MATH. SCAND. 58 (1986), 69–76

COVERINGS OF FOLIATIONS AND ASSOCIATED C*-ALGEBRAS

MOTO O'UCHI

This paper is dedicated to Professor O. Takenouchi on his sixtieth birthday

Introduction.

For an irrational number θ, let (V,F_θ) be the Kronecker foliation on $V = \mathsf{R}^2/\mathsf{Z}^2$ with the slope θ. For a natural number $n \geqq 2$, it follows from a result of M. A. Rieffel [9, Theorem 2.7] that the associated C*-algebras $C^*(V,F_\theta)$ and $C^*(V,F_{n\theta})$ are not isomorphic, but it is natural to think that there must be some relations between $C^*(V,F_\theta)$ and $C^*(V,F_{n\theta})$. In this paper, generalizing this question, we study the relations between covering maps of foliations and associated C*-algebras.

Let (V,F) and (V',F') be C^∞-foliations and $C^*(V,F)$ and $C^*(V',F')$ be associated C*-algebras. In section 1, we introduce the notion of a homogeneous covering map of (V,F) onto (V',F') with the structure group Z, and show that, if such a map exists, then there exists an action β of Z on $C^*(V,F)$ such that the reduced crossed product of $C^*(V,F)$ by β is isomorphic to $C^*(V',F')$. To prove this, we use essentially a result of M. Hilsum and G. Skandalis [6].

In section 2, we consider two examples of Anosov foliations. One of them is the Kronecker foliation mentioned above. It follows from the result of section 1 that $C^*(V,F_\theta)$ and $C^*(V,F_{n\theta})$ are crossed products of $C^*(V,F_{n\theta})$ and $C^*(V,F_\theta)$ respectively by actions of Z_n. The other example is a foliation obtained from the constant time suspension of an element of $SL(2,\mathsf{Z})$ acting on the two-torus, and we can show a similar result.

The author would like to express his hearty thanks to Professor O. Takenouchi for constant encouragement and helpful suggestions.

1. Coverings of foliations.

Let V and V' be C^∞-manifolds without boundary and (V,F) and (V',F') be $C^{\infty,0}$-foliations [4, Chapter VII]. A C^∞-map ψ of V onto V' is called a map of (V,F) onto (V',F') if, for each leaf L of (V,F), $\psi(L)$ is a leaf of (V',F'). A diffeomorphism of (V,F) onto (V',F') is an injective map of (V,F) onto (V',F') such that the inverse map is a map of (V',F') onto (V,F). The group of all diffeomorphisms of (V,F) onto itself is denoted by $\mathrm{Diffeo}(V,F)$. A map ψ of (V,F) onto (V',F') is said to be a covering map of

Received June 5, 1984; in revised form February 5, 1985.

(V,F) onto (V',F') if, for every point $x \in V'$, there exists an open neighborhood Ω of x such that the restriction of ψ to each connected component of $\psi^{-1}(\Omega)$ is a C^∞-diffeomorphism onto Ω.

Let G and G' be holonomy groupoids of (V,F) and (V',F') respectively [11, 5, § 5]. If ψ is a covering map of (V,F) onto (V',F'), one can define a homomorphism Ψ of G onto G' by

$$\Psi(\gamma)(t) = \psi(\gamma(t)) \quad t \in [0,1], \text{ for } \gamma \in G.$$

It is clear that Ψ is locally a diffeomorphism. We shall say that Ψ is the homomorphism associated with ψ.

A submanifold T of V is said to be a transverse submanifold of (V,F) if, for every $x \in T$, there exists a local coordinate $\Omega \cong D^p \times D^q$ of x in (V,F) such that $T \cap \Omega \cong D^k \times D^q$, where D^p is a unit ball of R^p and $\dim F = p$, $\dim T = k + q$, $0 \leqq k \leqq p$. A transverse submanifold T of (V,F) is said to be faithful if T meets every leaf of F. We define a subgroupoid G_T^T of G by $G_T^T = \{\gamma \in G;\ s(\gamma), r(\gamma) \in T\}$.

DEFINITION 1.1. A covering map ψ of (V,F) onto (V',F') is called a homogeneous covering map with the structure group Z if it satisfies the following properties:

(i) there exist faithfully transverse submanifolds T and T' of (V,F) and (V',F') respectively such that the restriction $\psi | T$ of ψ to T is a diffeomorphism onto T',

(ii) there exists a homomorphism w of Z into Diffeo (V,F) such that

(a) this action is free, that is, if $w_j(x) = x$ for some $x \in V$, then $j = e$, where e is the unit of Z,

(b) $\psi^{-1}(\psi(x)) = \{w_j(x); j \in Z\}$,

(c) $\overline{w_j(T)} \cap [\bigcup_{j' \neq j} w_{j'}(T)]^- = \varnothing$ for all $j \in Z$,

(iii) let $X = \bigcup_{j \in Z} w_j(T)$ and let ψ be the homomorphism of G onto G' associated with ψ, then for all $x \in X$, the restriction $\Psi | (G_X^X)^x$ of Ψ to $(G_X^X)^x$ is one-to-one.

Then we have the following theorem:

THEOREM 1.2. *Suppose that there exists a homogeneous covering map of (V,F) onto (V',F') with the structure group Z. Then there exists an action β of Z on $\mathrm{C}^*(V,F)$ such that $\mathrm{C}^*(V,F) \times_{\beta r} Z$ is isomorphic to $\mathrm{C}^*(V',F')$.*

Let ψ be a homogeneous covering map of (V,F) onto (V',F') with the structure group Z. Note that Z is a countable group. For a holonomy groupoid G, Aut (G) denotes the group of all diffeomorphisms ρ of G onto itself such that ρ and ρ^{-1} are algebraically homomorphisms. We define a homomorphism W of Z into $\mathrm{Aut}(G)$ by

$$W_j(\gamma)(t) = w_j(\gamma(t)), \quad t \in [0,1], \quad \text{for } j \in Z \text{ and } \gamma \in G.$$

We write $j \cdot \gamma$ for $W_j(\gamma)$. Note that $\Psi(j \cdot \gamma) = \Psi(\gamma)$. The semi-direct product $G \times_W Z$ is defined as follows [cf. 8, Chapter I, Definition 1.7]; $G \times_W Z$ is $G \times Z$ as a set and (γ,j) and (γ',j') are composable if and only if γ and $\gamma_1 = j \cdot \gamma'$ are composable,

$$(\gamma,j)(j^{-1} \cdot \gamma_1,j') = (\gamma\gamma_1,jj'), \quad (\gamma,j)^{-1} = (j^{-1} \cdot \gamma^{-1},j^{-1}).$$

The unit space $(G \times_W Z)^{(0)}$ may be identified with V. In the following, we set $H = G_X^X$ and $H' = G'^{T'}_{T'}$. We may define the semi-direct product $H \times_W Z$. To prove the theorem we prepare a lemma.

LEMMA 1.3. *Define a discrete groupoid $I_Z = Z \times Z$ as follows: (j_1,j_2) and (j'_1,j'_2) are composable if and only if*

$$j_2 = j'_1, \quad (j_1,j_2)(j_2,j'_2) = (j_1,j'_2), \quad (j_1,j_2)^{-1} = (j_2,j_1).$$

Then there exists a diffeomorphism Ψ' of $H \times_W Z$ onto the product groupoid $H' \times I_z$ which is algebraically an isomorphism, where Z is considered as a discrete group.

PROOF. We set $T(j) = w_j(T)$. Define a map Ψ' of $H \times_W Z$ into $H' \times I_Z$ by

$$\Psi'(\gamma,j) = (\Psi(\gamma), (j_1,j^{-1}j_2))$$

for $(\gamma,j) \in H \times_W Z$ with $r(\gamma) \in T(j_1)$, $s(\gamma) \in T(j_2)$. From the condition (c) of (ii) in Definition 1.1, Ψ' is locally a diffeomorphism.

We show that Ψ' is surjective. For $(\gamma', (j_1,j_2)) \in H' \times I_Z$, there exists $\gamma \in H$ such that $\Psi(\gamma) = \gamma'$. By taking $j \cdot \gamma$ if necessary, we may suppose that $r(\gamma) \in T(j_1)$. If $s(\gamma) \in T(j'_2)$, then we have

$$\Psi'(\gamma,j'_2 j_2^{-1}) = (\gamma', (j_1,j_2)).$$

The map Ψ' is a homomorphism. In fact, for $\bar{\gamma} = (\gamma,j)$, $\bar{\gamma}' = (j^{-1} \cdot \gamma',j') \in H \times_W Z$ with $s(\gamma) = r(\gamma')$, $r(\gamma) \in T(j_1)$, $s(\gamma) \in T(j_2)$, and $s(\gamma') \in T(j'_2)$, we have

$$\begin{aligned}\Psi'(\bar{\gamma})\,\Psi'(\bar{\gamma}') &= (\Psi(\gamma), (j_1,j^{-1}j_2))\,(\Psi(j^{-1} \cdot \gamma'), (j^{-1}j_2,j'^{-1}(j^{-1}j'_2))) \\ &= (\Psi(\gamma\gamma'),(j_1,(jj')^{-1}j'_2)) \\ &= \Psi'(\bar{\gamma}\,\bar{\gamma}'),\end{aligned}$$

and we have

$$\begin{aligned}\Psi'(\bar{\gamma}^{-1}) &= (\Psi(j^{-1}\cdot\gamma^{-1}), (j^{-1}j_2,j_1)) \\ &= (\Psi(\gamma)^{-1}, (j^{-1}j_2,j_1)) \\ &= \Psi'(\bar{\gamma})^{-1}.\end{aligned}$$

Let (γ,j), $(\gamma',j') \in H \times_W Z$ be such that $\Psi'(\gamma,j) = \Psi'(\gamma',j') = (\tilde{\gamma}, (j_1,j_2))$. Then we have $r(\gamma), r(\gamma') \in T(j_1)$ and $\psi(r(\gamma)) = \psi(r(\gamma'))$. From the condition (i) in Definition 1.1, we have $r(\gamma) = r(\gamma')$, and then, from the condition (iii), we have $\gamma = \gamma'$. It follows that Ψ' is one-to-one. Finally we show that Ψ'^{-1} is a homomorphism. For $\bar{\gamma}' = (\gamma', (j_1,j_2))$, $\bar{\gamma}'' = (\gamma'', (j_2,j_3)) \in H' \times I_Z$ with $s(\gamma') = r(\gamma'')$, let (γ,j) and (γ_0,j') be such that $\Psi'^{-1}(\bar{\gamma}') = (\gamma,j)$ and $\Psi'^{-1}(\bar{\gamma}'') = (\gamma_0,j')$. Since we have $s(\gamma), r(j\cdot\gamma_0) \in T(jj_2)$ and $\psi(s(\gamma)) = \psi(r(j\cdot\gamma_0))$, we have $s(\gamma) = r(j\cdot\gamma_0)$. It follows that (γ,j) and (γ_0,j') are composable and that

$$\Psi'^{-1}(\bar{\gamma}')\Psi'^{-1}(\bar{\gamma}'') = (\gamma(j\cdot\gamma_0), jj').$$

On the other hand, we have

$$\Psi'(\gamma(j\cdot\gamma_0), jj') = (\Psi(\gamma)\Psi(\gamma_0), (j_1,j_3)) = \bar{\gamma}'\bar{\gamma}''.$$

This implies that

$$\Psi'^{-1}(\bar{\gamma}'\bar{\gamma}'') = \Psi'^{-1}(\bar{\gamma}')\Psi'^{-1}(\bar{\gamma}'').$$

Since we have

$$\Psi'^{-1}(\bar{\gamma}')^{-1} = (j^{-1}\cdot\gamma^{-1}, j^{-1})$$

and

$$\Psi'(j^{-1}\cdot\gamma^{-1}, j^{-1}) = (\Psi(\gamma)^{-1}, (j_2,j_1)) = \bar{\gamma}'^{-1},$$

we have

$$\Psi'^{-1}(\bar{\gamma}'^{-1}) = \Psi'^{-1}(\bar{\gamma}')^{-1}.$$

Thus Ψ'^{-1} is a homomorphism.

PROOF OF THE THEOREM. We consider the foliation $(H,\mathscr{F})$ defined as in [11, 4, Chapter VII]. We also consider the foliation (X, F_X), where F_X is the set of connected components of $X \cap L$ for $L \in F$. Let $\Omega^{1/2}$ be the half density bundle of the tangent bundle of $(H,\mathscr{F})$ [5, § 5]. Since w_j can be considered as an element of Diffeo (X, F_X), there is an isomorphism $\Gamma(j,\gamma)$ of $\Omega^{1/2}_{j\cdot\gamma}$ onto $\Omega^{1/2}_{\gamma}$ associated with w_j for all $\gamma \in H$. For $j,j' \in Z$, we have

$$\Gamma(j,\gamma)\Gamma(j',j\cdot\gamma) = \Gamma(j'j,\gamma).$$

Let $C_c(H,\Omega^{1/2})$ be the involutive normed algebra defined as in [5, §5, §6]. For $j \in Z$, we define a map α_j of $C_c(H,\Omega^{1/2})$ into itself by

$$\alpha_j(f)(\gamma) = \Gamma(j^{-1},\gamma)(f(j^{-1}\cdot\gamma))$$

for $f \in C_c(H,\Omega^{1/2})$ and $\gamma \in H$. The completion of $C_c(H,\Omega^{1/2})$ is denoted by $C_r^*(H)$ [see 6]. The above map α_j is extended to a *-automorphism of $C_r^*(H)$, which is denoted again by α_j. Then the map $\alpha; j \in Z \mapsto \alpha_j \in \mathrm{Aut}(C_r^*(H))$ is an action of Z on $C_r^*(H)$.

We consider foliations $(H \times_W Z, F_1)$ and $(H' \times I_Z, F_2)$, where leaves of F_1 are connected components of $\{(\gamma,j_0) \in H \times_W Z; r(\gamma) \in L\}$ for $L \in F_X$, and leaves of F_2 are defined by a similar way. We denote again by $\Omega^{1/2}$ the half density bundles of the tangent bundles of these foliations. We form $C_c(H \times_W Z, \Omega^{1/2})$ and $C_c(H' \times I_Z, \Omega^{1/2})$, and then define $C_r^*(H \times_W Z)$ and $C_r^*(H' \times I_Z)$ as before. It follows from Lemma 1.3 that $C_r^*(H \times_W Z)$ and $C_r^*(H' \times I_Z)$ are isomorphic. It is clear that the reduced crossed product $C_r^*(H) \times_{\alpha r} Z$ of $C_r^*(H)$ by α is isomorphic to $C_r^*(H \times_W Z)$. By [6, Corollary 6], $C^*(V,F)$ (respectively $C^*(V',F')$) is isomorphic to $C_r^*(H) \otimes \mathscr{K}$ (respectively $C_r^*(H') \otimes \mathscr{K}$). We define an action β of Z on $C^*(V,F)$ by $\beta_j = \alpha_j \otimes \iota$, where ι is the trivial automorphism of $\mathscr{K}$. Since we have

$$C_r^*(H' \times I_Z) \cong C_r^*(H') \otimes \mathscr{K}(l^2(Z)),$$

$C^*(V,F) \times_{\beta r} Z$ is isomorphic to $C^*(V',F')$.

2. Examples.

In this section, we consider two examples of Anosov foliations.

1°. For an irrational nimber θ, let (V,F_θ) be the Kronecker foliation on $V = \mathsf{R}^2/\mathsf{Z}^2$, that is, the leaf through (x,y) is $\{(x+t, y+\theta t) \in V; t \in \mathsf{R}\}$. For a natural number $n \in \mathsf{N}$, we define a map ψ of $(V,F_{n\theta})$ onto (V,F_θ) by $\psi(x,y) = (nx,y)$. A submanifold $T = \{0\} \times \mathsf{R}/\mathsf{Z}$ of V is faithfully transverse to both $(V,F_{n\theta})$ and (V,F_θ). We define a homomorphism w of Z_n into $\mathrm{Diffeo}(V,F_{n\theta})$ by

$$w_j(x,y) = (x + j/n, y) \quad (j \in \mathsf{Z}_n).$$

Then ψ is a homogeneous covering map with the structure group Z_n. Similarly, if we define a map ψ' of (V,F_θ) onto $(V,F_{n\theta})$ by $\psi'(x,y) = (x,ny)$, then ψ' is a homogeneous covering map with the structure group Z_n. Thus we have:

THEOREM 2.1. (a) *There exists an action β of Z_n on* $C^*(V,F_{n\theta})$ *such that* $C^*(V,F_{n\theta}) \times_{\beta r} \mathsf{Z}_n$ *is isomorphic to* $C^*(V,F_\theta)$.

(b) *There exists an action β' of Z_n on* $C^*(V,F_\theta)$ *such that* $C^*(V,F_\theta) \times_{\beta' r} \mathsf{Z}_n$ *is isomorphic to* $C^*(V,F_{n\theta})$.

If θ is rational, the above ψ and ψ' are not in general homogeneous covering maps. The C*-algebras discussed here are completely classified by M. A. Rieffel [9, Theorem 2.7]. It follows from his result that $C^*(V,F_\theta)$ and $C^*(V,F_{n\theta})$ are not isomorphic if $n \neq 1$.

2°. Let

$$A_{m,n} = \begin{pmatrix} 1 & n \\ m & mn+1 \end{pmatrix}$$

be an element of $\mathrm{SL}(2,\mathsf{Z})$ and λ_1, λ_2 be eigenvalues of $A_{m,n}$ such that $0 < \lambda_2 < 1 < \lambda_1$. We define a Riemannian metric on $T^2 \times \mathsf{R}$ by

$$ds^2 = \lambda_1^{-2u}[m\lambda_1 dx + (1-\lambda_1)dy]^2 + \lambda_2^{-2u}[m\lambda_2 dx + (1-\lambda_2)dy]^2 + du^2$$

for $(x,y,u) \in T^2 \times \mathsf{R}$. Let $\{\phi_t ; t \in \mathsf{R}\}$ be a flow on $T^2 \times \mathsf{R}$ such that

$$\phi_t(x,y,u) = (x,y,u+t)$$

and α be an action of Z on $T^2 \times \mathsf{R}$ such that

$$\alpha_k(x,y,u) = (A_{m,n}^k(x,y), u-k),$$

where $A_{m,n}(x,y) = (x+ny, mx+(mn+1)y)$. We define an equivalence relation $\sim$ on $T^2 \times \mathsf{R}$ as follows: $a \sim b$ if and only if there exists $k \in \mathsf{Z}$ such that $b = \alpha_k(a)$, and we set $V_{m,n} = T^2 \times \mathsf{R}/\sim$. As the metric ds^2 is invariant under α, we consider it as a metric on $V_{m,n}$. We also consider (ϕ_t) as a flow on $V_{m,n}$. We define subspaces X_a, Y_a, Z_a of the tangent space $T_a(V_{m,n})$ at $a \in V_{m,n}$ as follows: X_a is generated by $n(\partial/\partial x)_a + (\lambda_2 - 1)(\partial/\partial y)_a$, Y_a is generated by $n(\partial/\partial x)_a + (\lambda_1 - 1)(\partial/\partial y)_a$, Z_a is generated by $(\partial/\partial u)_a$. Then we have

$$\begin{aligned} T_a(V_{m,n}) &= X_a \oplus Y_a \oplus Z_a, \\ \|(\varphi_t)^*\xi\|^2 &= \lambda_1^{-2t}\|\xi\|^2 \quad \text{for } \xi \in X_a, \\ \|(\varphi_t)^*\xi\|^2 &= \lambda_2^{-2t}\|\xi\|^2 \quad \text{for } \xi \in Y_a. \end{aligned}$$

Let $(V_{m,n}, F^{ws})$ (respectively $(V_{m,n}, F^{wu})$) be the foliation such that the tangent space of the leaf through a is $X_a \oplus Z_a$ (respectively $Y_a \oplus Z_a$). For the distance d on $V_{m,n}$ associated with ds^2, we set

$$\begin{aligned}
L^s(a) &= \{b \in V_{m,n}\,; d(\phi_t(a), \phi_t(b)) \to 0 \quad \text{as } t \to \infty\},\\
L^u(a) &= \{b \in V_{m,n}\,; d(\phi_t(a), \phi_t(b)) \to 0 \quad \text{as } t \to -\infty\},\\
L^{ws}(a) &= \bigcup\{L^s(\phi_t(a))\,; t \in (-\infty, +\infty)\},\\
L^{wu}(a) &= \bigcup\{L^u(\phi_t(a))\,; t \in (-\infty, +\infty)\}.
\end{aligned}$$

Then we have $F^i = \{L^i(a)\,; a \in V_{m,n}\}$ $i = ws, wu$. As for the above discussion, see [1, §13, 3, §2].

Let p be a divisor of m. We define a map ψ of $(V_{m,n}, F^i)$ onto $(V_{m/p,np}, F^i)$ by $\psi(x,y,u) = (px,y,u)$. Let T (respectively T') be a submanifold of $V_{m,n}$ (respectively $V_{m/p,np}$) which is the image of $\{0\} \times \boldsymbol{T} \times \{0\}$ under the quotient map $\boldsymbol{T}^2 \times \mathrm{R} \to V_{m,n}$ (respectively $\boldsymbol{T}^2 \times \mathrm{R} \to V_{m/p,np}$). We define a homomorphism w of Z_p into $\mathrm{Diffeo}(V_{m,n}, F^i)$ by

$$w_j(x,y,u) = (x + j/p, y, u) \quad (j \in \mathrm{Z}_p).$$

Then one can prove that ψ is a homogeneous covering map with the structure group Z_p. Let q be a divisor of n. If we define a map ψ' of $(V_{m,n}, F^i)$ onto $(V_{mq,n/q}, F^i)$ by $\psi'(x,y,u) = (x,qy,u)$, then ψ' is a homogeneous covering map with the structure group Z_q. Then we have:

THEOREM 2.2. (a) *If p is a divisor of m, then there exists an action β of Z_p on $\mathrm{C}^*(V_{m,n},F^i)$ such that $\mathrm{C}^*(V_{m,n},F^i) \times_{\beta r} \mathrm{Z}_p$ is isomorphic to $\mathrm{C}^*(V_{m/p,np}, F^i)$ $(i = ws, wu)$.*

(b) *If q is a divisor of n, then there exists an action β' of Z_q on $\mathrm{C}^*(V_{m,n}, F^i)$ such that $\mathrm{C}^*(V_{m,n}, F^i) \times_{\beta' r} \mathrm{Z}_q$ is isomorphic to $\mathrm{C}^*(V_{mq,n/q}, F^i)$ $(i = ws, wu)$.*

It follows from a result of H. Takai [10, Theorem 4.2] that $KK(\mathrm{C}^*(V_{m,n}, F^i))$ $(m,n = 1,2,\ldots)$ are isomorphic to one another, but it is not known whether $\mathrm{C}^*(V_{m,n}, F^i)$ $(m,n = 1,2,\ldots)$ are isomorphic to one another.

REFERENCES

1. V. I. Arnold and A. Avez, *Ergodic problems of classical mechanics*, W. A. Benjamin, Inc., 1968.
2. R. J. Blattner, *Quantization and representation theory*, Proc. Sympos. Pure Math. 26 (1974), 147–165.
3. R. Bowen, *Anosov foliations are hyperfinite*, Ann. of Math. 106 (1977), 549–565.
4. A. Connes, *Sur la théorie non commutative de l'integration* in *Algèbres d'operateurs* (Seminaire, Les Plans-sur-Bex, Suisse, 1978), ed. P. de la Harpe, (Lecture Notes in Math. 725), pp. 19–143. Springer-Verlag, Berlin - Heidelberg - New York, 1979.
5. A. Connes, *A survey of foliations and operator algebras*, Proc. Sympos. Pure Math. 38 (1982), 521–628.
6. M. Hilsum and G. Skandalis, *Stabilité des C*-algèbres de feuilletages*, Ann. Inst. Fourier (Grenoble) 33 (1983), 201–208.

7. G. K. Pedersen, *C*-algebras and their automorphism groups* (London Math. Soc. Monographs 14), Academic Press, London, 1979.
8. J. Renault, *A groupoid approach to C*-algebras* (Lecture Notes in Math. 793), Springer-Verlag, Berlin - Heidelberg - New York, 1980.
9. M. A. Rieffel, *C*-algebras associated with irrational rotations*, Pacific J. Math. 93 (1981), 415–429.
10. H. Takai, *C*-algebras of Anosov foliations*, preprint, University of Warwick, 1983.
11. H. E. Winkelnkemper, *The graph of a foliation*, preprint.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
EHIME UNIVERSITY
MATSUYAMA, 790
JAPAN