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CONVERGENCE OF DERIVATIONS ON NEST ALGEBRAS

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1. Introduction.

The main result of this paper is that if a sequence of derivations on a nest algebra converges in the strong topology, then it actually converges in norm.

By proving a distance formula for an amplification of a nest algebra, the result is extended to certain tensor products of operator algebras.

If the Hilbert spaces under consideration are separable the result can be found without using a distance formula.

2. Notation.

We will let H denote a complex Hilbert space and let $B(H)$ denote the algebra of bounded operators on H.

Let $\mathscr{P}$ be a set of orthogonal projections in $B(H)$, then $\mathrm{Alg}\,\mathscr{P}$ is the algebra of operators which leave any closed subspace $p(H)$ with $p \in \mathscr{P}$, invariant. If N is an algebra of operators in $B(H)$, then $\mathrm{Lat}\,N$ is the set of invariant orthogonal projections.

An algebra N is said to be reflexive if $N = \mathrm{Alg}(\mathrm{Lat}(N))$. A reflexive algebra N with $\mathrm{Lat}\,N$ commutative is called a CSL-algebra, and with $\mathrm{Lat}\,N$ totally ordered it is called a nest algebra.

Let S be a subset of $B(H)$, then S' will denote the commutant of S in $B(H)$.

A linear map $\delta : N \to B(H)$ is called a derivation if

$$\delta(nm) = n\delta(m) + \delta(n)m \quad \text{for all } n,m \text{ in } N.$$

If $x \in B(H)$ then $\mathrm{ad}(x) : N \to B(H)$ will denote the derivation $n \to xn - nx$.

3. Sequences of derivations.

The following theorem is similar to some well known results [1], [5], [6] for von Neumann algebras of type I, III and most algebras of type II.

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3.1. THEOREM. *Let $N \subseteqq B(H)$ be a nest algebra, and let $\delta_n : N \to B(H)$ be a sequence of derivations. If*

$$\lim_{n\to\infty} \|\delta_n(x)\| = 0 \quad \textit{for each } x \in N,$$

then $\lim_{n\to\infty} \|\delta_n\| = 0$.

PROOF. Let δ'_n be the restriction of δ_n to the diagonal $N \cap N^*$. Since this is a type I von Neumann algebra, we have from [1, Theorem 3.1] that δ'_n converges to zero in norm. A standard fixed point argument [10, Theorem 4.1.6] shows that δ'_n is implemented by an a_n, with $\|a_n\| \leqq \|\delta'_n\|$.

So by substituting δ_n with $\delta_n - \mathrm{ad}(a_n)$, we can assume that δ_n vanishes on the diagonal $N \cap N^*$. We know from [2, Corollary 3.11] that δ_n is implemented by a b_n, and from above we have that $b_n \in (N \cap N^*)' = (\mathrm{Lat}(N))''$.

For $e \in (\mathrm{Lat}(N))''$ define $\partial(e,n) = \mathrm{ad}(b_n e)$ on $B(eH) = eB(H)e$ and let

$$\mathscr{P} = \{p \in \mathrm{Lat}\, N \mid \lim_{n\to\infty} \|\partial(p,n)\| = 0\}$$

and

$$\mathscr{Q} = \{q \in \mathrm{Lat}\, N \mid \lim_{n\to\infty} \|\partial(1-q,n)\| = 0\}.$$

If $p \in \mathrm{Lat}\, N$ then we can find a partial isometry $u \in N$, such that either

$$u^*u \leqq 1 - p \quad \text{and} \quad uu^* = p$$

or

$$u^*u = 1 - p \quad \text{and} \quad uu^* \leqq p.$$

If we are in the first situation, then for each $y \in pB(H)p$ we have

$$\partial(p,n)(y) = p \cdot \partial(p,n)(y) \cdot p = p \cdot \partial_n(yu)u^* - py\delta_n(u)u^*$$

and since $yu \in N$ and $u \in N$, we have that $\lim_{n\to\infty} \|\partial(p,n)(y)\| = 0$.

Hence from [1, Theorem 3.1],

$$\lim_{n\to\infty} \|\partial(p,n)\| = 0, \quad \text{and} \quad p \in \mathscr{P}.$$

In the second situation we conclude in a similar way that $p \in \mathscr{Q}$, and hence we have

$$\mathscr{P} \cup \mathscr{Q} = \operatorname{Lat} N.$$

Put

$$p = \sup \mathscr{P} \quad \text{and} \quad q = \inf \mathscr{Q}.$$

Let $p_\alpha \in \mathscr{P}$, $\alpha \in \Lambda$ be an increasing subnet of $(\mathscr{P}, \leqq)$, such that each element p_α has a successor $p_{\alpha+1} > p_\alpha$ and such that $p = \sup p_\alpha$. If we let $e_\alpha = p_{\alpha+1} - p_\alpha$, we have that

$$\lim_{n\to\infty} \partial(e_\alpha, n) = 0,$$

and by [11, Theorem 4] we can find $\lambda(\alpha,n) \in \mathbf{C}$ such that

$$\| b_n e_\alpha - \lambda(\alpha,n) \| \leqq \inf\{ \| b_n e_\alpha - \lambda e_\alpha \| \mid \lambda \in \mathbf{C} \} + \frac{1}{n} \leqq \| \partial(p_\alpha, n) \| + \frac{1}{n}.$$

Let

$$t_n = \| b_n p - \sum_{\alpha\in\Lambda} \lambda(\alpha,n) e_\alpha \|$$

and suppose that $\limsup t_n = t > 0$.

For fixed $\gamma \in \Lambda$, we have

$$\left\| \sum_{\alpha \leqq \gamma} \bigl(\lambda(\alpha,n) - b_n\bigr) e_\alpha \right\| \leqq \| \partial(p_\gamma, n) \| + \frac{1}{n} \to 0 \quad \text{for } n \to \infty,$$

so we can choose $n_k > n_{k-1}$ and $\alpha_k > \alpha_{k-1}$ such that

$$\| \bigl(\lambda(\alpha_k, n_k) - b_{n_k}\bigr) e_{\alpha_k} \| \geqq \frac{t}{2}, \quad k \in \mathbf{N}.$$

If $\delta(e_{\alpha_k}, n_k)$ denotes the restriction of δ_{n_k} to the nest algebras $e_{\alpha_k} N e_{\alpha_k}$, we have from [2, Lemma 3.7], that

$$\| \delta(e_{\alpha_k}, n_k) \| \geqq \tfrac{1}{3} \inf\{ \| b_{n_k} e_{\alpha_k} - \lambda e_{\alpha_k} \| \mid \lambda \in \mathbf{C} \} \geqq \frac{t}{2} - \frac{1}{n_k}.$$

Hence we can find $x_k \in e_{\alpha_k} N e_{\alpha_k}$ with $\| x_k \| \leqq 1$, such that

$$\| \delta(e_{\alpha_k}, n_k)(x_k) \| \geqq \frac{t}{2} - \frac{2}{n_k},$$

and if we define $x = \sum x_k \in N$, we have $\| x \| \leqq 1$ and

$$\| \delta_{n_k}(x) \| \geqq \| \partial(e_{\alpha_k}, n_k)(x_k) \| \geqq \frac{t}{2} - \frac{2}{n_k},$$

contradicting the fact that $\lim \| \delta_n(x) \| = 0$.

We conclude that we can assume that $b_n = \sum_{\alpha \in \Lambda} \lambda(\alpha,n)e_\alpha$. Let

$$r_n = \sup\{|\lambda(\alpha,n) - \lambda(\beta,n)| \mid \alpha,\beta \in \Lambda\}$$

and assume that $\limsup r_n \geqq r > 0$.

Now choose $n_1 \in \mathbf{N}$ and $\alpha_1 < \beta_1$, in Λ such that

$$|\lambda(\alpha_1,n_1) - \lambda(\beta_1,n_1)| > \frac{r}{2}.$$

For fixed $\gamma \in \Lambda$ we have that

$$r_n(\gamma) = \sup\{|\lambda(\alpha,n) - \lambda(\beta,n)| \mid \alpha,\beta \leqq \gamma\} \leqq \|\partial(p_\gamma,n)\| + \frac{2}{n} \to 0 \quad \text{for } n \to \infty,$$

so we can choose $n_2 > n_1$ and $\beta_2 > \alpha_2 > \beta_1 > \alpha_1$ such that

$$|\lambda(\alpha_2,n_2) - \lambda(\beta_2,n_2)| > \frac{r}{2}.$$

By induction we obtain

$$n_j > n_{j-1},$$

$$\beta_j > \alpha_j > \beta_{j-1} > \alpha_{j-1}, \qquad \text{and}$$

$$|\lambda(\alpha_j,n_j) - \lambda(\beta_j,n_j)| > \frac{r}{2}.$$

Now we can choose rank one partial isometries $w_j \in N$ which maps from $e_{\beta_j}H$ to $e_{\alpha_j}H$. Then

$$w = \sum_{j=1}^{\infty} w_j$$

is a partial isometry in N, and we observe that

$$\|\delta_{n_j}(w)\| = \|\delta(p,n_j)(w)\| \geqq |\lambda(\alpha_j,n_j) - \lambda(\beta_j,n_j)| > \frac{r}{2}$$

contradicting the fact that $\lim \|\delta_n(w)\| = 0$.

Since $\lim r_n = 0$, we conclude that for an arbitrary $\alpha_0 \in \Lambda$ we have

$$\lim \|b_n p - \lambda(\alpha_0,n)\cdot p\| = 0$$

and hence that $p \in \mathscr{P}$.

Similarly we conclude that $q \in \mathscr{Q}$, and we are in one of the following cases:

i) $p < q$ or ii) $q \leqq p$.

In the case i) $q-p$ must be an atom, since $\mathscr{P} \cup \mathscr{Q} = \mathrm{Lat}(N)$. Since $(q-p)B(H)(q-p) \subseteqq N$, we can assume that there exists $\lambda_n, \mu_n \in \mathbf{C}$ such that

$$\lim \| b_n q - (\lambda_n p + \mu_n (q-p) \| = 0.$$

By a rank one partial isometry $v \in N$ from $(q-p)H$ to pH we conclude that

$$\lim |\lambda_n - \mu_n| = 0$$

and hence $q \in \mathscr{P}$, contradicting i) so we must be in case ii) since $p \in \mathscr{P} \cap \mathscr{Q}$ we can assume that there exists $\lambda_n, \mu_n \in \mathbf{C}$ such that

$$\lim \| b_n - (\lambda_n p + \mu_n (1-p)) \| = 0.$$

Again by a rank one partial isometry in N from $(1-p)H$ to pH we conclude that

$$\lim |\lambda_n - \mu_n| = 0$$

and hence $1 \in \mathscr{P}$, and we have obtained the conclusion of the theorem.

For two weakly closed algebras $N \subseteqq B(H)$ and $M \subseteqq B(K)$, we have a representation of the algebraic tensorproduct $N \odot M$ on $B(H \otimes K)$, by $N \otimes M$ we understand the weak closure of $N \odot M$ in $B(H \otimes K)$.

Before considering sequences of derivations on tensorproducts, we need some lemmas. The first is a distance formula, which is a generalization of [2, Lemma 3.7] in content and in proof.

3.2 Lemma. *Let $N \otimes M \subseteqq B(H \otimes K)$, where N is a nest algebra in $B(H)$ and M an injective von Neumann subalgebra of $B(K)$, Then for any operator $x \in B(H \otimes K)$*

$$d(x, (N \otimes M)') \leqq 4 \| \mathrm{ad}(x) \|$$

where $\mathrm{ad}(x) : N \otimes M \to B(H \otimes K)$.

Proof. Let $x \in B(H \otimes K)$ be fixed. We can assume that $\mathrm{Lat}(N)$ is not trivial, since otherwise it would be a consequence of [3, Theorem 2.3]. So let p be a nontrivial projection in $\mathrm{Lat}(N)$.

Define the injective von Neumann algebras

$$\mathscr{A} = (\mathrm{Lat}(N))'' \otimes M', \quad \mathscr{A}' = (N \cap N^*) \otimes M.$$

From [4] and [7] we have that $\mathscr{A}$ and $\mathscr{A}'$ have Schwartz property P, so as in [2] there is a point y in the intersection of $\mathscr{A}$ and the ultraweakly closed convex hull of the set $\{uxu^* \mid u \text{ unitary in } \mathscr{A}'\}$, and it satisfies

$$\|\operatorname{ad}(y)\| \leqq k = \|\operatorname{ad}(x)\|$$

and

$$\|x - y\| \leqq k.$$

For $\xi,\eta \in H$, let $T_{\xi,\eta}$ denote the operator

$$T_{\xi,\eta}(\gamma) = (\gamma|\xi)\eta, \quad \gamma \in H$$

and let $\omega_{\xi,\eta}$ denote the functional

$$\omega_{\xi,\eta}(x) = (x\xi|\eta), \quad x \in B(H).$$

For an ultraweakly continuous functional φ on $B(H)$, we define the slice map (see [12])

$$R_\varphi : B(H) \otimes B(K) \to B(K)$$

by

$$R_\varphi\left(\sum_{i=1}^{n} x_i \otimes m_i\right) = \sum_{i=1}^{n} \varphi(x_i)m_i.$$

Let $\xi \in (1-p)H$ and $\eta \in pH$, then $T_{\xi,\eta} \otimes 1 \in N \otimes M$, and we observe that

$$\|R_{\omega_{\xi,\eta}}(\operatorname{ad}(y)(T_{\xi,\eta} \otimes 1))\| \leqq k\|\xi\|^2\|\eta\|^2. \tag{1}$$

If $\|\xi\| = \|\eta\| = 1$ we find the formula

$$R_{\omega_{\xi,\eta}}(\operatorname{ad}(y)(T_{\xi,\eta} \otimes 1)) = R_{\omega_{\eta,\eta}}(y) - R_{\omega_{\xi,\xi}}(y),$$

with $R_{\omega_{\eta,\eta}}(y)$, $R_{\omega_{\xi,\xi}}(y) \in M'$. The formula is valid for elements in the algebraic tensor product, and hence for y, since the slice maps are ultraweakly continuous.

So we obtain

$$\|R_{\omega_{\eta,\eta}}(y) - R_{\omega_{\xi,\xi}}(y)\| \leqq k. \tag{2}$$

Let $m_1 = R_{\omega_{\xi_1,\xi_1}}(y)$, where $\xi_1 \in (1-p)H$ with $\|\xi_1\| = 1$ is fixed. Then for $\gamma \in H$ arbitrary with $\alpha = p\gamma$ we have

$$R_{\omega_{\gamma,\gamma}}(y(p \otimes 1) - p \otimes m_1) = R_{\omega_{\alpha,\alpha}}(y(p \otimes 1) - p \otimes m_1).$$

Hence we get from [2], that

(3) $$\| R_{\omega_{\gamma,\gamma}}(y(p \otimes 1) - p \otimes m_1) \| \leqq k$$

for all $\gamma \in H$ with $\|\gamma\| = 1$.

Since $(\mathrm{Lat}(N))''$ is commutative, elements in $\mathscr{A}$ can be approximated by simple step functions on the spectrum of $(\mathrm{Lat}(N))''$ with values in M'.

For a simple step function z, we can find $\gamma \in H$ with $\|\gamma\| = 1$, such that

$$\| z \| = \| R_{\omega_{\gamma,\gamma}}(z) \|,$$

hence (3) implies

(4) $$\| y(p \otimes 1) - p \otimes m_1 \| \leqq k.$$

Similarly we get for fixed $\eta_1 \in pH$ with $\|\eta_1\| = 1$ and $n_1 = R_{\omega_{\eta_1,\eta_1}}(y) \in M$, that

(5) $$\| y(1-p) \otimes 1 - (1-p) \otimes n_1 \| \leqq k.$$

Then from (2), (4) and (5) we get that

$$\| y - 1 \otimes m_1 \| \leqq 3k.$$

Finally we conclude that

$$d(x, M') \leqq 4k$$

as wished.

From [2] we know that the continuous and the algebraic cohomology groups are identical for CSL-algebras, i.e.

$$H^n_C(N, B(H)) = H^n(N, B(H)).$$

By proofs identical with those of [8], [9], except for obvious changes one obtains that if $N = N_1 \otimes \ldots \otimes N_k \subseteqq B(H)$ is a tensor product of nest algebras, then

$$H^n_C(N, B(H)) = 0.$$

Especially derivations on these algebras are implemented by bounded operators.

The following lemma shows that also derivations on an amplification of a nest algebra are inner.

3.3. Lemma. *Let $N \subseteqq B(H)$ be a nest algebra, then*

$$H^1(N \otimes \mathbf{C} \cdot I, B(H \otimes K)) = 0.$$

PROOF. Let δ be a bounded derivation. For any finite dimensional projection $q \in B(K)$, we can define the derivation

$$\delta_q : N \otimes \mathbf{C} \cdot I \to B(H \otimes K)$$

by

$$\delta_q(x) = (1 \otimes q)\delta(x)(1 \otimes q).$$

Since $H_C^1(N, B(H)) = 0$ and q is finite dimensional there is an $b_q \in B(H \otimes K)$, such that

$$\delta_q = \mathrm{ad}(b_q).$$

From Lemma 3.2 we can assume that

$$\|b_q\| \leqq 5 \cdot \|\delta_q\| \leqq 5 \cdot \|\delta\|.$$

Let b be a weak limit point of $(b_q)_q$, then since $1 \otimes q \in (N \otimes \mathbf{C} \cdot I)'$, we have

$$\delta = \mathrm{ad}(b)$$

which finishes the proof.

3.4. PROPOSITION. *Let N be a nest algebra, and let $\delta_k : N \otimes \mathbf{C} \cdot I \to B(H \otimes K)$ be a sequence of derivations which converges pointwise to zero, then* $\lim \|\delta_n\| = 0$.

PROOF. With the aid of Lemma 3.2 and Lemma 3.3, the proof of Theorem 3.1 carries over ad verbitum, except for the obvious changes, i.e. instead of p we consider $p \otimes 1$, we find $\lambda(n,\alpha) \in M'$, and the absolute value is to be interpreted as the norm. Instead of [9, Theorem 4] we need [3, Theorem 2.4].

Let now M be an ultraweakly closed subalgebra of $B(K)$ with unit, and with the property that if a sequence of derivations converges pointwise it converges in norm.

3.5 COROLLARY. *With M as above and N a nest algebra, then if a sequence of ultraweakly continuous derivations on $N \otimes M$ converges pointwise, it converges in norm.*

PROOF. Let $\delta_n : N \otimes M \to B(H \otimes K)$ be the sequence of ultraweakly continuous derivations converging pointwise to zero. From Lemma 3.3 and Proposition 3.4 we can assume that δ_n vanishes on $N \otimes \mathbf{C} \cdot I$. The derivation property and the fact that $N' = \mathbf{C} \cdot I$, then yields that δ_n maps $\mathbf{C} \cdot I \otimes M$ into $\mathbf{C} \cdot I \otimes B(K)$. Since δ_n is ultraweakly continuous it is of the form $1 \otimes \rho_n$, where $\rho_n : M \to B(K)$ is a derivation, and the corollary follows.

3.6. Corollary. *The tensor products* $N_1 \otimes \ldots \otimes N_k \otimes \mathbf{C} \cdot I \subseteqq B(H)$, *where* $N_1, \ldots, N_k$ *are nest algebras, have the above property.*

It is easy to give examples of CSL-algebras N with $H_C^1(N, B(H)) \neq 0$, but in the cases I know of the derivations are ultraweakly continuous, and every sequence of derivations which converges pointwise will also converge in norm.

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