MATH. SCAND. 58 (1986), 55–68

ANALYTIC CROSSED PRODUCTS AND OUTER CONJUGACY CLASSES OF AUTOMORPHISMS OF VON NEUMANN ALGEBRAS

PAUL S. MUHLY* AND KICHI-SUKE SAITO**

1. Introduction.

In [1] Arveson associated a non-self-adjoint operator algebra, $\mathfrak{A}(X,m,\tau)$, with each ergodic measure preserving transformation, τ, on a probability measure space, (X,m), and he showed that the unitary equivalence class of $\mathfrak{A}(X,m,\tau)$ determines the conjugacy class of τ and vice versa. That is, he showed that $\mathfrak{A}(X,m,\tau)$ and $\mathfrak{A}(X',m',\tau')$ are unitarily equivalent if and only if τ and τ' are conjugate. The algebra $\mathfrak{A}(X,m,\tau)$ is closely related to a subalgebra of the group-measure von Neumann algebra $W^*(X,m,\tau)$ constructed from $L^\infty(X)$ and τ. This subalgebra of $W^*(X,m,\tau)$ was formally defined in [7] and subsequently was studied systematically by McAsey and the authors [8–10] (and by others) under the name "non-self-adjoint crossed product"; nowadays, for a variety of reasons, we call it an "analytic crossed product". The important thing to keep in mind is that while the unitary equivalence class of $\mathfrak{A}(X,m,\tau)$ constitutes a complete set of conjugacy invariants for τ, as does the related analytic crossed product, $W^*(X,m,\tau)$ contains no information about the conjugacy class of τ. The reason, quite simply, is that for each such τ, $W^*(X,m,\tau)$ is a hyperfinite II_1 factor and all such factors are isomorphic (under the appropriate separability assumption, of course).

In a subsequent paper [2], Arveson and Josephson improved on [1] by showing that the *isomorphism* class of $\mathfrak{A}(X,m,\tau)$ determined τ up to conjugacy. Our objective in this paper is to extend their analysis and to relate the outer conjugacy class of an automorphism of a von Neumann algebra to the isomorphism class of the analytic crossed product constructed from the automorphism. More precisely, let M (respectively N) be a von Neumann algebra, let α (respectively β) be a $*$-automorphism

Received July 16, 1984.

* The first author was supported in part by a grant from the National Science Foundation.
** The second author was supported in part by a Grant-in-Aid for Scientific Research from the Japanese Ministry of Education.

of M (respectively N), and let $M \times_\alpha Z_+$ (respectively $N \times_\beta Z_+$) be the analytic crossed product determined by M and α (respectively N and β) (see section 2 for the definition of analytic crossed product). One says that α and β are *outer conjugate* if there is a $*$-isomorphism γ of M onto N and an inner $*$-automorphism σ of M such that $\gamma \circ \sigma \circ \alpha = \beta \circ \gamma$. It is not difficult to see that if α and β are outer conjugate, then $M \rtimes_\alpha Z_+$ and $N \rtimes_\beta Z_+$ are isometrically isomorphically by an isomorphism that is a σ-weak homeomorphism. Consequently, our interest centers on the converse assertion. In section 3, we show that if α is properly outer on M, if β is aperiodic on N, and if there exists an isomorphism Φ of $M \rtimes_\alpha Z_+$ onto $N \rtimes_\beta Z_+$ such that the restriction of Φ (respectively Φ^{-1}) to the self-adjoint part of $M \rtimes_\alpha Z_+$) (respectively $N \rtimes_\beta Z_+$) is $*$-preserving (respectively σ-weakly continuous), then α and β are outer conjugate. (Recall that to say that α is *properly outer* on M is to say that for no α-invariant projection p *in* M is $\alpha|pMp$ inner; α is *aperiodic* if α^n is properly outer for every $n \in Z$.)

We note that the notions of being properly outer and aperiodic are the proper replacements for the ergodic theoretic assumptions on τ in [1] and [2] although it should be emphasized that in the result just stated no assumptions concerning invariant normal states are made or used, as they are in [1] and [2]. We note, too, that if α is inner, then $M \rtimes_\alpha Z_+$ is isomorphic to $M \otimes H^\infty(T)$, where $H^\infty(T)$ is the usual Hardy space of boundary values of bounded analytic functions on the unit disc viewed as a subalgebra of $L^\infty(T)$, and in this case, while it is possible to describe the automorphisms of $M \rtimes_\alpha Z_+$, the preceding result does not hold. The general case, where α and β are arbitrary, turns out to be rather complicated, and we hope to investigate it at another time. Finally, we note that the continuity assumptions on Φ are quite minimal. We do not assume that Φ is globally continuous in any topology, rather, we assume only that the restriction of Φ to the self-adjoint part of $M \rtimes_\alpha Z_+$ is star preserving, and therefore isometric there, and that the restriction of Φ^{-1} to the self-adjoint part of $N \rtimes_\beta Z_+$ is σ-weakly continuous. While this last assumption may appear a bit strange, we need it for our arguments. Nevertheless, we expect that it is superfluous. It turns out that it is superfluous both in the case when N is a type II_1 factor and in the case when N is a type III factor.

Our most complete generalization of [1] and [2] occurs in section 4, where we assume that M is injective and N carries a faithful, normal, β-invariant state. Under these assumptions we prove that if α is properly outer and β is aperiodic, then α and β are outer conjugate if and only if there exists an isomorphism Φ of $M \rtimes_\alpha Z_+$ onto $N \rtimes_\beta Z_+$ such that the restriction of Φ (respectively Φ^{-1}) to the self-adjoint part of $M \rtimes_\alpha Z_+$

(respectively $N \rtimes_{\beta} \mathsf{Z}_{+}$) is σ-strongly continuous (respectively σ-weakly continuous).

The theory of isomorphisms of function algebras has been an actively studied area since the appearance of Nagasawa's paper [12]. An analytic crossed product may be viewed as a certain type of noncommutative function algebra determined by a one-parameter subgroup of the automorphism group of a von Neumann algebra. This is the perspective developed in [7] where such algebras are called analytic subalgebras of von Neumann algebras. In [11], the first author determined the isomorphisms between analytic subalgebras of abelian von Neumann algebras. It is our hope that the present study will point the way to finding the isomorphisms between analytic subalgebras of arbitrary von Neumann algebras.

2. Preliminaries: Analytic crossed products.

Throughout this paper, M will be a von Neumann algebra on a Hilbert space H and α will be a $*$-automorphism of M. Recall that the crossed product $M \rtimes_{\alpha} \mathsf{Z}$ of M by the automorphism group $\{\alpha^n\}_{n \in \mathsf{Z}}$ is the von Neumann algebra on the Hilbert space $l^2(\mathsf{Z}, H)$ generated by the operators $\pi^{\alpha}(x)$, $x \in M$, and L_{α} defined by the equations

$$(\pi^{\alpha}(x)\xi)(n) = \alpha^{-n}(x)\xi(n), \quad \xi \in l^2(\mathsf{Z},H), \quad n \in \mathsf{Z},$$

and

$$(L_{\alpha}\xi)(n) = \xi(n-1), \quad \xi \in l^2(\mathsf{Z},H), \quad n \in \mathsf{Z}.$$

The automorphism group $\{\hat{\alpha}_t\}_{t \in \mathsf{R}}$ of $M \rtimes_{\alpha} \mathsf{Z}$ dual to $\{\alpha^n\}_{n \in \mathsf{Z}}$ in the sense of Takesaki [16] is implemented by the unitary representation of R, $\{V_t\}_{t \in \mathsf{R}}$, defined by the formula

$$(V_t\xi)(n) = e^{2\pi int}\xi(n), \quad \xi \in l^2(\mathsf{Z},H);$$

that is, $\hat{\alpha}_t(T) = V_t T V_t^*$, $T \in M \rtimes_{\alpha} \mathsf{Z}$. For every $n \in \mathsf{Z}$, we define a σ-weakly continuous linear map ε_n^{α} on $M \rtimes_{\alpha} \mathsf{Z}$ by the integral

$$\varepsilon_n^{\alpha}(T) = \int_0^1 e^{-2\pi int}\,\hat{\alpha}_t(T)\,dt, \quad T \in M \rtimes_{\alpha} \mathsf{Z}.$$

Recall that ε_0^{α} is a faithful, normal, $\{\hat{\alpha}_t\}_{t \in \mathsf{R}}$-invariant conditional expectation of $M \rtimes_{\alpha} \mathsf{Z}$ onto $\pi^{\alpha}(M)$. We define $M \rtimes_{\alpha} \mathsf{Z}_{+}$ to be $\{T \in M \rtimes_{\alpha} \mathsf{Z} : \varepsilon_n^{\alpha}(T) = 0, \text{ for all } n < 0\}$, and call $M \rtimes_{\alpha} \mathsf{Z}_{+}$ the *analytic crossed product* determined by M and α. (In previous papers, we called this algebra a non-self-adjoint crossed product.) It is clear that $M \rtimes_{\alpha} \mathsf{Z}_{+}$ is a σ-weakly closed subalgebra of $M \rtimes_{\alpha} \mathsf{Z}$ which is generated by $\pi^{\alpha}(M)$ and L_{α}. As is shown in [7] and [13], the algebra $M \rtimes_{\alpha} \mathsf{Z}_{+}$ is a maximal subdiagonal algebra in $N \rtimes_{\alpha} \mathsf{Z}$ with respect to ε_0^{α} and so, in particular, ε_0^{α} is a homomorphism of $M \rtimes_{\alpha} \mathsf{Z}_{+}$ onto $\pi^{\alpha}(M)$.

3. Isomorphisms.

We continue with the notation of section 2, and we let N be a von Neumann algebra with a $*$-automorphism β. Suppose that α and β are outer conjugate, that is, suppose that there is a $*$-isomorphism γ of M onto N and an inner automorphism $\mathrm{Ad}(u)$ of M, for some unitary operator u in M, such that $\gamma \circ (\mathrm{Ad}(u)) \circ \alpha = \beta \circ \gamma$. Then, by [16, Propositions 3.4 and 3.5] or [17], there exists a unique $*$-isomorphism Φ of $M \rtimes_\alpha \mathbf{Z}$ onto $N \rtimes_\beta \mathbf{Z}$ such that

$$\varphi(\pi^\alpha(x)) = \pi^\beta(\gamma(x)), \text{ for } x \in M,$$

and

$$\Phi(L_\alpha) = \pi^\beta(\gamma(u^*))\, L_\beta.$$

Further, Φ takes $M \rtimes_\alpha \mathbf{Z}_+$ isometrically and isomorphically on $N \rtimes_\beta \mathbf{Z}_+$. Therefore, we have the following

PROPOSITION 3.1. *If α and β are outer conjugate, then there exists an isometric isomorphism from $M \rtimes_\alpha \mathbf{Z}_+$ onto $N \rtimes_\beta \mathbf{Z}_+$ that is a homeomorphism with respect to the σ-weak topologies on $M \rtimes_\alpha \mathbf{Z}_+$ and $N \rtimes_\beta \mathbf{Z}_+$.*

Our aim in this section is to prove a converse to this proposition. We assume, unless otherwise stated, that Φ is an algebraic isomorphism from $M \rtimes_\alpha \mathbf{Z}_+$ onto $N \rtimes_\beta \mathbf{Z}_+$ such that the restriction, $\Phi|_{\pi^\alpha(M)}$, of Φ to $\pi^\alpha(M)$ is $*$-preserving and that the restriction, $\Phi^{-1}|_{\pi^\beta(N)}$, of Φ^{-1} to $\pi^\beta(N)$ is σ-weakly continuous.

LEMMA 3.2.

$$\bigcap_{n=0}^{\infty} L_\alpha^n(M \rtimes_\alpha \mathbf{Z}_+) = \{0\}.$$

PROOF. Let

$$X \in \bigcap_{n=0}^{\infty} L_\alpha^n(M \rtimes_\alpha \mathbf{Z}_+).$$

Then for each $n \in \mathbf{Z}_+$ there is an element $Y_n \in M \rtimes_\alpha \mathbf{Z}_+$ such that $X = L_\alpha^n Y_n$. Thus it is clear that $\varepsilon_n^\alpha(X) = 0$ for all $n \in \mathbf{Z}$. By [13, Theorem 3.1], we have $X = 0$. This completes the proof.

We put $M \rtimes_\alpha \mathbf{Z}_{+0} = L_\alpha(M \rtimes_\alpha \mathbf{Z}_+)$ and $N \rtimes_\beta \mathbf{Z}_{+0} = L_\beta(N \rtimes_\beta \mathbf{Z}_+)$. Then we have the following lemma whose proof is the primary place our continuity hypotheses are used.

LEMMA 3.3. *If α is properly outer on M, then $\varepsilon_0^\beta(\Phi(L_\alpha)) = 0$.*

PROOF. Since Φ is $*$-preserving on $\pi^\alpha(M)$, we have

$$\Phi(\pi^\alpha(M)) = \Phi(\pi^\alpha(M)) \cap \Phi(\pi^\alpha(M))^* \subset (N \rtimes_\beta \mathbf{Z}_+) \cap (N \rtimes_\beta \mathbf{Z}_+)^* = \pi^\beta(N).$$

Thus we have $\Phi(\pi^\alpha(M)) \subsetneq \pi^\beta(N)$. For all $x \in M$, we have

$$\varepsilon_0^\beta(\Phi(L_\alpha \pi^\alpha(x))) = \varepsilon_0^\beta(\Phi(\pi^\alpha(\alpha(x)) L_\alpha))$$

and so

$$\varepsilon_0^\beta(\Phi(L_\alpha))\, \Phi(\pi^\alpha(x)) = \Phi(\pi^\alpha(\alpha(x)))\, \varepsilon_0^\beta(\Phi(L_\alpha)).$$

Since Φ is isomorphic, we have

$$\Phi^{-1}(\varepsilon_0^\beta(\Phi(L_\alpha)))\pi^\alpha(x) = \pi^\alpha(\alpha(x))\Phi^{-1}(\varepsilon_0^\beta(\Phi(L_\alpha))).$$

Since ε_0^α is a conditional expectation of $M \rtimes_\alpha \mathbf{Z}$ onto $\pi^\alpha(M)$,

$$\varepsilon_0^\alpha(\Phi^{-1}(\varepsilon_0^\beta(\Phi(L_\alpha))))\pi^\alpha(x) = \pi^\alpha(\alpha(x))\varepsilon_0^\alpha(\Phi^{-1}(\varepsilon_0^\beta(\Phi(L_\alpha)))).$$

Since α is properly outer on M, by [15, Proposition 17.4],

$$\varepsilon_0^\alpha(\Phi^{-1}(\varepsilon_0^\beta(\Phi(L_\alpha)))) = 0$$

and so $\Phi^{-1}(\varepsilon_0^\beta(\Phi(L_\alpha))) \in M \rtimes_\alpha \mathbf{Z}_{+0}$. It is clear that

$$\varepsilon_0^\beta(\Phi(M \rtimes_\alpha \mathbf{Z}_{+0})) = \varepsilon_0^\beta(\Phi(L_\alpha))\pi^\beta(N) = \pi^\beta(N)\varepsilon_0^\beta(\Phi(L_\alpha)).$$

Thus $\varepsilon_0^\beta(\Phi(M \rtimes_\alpha \mathbf{Z}_{+0}))$ is a two-sided ideal of $\pi^\beta(N)$. Put B equal to the σ-weak closure of $\varepsilon_0^\beta(\Phi(M \rtimes_\alpha \mathbf{Z}_{+0}))$ in $\pi^\beta(N)$. Then there exists a central projection p in N such that $B = \pi^\beta(p)\pi^\beta(N)$. Since B is the σ-weak closure of $\pi^\beta(N)\varepsilon_0^\beta(\Phi(L_\alpha))$, there exists a net $\{z_\lambda\}_{\lambda \in \Lambda}$ in N such that the σ-weak limit of $\{\pi^\beta(z_\lambda)\varepsilon_0^\beta(\Phi(L_\alpha))\}_{\lambda \in \Lambda}$ is $\pi^\beta(p)$. Since the restriction, $\Phi^{-1}|_{\pi^\beta(N)}$, of Φ^{-1} to $\pi^\beta(N)$ is σ-weakly continuous, we see that $\Phi^{-1}(\pi^\beta(p))$ is the σ-weak limit of $\{\Phi^{-1}(\pi^\beta(z_\lambda)\varepsilon_0^\beta(\Phi(L_\alpha)))\}_{\lambda \in \Lambda}$. Since

$$\Phi^{-1}(\varepsilon_0^\beta(\Phi(L_\alpha))) \in M \rtimes_\alpha \mathbf{Z}_{+0},$$

$\Phi^{-1}(\pi^\beta(z_\lambda)\varepsilon_0^\beta(\Phi(L_\alpha)))$ belongs to $M \rtimes_\alpha \mathbf{Z}_{+0}$ and so

$$\Phi^{-1}(\pi^\beta(p)) \in M \rtimes_\alpha \mathbf{Z}_{+0}.$$

Since p is a projection in N, $p^n = p$ for all $n > 0$. Thus, for $n > 0$,

$$\Phi^{-1}(\pi^\beta(p)) = \Phi^{-1}(\pi^\beta(p))^n \in (L_\alpha(M \rtimes_\alpha \mathbf{Z}_+))^n = L_\alpha^n(M \rtimes_\alpha \mathbf{Z}_+).$$

Thus,

$$\Phi^{-1}(\pi^\beta(p)) \in \bigcap_{n=0}^{\infty} L_\alpha^n(M \rtimes_\beta Z_+).$$

By Lemma 3.2, $\Phi^{-1}(\pi^\beta(p)) = 0$ and so $p = 0$. This implies that $\varepsilon_0^\beta(\Phi(L_\alpha)) = 0$ and completes the proof.

PROPOSITION 3.4. *Suppose that α is properly outer on M. Then*

(i) $\varepsilon_0^\beta(\Phi(L_\alpha(M \rtimes_\alpha Z_+))) = \{0\}$.

(ii) $\varepsilon_0^\beta \circ \Phi = \Phi \circ \varepsilon_0^\alpha$ on $M \times_\alpha Z_+$.

(iii) $\Phi(\pi^\alpha(M)) = \pi^\beta(N)$.

(iv) $\Phi(M \rtimes_\alpha Z_{+0}) = N \rtimes_\beta Z_{+0}$.

PROOF. (i) is clear from Lemma 3.3.

(ii) Let $T \in M \rtimes_\alpha Z_+$ and put $\pi^\alpha(x) = \varepsilon_0^\alpha(T) \in \pi^\alpha(M)$. Then there is an element T_1 in $M \rtimes_\alpha Z_+$ such that $T = \pi^\alpha(x) + L_\alpha T_1$. So, by (i),

$$\begin{aligned}\varepsilon_0^\beta(\Phi(T)) &= \varepsilon_0^\beta(\Phi(\pi^\alpha(x))) + \varepsilon_0^\beta(\Phi(L_\alpha T_1)) \\ &= \Phi(\pi^\alpha(x)) + 0 = \Phi(\varepsilon_0^\alpha(T)).\end{aligned}$$

Therefore, $\varepsilon_0^\beta \circ \Phi = \Phi \circ \varepsilon_0^\alpha$ on $M \rtimes_\alpha Z_+$.

(iii) From (ii),

$$\begin{aligned}\Phi(\pi^\alpha(M)) &= \Phi(\varepsilon_0^\alpha(M \rtimes_\alpha Z_+)) = \varepsilon_0^\beta(\Phi(M \rtimes_\alpha Z_+)) \\ &= \varepsilon_0^\beta(N \rtimes_\beta Z_+) = \pi^\beta(N).\end{aligned}$$

Hence we have (iii).

(iv) From (i), we have $\Phi(M \rtimes_\alpha Z_{+0}) \subset N \rtimes_\beta Z_{+0}$. Conversely, for every $T \in N \rtimes_\beta Z_{+0}$, there exists an element S in $M \rtimes_\alpha Z_+$ such that $\Phi(S) = T$. From (ii),

$$\Phi(\varepsilon_0^\alpha(S)) = \varepsilon_0^\beta(\Phi(S)) = \varepsilon_0^\beta(T) = 0$$

and so $\varepsilon_0^\alpha(S) = 0$. This implies that $S \in M \rtimes_\alpha Z_{+0}$. Hence we have $\Phi(M \rtimes_\alpha Z_{+0}) = N \rtimes_\beta Z_{+0}$. This completes the proof.

Every $*$-automorphism $\rho \in \mathrm{Aut}(M)$ defines an action of Z on M: $n \to \sigma^n \in \mathrm{Aut}(M)$. Clearly, this action is ergodic if and only if the $*$-automorphism σ is ergodic. We shall say that the $*$-automorphism σ is aperiodic if the action σ is properly outer, i.e., σ^n is properly outer on N for all $n \neq 0$. Then, by [15, Proposition 17.7], if M is a von Neumann algebra without minimal projections, and σ is ergodic, then σ is aperiodic. The following lemma is well known as the relative commutant theorem (cf. [15, Theorem 22.3]).

LEMMA 3.5. *Let β be a $*$-automorphism of N. Then β is aperiodic on N if and only if*

$$\pi^{\beta}(N)' \cap (N \rtimes_{\beta} \mathbf{Z}) = \pi^{\beta}(\mathfrak{Z}(N)),$$

where $\mathfrak{Z}(N)$ is the center of N.

LEMMA 3.6. *Suppose that α is properly outer on M and β is aperiodic on N. Then*

$$\Phi(L_{\alpha})^{*}\Phi(L_{\alpha}) \in \pi^{\beta}(\mathfrak{Z}(N)).$$

PROOF. For all $x \in M$,

$$\Phi(L_{\alpha})\Phi(\pi^{\alpha}(x)) = \Phi(L_{\alpha}\pi^{\alpha}(x)) = \Phi(\pi^{\alpha}(\alpha(x)))\Phi(L_{\alpha}).$$

Thus we have

$$\Phi(\pi^{\alpha}(x))^{*}\Phi(L_{\alpha})^{*} = \Phi(L_{\alpha})^{*}\,\Phi(\pi^{\alpha}(\alpha(x)))^{*}.$$

Since $\Phi_{|\pi^{\alpha}(M)}$ is *-preserving,

$$\Phi(\pi^{\alpha}(x))\,\Phi(L_{\alpha})^{*} = \Phi(L_{\alpha})^{*}\,\Phi(\pi^{\alpha}(\alpha(x)))$$

and so

$$\Phi(\pi^{\alpha}(\Phi^{-1}(x)))\,\Phi(L_{\alpha})^{*} = \Phi(L_{\alpha})^{*}\,\Phi(\pi^{\alpha}(x)).$$

Therefore we have

$$\begin{aligned}\Phi(L_{\alpha})^{*}\,\Phi(L_{\alpha})\,\Phi(\pi^{\alpha}(x)) &= \Phi(L_{\alpha})^{*}\,\Phi(\pi^{\alpha}(\alpha(x)))\,\Phi(L_{\alpha})\\ &= \Phi(\pi^{\alpha}(x))\,\Phi(L_{\alpha})^{*}\,\Phi(L_{\alpha}).\end{aligned}$$

This implies that $\Phi(L_{\alpha})^{*}\,\Phi(L_{\alpha}) \in \pi^{\beta}(N)' \cap (N \rtimes_{\beta} \mathbf{Z})$. By Lemma 3.5,

$$\Phi(L_{\alpha})^{*}\,\Phi(L_{\alpha}) \in \pi^{\beta}(\mathfrak{Z}(N)).$$

This completes the proof.

LEMMA 3.7. *Suppose that α is properly outer on M and β is aperiodic on N. Then*

$$\Phi(\pi^{\alpha}(M)L_{\alpha}) = \pi^{\beta}(N)L_{\beta}.$$

PROOF. By Proposition 3.4 (iv), we have

$$L_{\beta}(N \rtimes_{\beta} \mathbf{Z}_{+}) = N \rtimes_{\beta} \mathbf{Z}_{+0} = \Phi(M \rtimes_{\alpha} \mathbf{Z}_{+0}) = \Phi(L_{\alpha})(N \rtimes_{\beta} \mathbf{Z}_{+}).$$

Thus

$$L_{\beta}{}^{*}\Phi(L_{\alpha})(N \rtimes_{\beta} \mathbf{Z}_{+}) = N \rtimes_{\beta} \mathbf{Z}_{+}$$

and so

$$L_{\beta}{}^{*}\Phi(L_{\alpha}) \in N \rtimes_{\beta} \mathbf{Z}_{+}.$$

On the other hand, since $L_\beta(N \rtimes_\beta \mathbf{Z}_+) = \Phi(L_\alpha)(N \rtimes_\beta \mathbf{Z}_+)$, there exists an element S in $N \rtimes_\beta \mathbf{Z}_+$ such that $L_\beta = \Phi(L_\alpha)S$. Then we have

$$\Phi(L_\alpha)^* L_\beta = \Phi(L_\alpha)^* \Phi(L_\alpha)S \in \pi^\beta(\mathfrak{Z}(N))(N \rtimes_\beta \mathbf{Z}_+) \subset N \rtimes_\beta \mathbf{Z}_+.$$

Thus

$$L_\beta^* \Phi(L_\alpha) \in (N \rtimes_\beta \mathbf{Z}_+)^*.$$

This implies that $L_\beta^* \Phi(L_\alpha) \in \pi^\beta(N)$ and so

$$\Phi(\pi^\alpha(M)L_\alpha) \subset \pi^\beta(N)L_\beta.$$

Since $L_\beta \in \Phi(M \rtimes_\alpha \mathbf{Z}_{+0})$, there exists an element T in $M \rtimes_\alpha \mathbf{Z}_+$ such that $L_\beta = \Phi(L_\alpha T)$. Since $L_\beta^* \Phi(L_\alpha) \in \pi^\beta(N)$, we have

$$\begin{aligned} L_\beta^* \Phi(L_\alpha)\Phi(\varepsilon_0^\alpha(T)) &= L_\beta^* \Phi(L_\alpha)\varepsilon_0^\beta(\Phi(T)) \\ &= \varepsilon_0^\beta(L_\beta^* \Phi(L_\alpha)\Phi(T)) = \varepsilon_0^\beta(L_\beta^* L_\beta) = 1. \end{aligned}$$

Thus

$$L_\beta = \Phi(L_\alpha)\Phi(\varepsilon_0^\alpha(T)) \in \Phi(L_\alpha \pi^\alpha(M)),$$

which implies that $\Phi(\pi^\alpha(M)L_\alpha) = \pi^\beta(N)L_\beta$. This completes the proof.

By Lemma 3.7, $\Phi(L_\alpha)L_\beta^* \in \pi^\beta(N)$. Thus there exists an element b in N such that $\Phi(L_\alpha)L_\beta^* = \pi^\beta(b)$. This observation leads to the following lemma.

LEMMA 3.8. *With the notation and assumptions as above, b is invertible in N.*

PROOF. Since $\Phi^{-1}(L_\beta)L_\alpha^* \in \pi^\alpha(M)$ by Lemma 3.7, there exists an element c in N such that

$$\pi^\beta(c) = \Phi(\Phi^{-1}(L_\beta)L_\alpha^*) \in \pi^\beta(N).$$

Then

$$\pi^\beta(b)\pi^\beta(c) = \Phi(L_\alpha)L_\beta^* \Phi(\Phi^{-1}(L_\beta)L_\alpha^*).$$

Applying Φ^{-1}, we have

$$\begin{aligned} \Phi^{-1}(\pi^\beta(b)\pi^\beta(c)) &= \Phi^{-1}(\Phi(L_\alpha)L_\beta^*)\Phi^{-1}(L_\beta)L_\alpha^* \\ &= \Phi^{-1}(\Phi(L_\alpha)L_\beta^* L_\beta)L_\alpha^* = 1, \end{aligned}$$

and so $\pi^\beta(b)\pi^\beta(c) = 1$. On the other hand,

$$\begin{aligned} \pi^\beta(c)\pi^\beta(b) &= \Phi(\Phi^{-1}(L_\beta)L_\alpha^*)\Phi(L_\alpha)L_\beta^* \\ &= \Phi(\Phi^{-1}(L_\beta)L_\alpha^* L_\alpha)L_\beta^* = 1. \end{aligned}$$

Therefore $bc = cb = 1$. This implies that b is invertible in N, and completes the proof.

We consider the polar decomposition of b in N. That is, write $b = v|b|$. Since b is invertible in N, v is a unitary operator in N and $|b| = (b^*b)^{1/2} \in \mathfrak{Z}(N)$, because

$$\begin{aligned}\pi^\beta(b^*b) &= \pi^\beta(b)^*\pi^\beta(b) \\ &= L_\beta \Phi(L_\alpha)^*\Phi(L_\alpha)L_\beta^* \in L_\beta\pi^\beta(\mathfrak{Z}(N))L_\beta^* = \pi^\beta(\mathfrak{Z}(N)).\end{aligned}$$

Therefore there exists a unitary operator v_0 in M such that $\Phi(\pi^\alpha(v_0)) = \pi^\beta(v^*)$, and we have

$$\begin{aligned}\Phi(\pi^\alpha(v_0)L_\alpha) &= \Phi(\pi^\alpha(v_0))\Phi(L_\alpha) = \pi^\beta(v^*)\pi^\beta(b)L_\beta \\ &= \pi^\beta(|b|)L_\beta.\end{aligned}$$

Our goal in this section is the following theorem.

THEOREM 3.9. *Suppose that α is properly outer on M and β is aperiodic on N. If there is an isomorphism Φ of $M \rtimes_\alpha \mathbf{Z}_+$ onto $N \rtimes_\beta \mathbf{Z}_+$ such that the restriction, $\Phi|_{\pi^\alpha(M)}$, of Φ to $\pi^\alpha(M)$ is $*$-preserving and the restriction, $\Phi^{-1}|_{\pi^\beta(N)}$, of Φ^{-1} to $\pi^\beta(N)$ is σ-weakly continuous, then α is outer conjugate to β, that is, there exist a $*$-isomorphism γ of M onto N and an inner automorphism σ of M such that $\gamma \circ \sigma \circ \alpha = \beta \circ \gamma$.*

PROOF. Since $\Phi(\pi^\alpha(M)) = \pi^\beta(N)$, there exists a $*$-isomorphism γ of M onto N such that $\Phi(\pi^\alpha(x)) = \pi^\beta(\gamma(x))$, $x \in M$. Put $\sigma(x) = v_0xv_0^*$, $x \in M$, where v_0 is defined above. Then σ is an inner automorphism of M. For every $x \in M$, we have

$$\begin{aligned}\Phi(\pi^\alpha(v_0)L_\alpha)\Phi(\pi^\alpha(x)) &= \Phi(\pi^\alpha(v_0)L_\alpha\pi^\alpha(x)) \\ &= \Phi(\pi^\alpha(v_0)\pi^\alpha(\alpha(x))L_\alpha) = \Phi(\pi^\alpha(\sigma\circ\alpha(x))\pi^\alpha(v_0)L_\alpha) \\ &= \Phi(\pi^\alpha(\sigma\circ\alpha(x))\Phi(\pi^\alpha(v_0)L_\alpha) = \Phi(\pi^\alpha(\sigma\circ\alpha(x))\pi^\beta(|b|)L_\beta.\end{aligned}$$

On the other hand, since $|b| \in \mathfrak{Z}(N)$,

$$\begin{aligned}\Phi(\pi^\alpha(v_0)L_\alpha)\Phi(\pi^\alpha(x)) &= \pi^\beta(|b|)L_\beta\Phi(\pi^\alpha(x)) \\ &= \pi^\beta(|b|)L_\beta\Phi(\pi^\alpha(x))L_\beta^*L_\beta = L_\beta\Phi(\pi^\alpha(x))L_\beta^*\pi^\beta(|b|)L_\beta.\end{aligned}$$

Therefore,

$$\left(\Phi(\pi^\alpha(\sigma\circ\alpha(x))) - L_\beta\Phi(\pi^\alpha(x))L_\beta^*\right)\pi^\beta(|b|)L_\beta = 0.$$

Since $|b|$ is an invertible element of N, we have

$$\Phi(\pi^\alpha(\sigma\circ\alpha(x))) = L_\beta\Phi(\pi^\alpha(x))L_\beta^*.$$

This implies that $\gamma \circ \sigma \circ \alpha = \beta \circ \gamma$, and completes the proof.

COROLLARY 3.10. *Suppose that α is properly outer on M and β is aperiodic on N. If there exists an isomorphism Φ of $M \rtimes_\alpha Z_+$ onto $N \rtimes_\beta Z_+$ such that $\Phi|_{\pi^\alpha(M)}$ is $*$-preserving and $\Phi^{-1}|_{\pi^\beta(N)}$ is σ-weakly continuous, then α is aperiodic on M.*

Combining Proposition 3.1 and Theorem 3.9, we have the following corollary.

COROLLARY 3.11. *Suppose that α is aperiodic on M and β is aperiodic on N. Then α is outer conjugate to β if and only if there exists an isomorphism Φ of $M \rtimes_\alpha Z_+$ onto $N \rtimes_\beta Z_+$ such that $\Phi|_{\pi^\alpha(M)}$ is $*$-preserving and $\Phi^{-1}|_{\pi^\beta(N)}$ is σ-weakly continuous.*

If N is a type II_1 factor or a type III factor, then we do not need the assumption that $\Phi^{-1}|_{\pi^\alpha(M)}$ is σ-weakly continuous. In fact, we have

THEOREM 3.12. *Let N be a type II_1 factor or a type III factor, let α be properly outer on M, and let β be aperiodic on N. If there exists an isomorphism Φ of $M \rtimes_\alpha Z_+$ onto $N \rtimes_\beta Z_+$ such that $\Phi|_{\pi^\alpha(M)}$ is $*$-preserving, then α is outer conjugate to β.*

PROOF. It suffices to prove that Lemma 3.3 holds under our hypotheses. In the proof of Lemma 3.3 we observed that $\dot{\varepsilon}_0^\beta(\Phi(M \times_\alpha Z_{+0}))$ is a two-sided ideal in $\pi^\beta(N)$. Since finite and purely infinite factors are algebraically simple, we conclude that either this ideal is the zero ideal, in which case we are done, or that it is all of $\pi^\beta(N)$. In this case, there is a $z \in N$ such that $\pi^\beta(z)\dot{\varepsilon}_0^\beta(\Phi(L_\alpha)) = \pi^\beta(1)$ showing that $1 = \Phi^{-1}(\pi^\beta(1)) \in M \rtimes_\alpha Z_{+0}$. This contradiction shows that $\dot{\varepsilon}_0^\beta(\Phi(L_\alpha)) = 0$, which is what we wanted to prove.

4. Injective von Neumann algebras.

Throughout this section we suppose that M is an injective von Neumann algebra on a Hilbert space H and that there exists a faithful normal state ψ on N such that $\psi \circ \beta = \psi$. Our objective is to prove.

THEOREM 4.1. *Suppose that α is properly outer on M and β is aperiodic on N. Then α is outer conjugate to β if and only if there exists an isomorphism Φ of $M \rtimes_\alpha Z_+$ onto $N \rtimes_\beta Z_+$ such that $\Phi|_{\pi^\alpha(M)}$ is σ-strongly continuous and $\Phi^{-1}|_{\pi^\beta(N)}$ is σ-weakly continuous.*

The following proposition is the key to the proof of Theorem 4.1, and is modelled on some arguments in [2].

PROPOSITION 4.2. *If there exists an isomorphism Φ of $M \rtimes_{\alpha} \mathbf{Z}_+$ onto $N \rtimes_{\beta} \mathbf{Z}_+$ such that $\Phi|_{\pi^\alpha(M)}$ is σ-strongly continuous and $\Phi^{-1}|_{\pi^\beta(N)}$ is σ-weakly continuous, then there exists an isomorphism Ψ of $M \rtimes_{\alpha} \mathbf{Z}_+$ onto $N \rtimes_{\beta} \mathbf{Z}_+$ such that $\Psi|_{\pi^\alpha(M)}$ is $*$-preserving and $\Psi^{-1}|_{\pi^\alpha(N)}$ is σ-weakly continuous.*

To prove this proposition, we use the amenability of the group $U(M)$ of all unitary operators in M. In [6], de la Harpe proved that a von Neumann algebra M on a separable Hilbert space is injective if and only if $U(M)$ is amenable in the sense that the space, $C_l^b(U(M))$, of left uniformly continuous functions on $U(M)$ admits a right invariant mean. Here $U(M)$ is considered as a topological group in the strong operator topology. The separability condition on H is not essential (cf. [5, Remark]).

LEMMA 4.3. *For any $\xi,\eta \in l^2(\mathbf{Z},H)$, we define a function $f_{\xi,\eta}$ on $U(M)$ by*

$$f_{\xi,\eta}(v) = \big(\Phi(\pi^\alpha(v))^* \Phi(\pi^\alpha(v))\xi,\eta\big), \quad v \in U(M).$$

Then $f_{\xi,\eta}$ is left uniformly continuous on $U(M)$.

PROOF. Suppose that ξ and η are non-zero. Since $\Phi|_{\pi^\alpha(M)}$ is σ-strongly continuous, $\Phi|_{\pi^\alpha(M)}$ is bounded. For any $\varepsilon > 0$, put

$$W_\varepsilon = \{v \in U(M) : \|(\Phi(\pi^\alpha(v)) - 1)\xi\| < \varepsilon/(2\,\|\Phi\|^2\,\|\eta\|)$$

and

$$\|(\Phi(\pi^\alpha(v)) - 1)\eta\| < \varepsilon/(2\,\|\Phi\|^2\,\|\xi\|),$$

where $\|\Phi\|$ is the norm of $\Phi|_{\pi^\alpha(M)}$. We shall prove that $|f_{\xi,\eta}(u) - f_{\xi,\eta}(v)| < \varepsilon$ if $v^*u \in W_\varepsilon$. If $v^*u \in W_\varepsilon$, then we have

$$\begin{aligned}
&|f_{\xi,\eta}(u) - f_{\xi,\eta}(v)| \\
&\quad \leqq |\big(\Phi(\pi^\alpha(v))^* \Phi(\pi^\alpha(v))\xi, \eta\big) - \big(\Phi(\pi^\alpha(v))^* \Phi(\pi^\alpha(u))\xi, \eta\big)| + \\
&\qquad + |\big(\Phi(\pi^\alpha(v))^* \Phi(\pi^\alpha(u))\xi, \eta\big) - \big(\Phi(\pi^\alpha(u))^* \Phi(\pi^\alpha(u))\xi, \eta\big)| \\
&\quad \leqq \|\Phi(\pi^\alpha(v))\xi - \Phi(\pi^\alpha(u))\xi\| \; \|\Phi(\pi^\alpha(v))\eta\| + \\
&\qquad + \|\Phi(\pi^\alpha(u))\xi\| \; \|\Phi(\pi^\alpha(v))\eta - \Phi(\pi^\alpha(u))\eta\| \\
&\quad \leqq \|\Phi(\pi^\alpha(v))\| \; \|\Phi(\pi^\alpha(v^*u))\xi - \xi\| \; \|\Phi(\pi^\alpha(v))\eta\| + \\
&\qquad + \|\Phi(\pi^\alpha(u))\xi\| \; \|\Phi(\pi^\alpha(v))\| \; \|\Phi(\pi^\alpha(v^*u))\eta - \eta\| \\
&\quad \leqq \|\Phi\|^2 \; \|\eta\| \; \|\Phi(\pi^\alpha(v^*u))\xi - \xi\| + \\
&\qquad + \|\Phi\|^2 \; \|\xi\| \; \|\Phi(\pi^\alpha(v^*u))\eta - \eta\| < \varepsilon.
\end{aligned}$$

This completes the proof.

PROOF OF PROPOSITION 4.2. Put

$$R = \sup\{\|\Phi(\pi^\alpha(v))\| : v \in U(M)\}.$$

Then from Lemma 4.3, $R < \infty$ and we have

$$R^{-2}\|\xi\|^2 \leqq \big(\Phi(\pi^\alpha(v))\xi, \Phi(\pi^\alpha(v))\xi\big) \leqq R^2\|\xi\|^2.$$

Since $\big(\Phi(\pi^\alpha(v))\xi, \Phi(\pi^\alpha(v))\eta\big)$ is left-uniformly continuous on $U(M)$ by Lemma 4.3, we may define

$$[\xi,\eta] = \int_{U(M)} \big(\Phi(\pi^\alpha(v))\xi, \Phi(\pi^\alpha(v))\eta\big)dm(v),$$

where m is a right invariant mean on $U(M)$ (such a mean exists by assumption). By a lemma of Riesz, there exists an invertible positive operator T such that $[\xi,\eta] = (T\xi,\eta)$. Then T belongs to the weakly closed convex hull of $\{\Phi(\pi^\alpha(u))^*\Phi(\pi^\alpha(u))\}_{u\in U(M)}$. Thus $T \in N \rtimes_\beta \mathsf{Z}$. Since T is an invertible positive operator in $N \rtimes_\beta \mathsf{Z}$, we may apply [10, Corollary 5.3], to conclude that there exists an element A in $(N \rtimes_\beta \mathsf{Z}_+) \cap (N \rtimes_\beta \mathsf{Z}_+)^{-1}$ such that $T = A^*A$. Then, by the right invariance of m, we see that for all $\xi \in l^2(\mathsf{Z},K)$,

$$\begin{aligned}
&\|A\Phi(\pi^\alpha(v))A^{-1}\xi\|^2 \\
&= \big(A^*A\Phi(\pi^\alpha(v))A^{-1}\xi, \Phi(\pi^\alpha(v))A^{-1}\xi\big) \\
&= \int_{U(M)} \big(\Phi(\pi^\alpha(uv)A^{-1}\xi, \Phi(\pi^\alpha(uv))A^{-1}\xi\big)dm(u) \\
&= \int_{U(M)} \big(\Phi(\pi^\alpha(u))A^{-1}\xi, \Phi(\pi^\alpha(u))A^{-1}\xi\big)dm(u) \\
&= (TA^{-1}\xi, A^{-1}\xi) = (A^*AA^{-1}\xi, A^{-1}\xi) = \|\xi\|^2.
\end{aligned}$$

This implies that $A\Phi(\pi^\alpha(v))A^{-1}$ is a unitary operator. Therefore, as in [2, Corollary of Theorem 2.4], we may define Ψ by the equation $\Psi(S) = A\Phi(S)A^{-1}$, $S \in M \rtimes_\alpha \mathsf{Z}_+$. Then Ψ is clearly an isomorphism of $M \rtimes_\alpha \mathsf{Z}_+$ onto $N \rtimes_\beta \mathsf{Z}_+$ such that the restriction, $\Psi|_{\pi^\alpha(M)}$, of Ψ to $\pi^\alpha(M)$ is $*$-preserving. Further, since

$$\Psi^{-1}(T) = \Phi^{-1}(A^{-1})\Phi^{-1}(T)\Phi^{-1}(A), \quad T \in N \rtimes_\beta \mathsf{Z}_+,$$

$\Psi^{-1}|_{\pi^\beta(N)}$ is σ-weakly continuous. This completes the proof.

Combining Proposition 4.2 and Theorem 3.9 proves Theorem 4.1. Moreover, we have the following corollary.

COROLLARY 4.4. *If α is properly outer on M, if β is aperiodic on N, and if there is an isomorphism Φ of $M \rtimes_\alpha \mathsf{Z}_+$ onto $N \rtimes_\beta \mathsf{Z}_+$ such that $\Phi|_{\pi^\alpha(M)}$ is σ-strongly continuous and $\Phi^{-1}|_{\pi^\beta(N)}$ is σ-weakly continuous, then α is aperiodic on M and N is injective.*

Since an ergodic automorphism on a von Neumann algebra without minimal projections is aperiodic, we have the following proposition.

PROPOSITION 4.5. *Suppose that M and N are von Neumann algebras without minimal projections. If α and β are ergodic, then α is outer conjugate to β if and only if there exists an isomorphism Φ of $M \rtimes_\alpha \mathbf{Z}_+$ onto $N \rtimes_\beta \mathbf{Z}_+$ such that $\Phi|_{\pi^\alpha(M)}$ is σ-strongly continuous and $\Phi^{-1}|_{\pi^\beta(N)}$ is σ-weakly continuous.*

If N is either a II_1-factor, or a factor of type III, then the following theorem is a consequence of Theorems 3.12 and 4.1.

THEOREM 4.5. *Let N be a II_1-factor or a factor of type III. Suppose that α is properly outer on M, that β is aperiodic on N, and that there exists a faithful normal state ψ on N such that $\psi \circ \beta = \psi$. Then α is outer conjugate to β if and only if there exists an isomorphism Φ of $M \rtimes_\alpha \mathbf{Z}_+$ onto $N \rtimes_\beta \mathbf{Z}_+$ such that $\Phi|_{\pi^\alpha(M)}$ is σ-strongly continuous.*

REFERENCES

1. W. B. Arveson, *Operator algebras and measure preserving transformations*, Acta Math. 118 (1967), 95–109.
2. W. B. Arveson and K. B. Josephson, *Operator algebras and measure preserving transformations* II, J. Funct. Anal. 4 (1969), 100–135.
3. A. Connes, *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. École Norm. Sup. 8 (1975), 383–419.
4. F. P. Greenleaf, *Invariant means of topological groups and their applications*, (Van Nostrand Math. Stud. 16) Van Nostrand, New York, London, 1969.
5. U. Haagerup, *All nuclear C*-algebras are amenable*, Invent. Math. 74 (1983), 305–319.
6. P. de la Harpe, *Moyennabilité du groupe unitaire et propriété* P *de Schwartz des algèbres de von Neumann* in *Algèbres d'operateurs*, (Sém., Les Plans-sur-Bec, Suisse, 1978), ed. P. de la Harpe. (Lecture Notes in Math. 725) pp. 220–227. Springer-Verlag, Berlin - Heidelberg - New York, 1979.
7. R. I. Loebl and P. S. Muhly, *Analyticity and flows in von Neumann algebras*, J. Funct. Anal. 29 (1978), 214–252.
8. M. McAsey, P. S. Muhly, and K.-S. Saito, *Nonselfadjoint crossed products (invariant subspaces and maximality)*, Trans. Amer. Math. Soc. 248 (1979), 381–409.
9. M. McAsey, P. S. Muhly, and K.-S. Saito, *Nonselfadjoint crossed products* II, J. Math. Soc. Japan 33 (1981), 485–495.
10. M. McAsey, P. S. Muhly, and K.-S. Saito, *Nonselfadjoint crossed products* III (infinite algebras), J. Operator Theory 12 (1984), 3–22.
11. P. S. Muhly, *Isometries of ergodic Hardy spaces*, Israel J. Math. 36 (1980), 50–74.
12. M. Nagasawa, *Isomorphisms between commutative Banach algebras with an application to rings of analytic functions*, Kōdai Math. Sem. Rep. 11 (1959), 182–188.
13. K.-S. Saito, *The Hardy spaces associated with a periodic flow on a von Neumann algebra*, Tohoku Math. J. 29 (1977), 69–75.

14. K.-S. Saito, *Automorphisms and nonselfadjoint crossed products*, Pacific J. Math. 102 (1982), 179–187.
15. S. Stratila, *Modular theory in operator algebras*, Abacus Press, Tunbridge Wells, 1981.
16. M. Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type* III, Acta Math. 131 (1973), 249–310.
17. M. Takesaki, *Structure of factors and automorphism groups*, CBMS Regional Conf. Ser. in Math. 51, American Mathematical Society, Providence, R.I., 1983.

P. S. MUHLY
DEPARTMENT OF MATHEMATICS
DIVISION OF MATHEMATICAL SCIENCES
UNIVERSITY OF IOWA
IOWA CITY, IOWA 52242
U.S.A.

KICHI-SUKE SAITO
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
NIIGATA UNIVERSITY
NIIGATA, 950-21,
JAPAN