(c,l)$ and $(c,e)\rightharpoonup(u,l^{-1})$. If $g\in\Pi(c,c)$, we get $lgl^{-1}\in\Pi(u,u)$ and, conversely, every element of $\Pi(u,u)$ can be represented in this way. Hence $l\Pi(c,c)l^{-1} = \Pi(u,u)$.

Remark. A slight modification of Example 2 will make $\Pi(1,1)$ a group ($=\mathsf{R}$): Introduce a negative element -2, say, and a transition probability $P(1,-2)>0$ and let $P(-2,1)=1$. Then $\Pi(x,y)=\mathsf{R}$ for all $x,y\in C$. Note that the assumption (B) below is not satisfied by the Markov chain in Example 2 even in its modified form.

For any (Feller) Markov chain, the essential classes are closed. If, for example, the group G is compact or $\Pi(c,c)=G$, we can then conclude that *all* elements of $C\times G$ are essential. This also holds, of course, for the case when

(B) $$C=\bigcup_{n\geqq 1}\text{support}\,(P^n(c,\cdot)).$$

Conversely, if (B) is true for all $c\in C$ we can argue as in [8] to prove that $\Pi(c,c)$ is a group (for one and thus for all $c\in C$) if and only if (c,g) is essential for the Z-chain (for some $g\in G$).

We summarize the preceding discussion in

Theorem 1. *Under assumption* (A), *if there is an essential element* (c,g) *for the Z-chain the set of essential elements contains the set*

$$\bigcup_{\Pi(c,u)\neq\emptyset} \{u\}\times G.$$

The invariance semigroup $\Pi(c,c)$ *is a closed subgroup of* G. *Furthermore, for* u *such that* $\Pi(c,u)\neq\emptyset$, $\Pi(u,u)$ *is a group conjugate to* $\Pi(c,c)$.

Corollary 1. *If, in addition, condition* (B) *holds for all* $c\in C$, *either all or none of the invariance semigroups* $\Pi(c,c)$ *are groups. Then all or none, respectively, of the elements of* $C\times G$ *are essential for the Z-chain.*

COROLLARY 2. *If $\Pi(c,c) = G$ or if G is compact, then the same holds for any $u \in C$ and all elements of $C \times G$ are essential.*

For compact C and G we refer to the thorough discussion in [13].

REMARK. It might be worth while to note that the (relative) compactness of $\Pi(c,c)$ will automatically make it a group, since a compact subsemigroup of a group is, in fact, a subgroup.

2.

If X is *stationary*, i.e. X has an invariant probability measure π as its initial distribution, then S is a special case of what we could call a *generalized random walk* on G, cf. [3], [7]. For a stationary X on a countable state space to admit a unique invariant probability distribution it is necessary and sufficient that X be ergodic, cf. [13, p. 136]. In the case of a general state space we will use the sufficient condition that X be positive Harris, see [12]. In this section, we will study the *recurrence* properties of such a walk. For the case $G = \mathsf{R}$ the criterion for recurrence is particularly simple [3]:

S is recurrent if and only if $E_\pi f(X_1)(= E_\pi S_1) = 0$

(E_x is expectation with respect to P_x and $E_\pi(\cdot) = \int E_x(\cdot\, \pi(dx).)$

If G is the direct product of a compact group K and R, we have a similar criterion, namely $E_\pi f_2(X_1) = 0$, where f_2 is the second component of f: If $f(x) = (k,r) \in K \times \mathsf{R}$, then $f_2(x) = r$. This result follows from the preceding, since the walk on the compact factor is automatically recurrent and the walk on the second factor R is itself nothing but a generalized random walk on R. For $G = \mathsf{C}^*$, the multiplicative group of non-zero complex numbers $\simeq T \times \mathsf{R}$, where T is the circle group, we thus have the following recurrence criterion

$$E_\pi \log|S_1| = 0.$$

Consider a group G. If all (ordinary) random walks generated by probability measures whose supports are not contained in any proper closed subgroups of G are transient, the group itself is called a *transient group*. It is well-known that R^m is transient for $m \geqq 3$. Under mild conditions Markov random walks are also transient on R^m, $m \geqq 3$, see [4]. Our aim is to show that the same holds true for all transient groups provided that some additional assumptions are satisfied.

Introduce the following condition on the transition kernel P of the Markov chain X, cf. [9, (1.7)]

(1) $$P(x,A) \geqq h(x)\nu(A), \quad x \in C, \quad A \in \mathscr{C},$$

where ν is a measure on $(C,\mathscr{C})$ and $h \geqq 0$ a function on C with $\int h(x)\pi(dx) > 0$.

Using the splitting technique described in [9], cf. also [1], we can decompose the chain $X = (X_1, X_2, \ldots)$ by (a.s. finite) random times $\tau(1), \tau(2), \ldots$ in such a way that the blocks $X_{\tau(i-1)+1}, \ldots, X_{\tau(i)}$ and $X_{\tau(i)+1}, \ldots, X_{\tau(i+1)}$ are independent and identically distributed (i.i.d.). (For a thorough exposition of the technique the reader should consult the fundamental [10]).

REMARK. For a Harris recurrent chain, P^k is minorized as in (1) for some positive integer k, see [9]. The choice of $k = 1$ is necessary in order to ensure the decomposition of $X_1, X_2, \ldots$ into *independent* blocks.

Suppose first that C is discrete and that X is positive recurrent and aperiodic. (1) is automatically satisfied with $h(x) = 1_{\{c\}}(x)$, $\nu = P(c, \cdot)$, where c is an arbitrary fixed element of C. The times $\tau(1), \tau(2), \ldots$ are the successive returns of X to c. They are all a.s. finite: $E_\pi \tau(1) = (\pi(c))^{-1}$.

$$X_{\tau(1)+1} X_{\tau(1)+2} \ldots X_{\tau(n)} = S_{\tau(1)}^{-1} S_{\tau(n)} \quad n = 1,2,\ldots$$

are products of i.i.d. random elements of $\Pi(c,c) \subseteqq G$. In other words, $(S_{\tau(1)}^{-1} S_{\tau(n)})_{n=1}^{\infty}$ is a random walk on $\Pi(c,c)$, generating all of $\Pi(c,c)$. Hence if $\Pi(c,c)$ is a transient group the random walk is transient. If $\Pi(c,c)$ is a subsemigroup of G which is not a group, then the random walk must be transient, too (because any recurrent random walk automatically generates a group, see [12]). The same holds for the process $(S_{\tau(n)})_{n=1}^{\infty}$, which differs from $(S_{\tau(1)}^{-1} S_{\tau(n)})_{n=1}^{\infty}$ only by its initial distribution. Now, by (an extension to the group case of) the Main Lemma of [1], we can conclude that the Markov random walk $(S_n)_{n=1}^{\infty}$ itself is transient. Hence the desired result – $\Pi(c,c)$ transient or no group at all $\Rightarrow S$ transient – holds for discrete C. (From the discussion in Section 1 we know that the invariance semigroups $\Pi(c,c)$ are all isomorphic if they are groups).

In the general case the times $\tau(i)$ are return times to the set, where the function h is positive, but, in general, not all of them. The difficulty lies in the problem of characterizing the set (subgroup), where the random walk $(S_{\tau(1)}^{-1} S_{\tau(n)})_{n=1}^{\infty}$ "lives", i.e. the smallest closed subgroup containing all the random variables $S_{\tau(1)}^{-1} S_{\tau(n)}$, $n = 1,2,\ldots$. Under certain conditions on f and ν, a characterization of this subgroup can be obtained. Suppose, for example, that the support of ν has non-empty interior and that the function f is an open mapping. Then the smallest closed subgroup that could possibly contain $f(\text{supp}(\nu))$ is the identity component of G. If G is connected and transient it follows that S is transient.

We can formulate this sufficient condition for transience slightly more generally. The distribution of the random group element $f(Z)$, where Z is distributed according to ν, is a measure νf^{-1} on the Borel sets of G. If the support of νf^{-1} generates all of G, then $(S_{\tau(1)}^{-1} S_{\tau(n)})_{n=1}^{\infty}$ lives on G proper, i.e. there is no closed subgroup a.s. containing all the products. Hence this random walk is transient if G is a transient group. As before, it follows that $(S_n)_{n=1}^{\infty}$ is transient, too.

We summarize our results in a

THEOREM 2. *Let X be positive Harris satisfying* (1). *If*

(i) *C is discrete and the invariance semigroups $\Pi(c,c)$, $c \in C$, are either no groups at all or transient groups or*
(ii) *the smallest closed subgroup containing the support of the measure νf^{-1} $(\nu f^{-1}(A) = \nu(f^{-1}(A)),\ A \in \mathcal{G})$ is G itself and G is transient,*

then S is transient.

We saw above that for $G = \mathsf{R}$ or $K \times \mathsf{R}$ the stationarity of X is enough to furnish us with a recurrence criterion, namely $E_\pi S_1 = 0$ and $E_\pi f_2(X) = 0$, respectively. For other groups, we will again use the decomposition of $X_1, X_2, \ldots$ into independent blocks. If $(S_{\tau(1)}^{-1} S_{\tau(n)})_{n=1}^{\infty}$ is recurrent, then a fortiori $(S_n)_{n=1}^{\infty}$ is.

A random walk S on R^2 is recurrent if and only if $ES_1 = 0$ and $ES_1^2 < \infty$, cf. [12, p. 98]. Hence our Markov random walk on R^2 will be recurrent if and only if

$$E_\pi(S_{\tau(2)} - S_{\tau(1)}) = 0 \quad \text{and} \quad E_\pi |S_{\tau(2)} - S_{\tau(1)}|^2 < \infty$$

(assuming, as before, that the process does not live on a smaller subgroup). The first condition reduces to $E_\pi S_1 = 0$, since $E_\pi S_\tau = (E_\pi \tau)(E_\pi S_1)$ (with $\tau = \tau(1)$). The second condition may be written

$$E_\nu \Big| \sum_{i=0}^{\tau} f(X_i) \Big|^2 < \infty.$$

If f is bounded, $E_\nu \tau^2 < \infty$ will guarantee the recurrence of S.

THEOREM 3. *Let X be as in Theorem 2. If*

(i) *C is discrete, the invariance semigroups $\Pi(c,c) = \mathsf{R}$ or R^2 and $E_\pi S_1 = 0$ and, in addition, in the R^2-case,*

$$E_c \Big| \sum_{i=1}^{\tau} f(X_i) \Big|^2 < \infty \quad \textit{for some } c \in C,$$

where τ is the first return time to c, or

(ii) $G = \mathsf{R}$ *and* $E_\pi S_1 = 0$ *or*
(iii) $G = \mathsf{R}^2$, $E_\pi S_1 = 0$ *and*

$$E_\nu \Big| \sum_{i=0}^{\tau(1)} f(X_i)\Big|^2 < \infty,$$

where $\tau(1), \tau(2), \ldots$ *are the times discussed above decomposing the chain* $X_1, X_2, \ldots$ *into* i.i.d. *blocks,*

then the Markov random walk S is recurrent.

The conditions given are also necessary (in (ii) and (iii) provided that the distribution of $S_{\tau(2)} - S_{\tau(1)}$ is not supported by a proper closed subgroup).

REMARK. If G is compact (or $\Pi(c,c)$ in the discrete case) then S is, of course, always recurrent.

3.

For all Harris chains condition (1) is valid for some P^k. However, it is crucial for the decomposition of X into i.i.d. blocks that $k = 1$. This limitation is a considerable drawback, when we want to investigate Markov random walks, where the increment $S_{n+1} - S_n$ is a function of two (or more) consecutive values, e.g., $S_{n+1} - S_n = f(X_n, X_{n+1})$.

The process S is still a Markov random walk: just consider $\hat{X}_n = (X_n, X_{n+1}) \in C \times C$, cf. [11], but property (1) is lost in general.

$$P(\hat{X}_{n+1} \in A \times B \mid \hat{X}_n = (x_1, x_2)) =$$
$$\delta_{x_2}(A) P(X_{n+2} \in B \mid X_{n+1} = x_2) \geqq \delta_{x_2}(A) h(x_2) \nu(B)$$

which cannot be written in the form (1) unless, of course, $\pi\{x_2\} > 0$. The two-step transition is again more regular,

$$P(\hat{X}_{n+2} \in A \times B \mid \hat{X}_n = (x_1, x_2)) \geqq h(x_2) \Big(\int_A \nu(dy) h(y)\Big) \nu(B).$$

Random walks on certain groups, notably the group of rigid motions on R^d, have been investigated using the theory of semi-Markov processes, cf. [4]. It can also be regarded as a Markov random walk on R^d. The essential features of random walks on the semigroup of real $n \times n$-matrices of rank $k < n$ may be studied by viewing (an embedded) Markov random walk on the general linear group $\mathrm{GL}(k, \mathsf{R})$, see [5] or the survey [6].

Our Example 3 is a simple example of such a walk, the product of random projections of R^2 on 1-dimensional subspaces. Almost all of these projections can be represented as matrices of the form

$$\begin{pmatrix} (1+\alpha\beta)^{-1} & (1+\alpha\beta)^{-1}\alpha \\ \beta(1+\alpha\beta)^{-1} & \beta(1+\alpha\beta)^{-1}\alpha \end{pmatrix}$$

where $\alpha,\beta \in \mathsf{R}$, $\alpha\beta \neq -1$. The projection is orthogonal if its nullspace, spanned by the vector $(-\alpha,1)$, and its range, spanned by $(1,\beta)$, are orthogonal; thus the criterion for orthogonality is $\alpha = \beta$.

The product of two projections P_1 and P_2 is not, in general, a projection itself (below, the subscripts i refer to the matrix P_i, $i = 1,2$):

$$P_1P_2 = (1+\alpha_1\beta_1)^{-1}(1+\alpha_1\beta_2)(1+\alpha_2\beta_2)^{-1}\begin{pmatrix} 1 & \alpha_2 \\ \beta_1 & \beta_1\alpha_2 \end{pmatrix}.$$

Suppose $X = (X_1,X_2,\ldots)$ is a stationary Markov chain on the set of matrices of the same type as P above. The first element of the product matrix $X_1X_2\ldots X_n$ is a Markov random walk on the multiplicative semigroup R. Call that element S_n, then

$$S_n = \prod_{k=1}^{n} f(X_{k-1},X_k), \quad S_0 = 1, \quad S_1 = (1+\alpha_1\beta_1)^{-1},$$

where

$$f(X_{k-1},X_k) = (1+\alpha_{k-1}\beta_k)(1+\alpha_k\beta_k)^{-1}, \quad k = 2,3,\ldots.$$

Assume that $P_\pi\big(f(X_k,X_{k+1}) = 0\big) = 0$ for all k (where π is the invariant probability measure for the chain X). Then S is a Markov random walk on the multiplicative group $\mathsf{R} \setminus \{0\}$. Hence it is recurrent if and only if

$$E_\pi \log|f(X_0,X_1)| = \int \log|f(x,y)|\,\pi(dx)P(x,dy) = 0.$$

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REFERENCES

1. H. Berbee, *Recurrence and transience for random walks with stationary increments*, Z. Wahrsch. Verw. Gebiete 56 (1981), 531–536.
2. L. Breiman, *Probability*, Addison-Wesley, Reading, Mass., 1968.
3. F. M. Dekking, *On transience and recurrence of generalized random walks*, Z. Wahrsch. Verw. Gebiete 61 (1982), 459–465.

4. H. Hennion, *Transience de certaines chaînes semimarkoviennes*, Ann. Inst. H. Poincaré Sect. B, 18 (1982), 277–291.
5. G. Högnäs and A. Mukherjea, *Recurrent random walks and invariant measures on semigroups of* $n \times n$ *matrices*, Math. Z. 173 (1980), 69–94.
6. G. Högnäs, *Measures on the semigroup of singular matrices*, Reports on Computer Science & Mathematics, Åbo Akademi, Ser. A, No. 20, 1982.
7. G. Högnäs, *Recurrence and transience of Markov random walks*, Acta Acad. Aboensis, Ser. B, 44, No. 4 (1984).
8. H. Muthsam, *Über die Summe Markoffscher Ketten auf Halbgruppen*, Monatsh. Math. 76 (1972), 43–54.
9. S. Niemi and E. Nummelin, *Central limit theorems for Markov random walks*, Soc. Sci. Fenn. Comment. Phys.-Math. No. 54 (1982).
10. E. Nummelin, *A splitting technique for Harris recurrent Markov chains*, Z. Wahrsch. Verw. Gebiete 43 (1978), 309–318.
11. G. L. O'Brien, *Limit theorems for sums of chain-dependent processes*, J. Appl. Probab. 11 (1974), 582–587.
12. D. Revuz, *Markov chains* (North-Holland Mathematical Library 11) North-Holland Publishing Company, Amsterdam - Oxford, 1975.
13. M. Wolff, *Über Produkte abhängiger zufälliger Veränderlicher mit Werten in einer kompakten Halbgruppe*, Z. Wahrsch. Verw. Gebiete 35 (1976), 253–264.

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