MATH. SCAND. 58 (1986), 77–104

DUAL ALGEBRAS

MARTIN E. WALTER[1]

Abstract.

We formulate in a symmetric fashion a notion of duality within the category of Banach algebras which generalizes the well known Pontriagin–Van Kampen duality for abelian locally compact groups. This paper is primarily devoted to the development of two concrete but reasonably different examples of this duality which have served as motivation for the more general theory. We show that $A(G)$, the Fourier algebra of locally compact group G, is dual to $L^1(G)$, the group algebra of absolutely left Haar integrable functions, in the sense that each occurs as a Banach algebra of completely bounded maps of the C*-completion of the other. We also exhibit a similar duality for $n \times n$ matrix algebras. Some of the material in the original preprint of this paper has been deleted at the request of the referee.

0. Introduction.

Some time ago we noted an analogy between the complex $n \times n$ matrices, M_n, and certain complex-valued functions on the real line, R. In particular there is a *local*, or pointwise, product defined for functions on R as well as for M_n. In the latter case it is called the componentwise, or the Schur (or Schur–Hadamard) product. There is a *global* product, called convolution, defined for absolutely Haar–Lebesgue integrable functions on R; and there is a global product, the usual matrix product, defined on M_n. There are also local notions of positivity, namely pointwise non-negative functions and componentwise non-negative matrices, as well as global notions of positivity, namely positive definite functions on R and Hermitian positive definite matrices.

Now in both situations the local product of globally positive objects is globally positive. Also in both situations the global product of locally positive objects is locally positive. Finally, at least in the case of R, there is a

[1] This research has been partially supported by a grant from the National Science Foundation. 1980 AMS subject classification 22D35, 22D25.

Received November 16, 1982; in revised form May 2, 1985.

well known Fourier transform which relates global products to local products and global positivity to local positivity. We note in passing that there is an involution associated with each notion of positivity.

This paper was in part inspired by the attempt to see this analogy from the standpoint of a general theory that encompasses both examples. In this we have succeeded, but now more questions have been raised – the answers to which must be postponed to a subsequent publication. We note in passing that Propositions 5 and 8, surprisingly enough, give apparently new facts about M_n. There are also, we believe, as yet unexplored relationships between [17] and the contents of this paper.

1. Preliminaries.

The following paper can be motivated by the following discussion of locally compact group G and some Banach algebras closely related to G. Recall first of all that if G is abelian then $G\hat{}$ is the collection of all continuous homomorphisms (called characters) of G into T, the "circle group" of complex numbers of length one. A famous result (about *abelian* locally compact group G) of Pontriagin and Van Kampen, cf. [7, p. 378], states that $G\hat{}$, called the dual group of G, is itself a locally compact abelian group (with pointwise multiplication of characters and the compact-open topology) and that G is topologically isomorphic to the dual of $G\hat{}$, namely, $G\hat{\hat{}}$.

Now for G a locally compact group (abelian or not) we define $M^1(G)$ to be the usual Banach algebra of (bounded) regular complex Borel measures with the product in this algebra being the usual convolution of measures, denoted $\mu * \nu$ for $\mu,\nu \in M^1(G)$, cf. [3, p. 252], [7, p. 266]. Recall also that $M^1(G) \cong C_0(G)'$, i.e., as a Banach space $M^1(G)$ is isometrically isomorphic to the Banach space of all continuous complex-valued linear functionals on $C_0(G)$, the continuous complex-valued functions on G which vanish at infinity, cf. [15, p. 131]. We denote the value of $\mu \in M^1(G)$ at $f \in C_0(G)$ by $\langle f,\mu\rangle$. In this notation we recall that $M^1(G)$ has an isometric involution $*$, where $\langle f,\mu^*\rangle = \overline{\langle f^\flat,\mu\rangle}$ with $f^\flat(x) = \bar{f}(x^{-1})$, $x \in G$, over-bar denoting complex-conjugation.

Contained in $M^1(G)$ is a two-sided ideal (invariant under involution $*$) of measures absolutely continuous with respect to left Haar measure λ. We denote this ideal $L^1(G)$, or $L^1(G,d\lambda)$, which thus inherits its Banach involution algebra structure from $M^1(G)$, with the usual formulas for convolution and involution, viz. for $x,y \in G$

$$f * g(x) = \int_G f(y)g(y^{-1}x)d\lambda(y) \quad \text{and} \quad f^\sharp(x) = \Delta^{-1}(x)f^\flat(x),$$

where $f,g \in L^1(G)$, or more precisely $fd\lambda$, $gd\lambda \in L^1(G)$. Note that Δ is the modular function of G determined by the formula $d\lambda(x^{-1}) = \Delta(x^{-1})d\lambda(x)$.

Now for the sake of motivation return for the moment to the case where G is abelian and let $M^1(G\hat{\ })$ be the Banach (convolution) algebra on the dual group $G\hat{\ }$. A theorem of Bochner, cf. [14, p. 19], says that a continuous complex-valued function p on G is the inverse Fourier transform of a positive measure $\mu \in M^1(G)$, that is $p = \mathscr{F}^{-1}(\mu)$, $\mu \geqq 0$, if and only if p is positive definite. Letting $\mathbf{C}$ be the complex numbers, recall that $p: G \to \mathbf{C}$ is positive definite by definition if

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j p(x_j^{-1} x_i) \geqq 0 \quad \text{for each } n = 1,2,3,\ldots$$

and each choice $\{x_1, x_2, \ldots x_n\} \subset G$ and each choice of complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. We shall denote by $P(G)$ the collection of all continuous positive definite functions on (abelian or nonabelian) G. Also recall that

$$\mathscr{F}^{-1}(\mu)(x) = \int_{\gamma \in G\hat{\ }} \langle x,\gamma \rangle d\mu(\gamma), \quad x \in G$$

where $\langle x,\gamma \rangle$ is the value of $\gamma \in G\hat{\ }$ at $x \in G$. Note that the formula for $\mathscr{F}$, the Fourier transform, is

$$\mathscr{F}(\mu)(\gamma) = \int_{x \in G} \overline{\langle x,\gamma \rangle} d\mu(x) \quad \text{for } \gamma \in G\hat{\ },\ \mu \in M^1(G).$$

Thus for abelian G we may define the Fourier–Stieltjes algebra of G, $B(G)$, to be $\mathscr{F}^{-1}(M^1(G\hat{\ }))$, where

$$\| b \|_{B(G)} = \| \mu \|_{M^1(G\hat{\ })} \quad \text{if } b = \mathscr{F}^{-1}(\mu).$$

Alternatively, by Bochner's theorem, $B(G)$ is, as a commutative algebra, the (finite) linear combinations of continuous positive definite functions with the algebra product now being pointwise product on G. A theorem of Wiener states that $\mathscr{F}^{-1}(L^1(G\hat{\ })) = A(G)$, where $A(G)$ is the norm closure in $B(G)$ of the functions in $B(G)$ with compact support. The algebra $A(G)$ is called the Fourier algebra of G.

Now by the above mentioned theorem of Pontriagin and Van Kampen the dual group $G\hat{\ }$ of abelian group G is a complete invariant of G in the sense that G can be recovered from $G\hat{\ }$. By theorems of Wendel, cf. [26], and Johnson, cf. [9], $M^1(G)$ and $L^1(G)$ are complete invariants of G and $M^1(G\hat{\ })$ and $L^1(G\hat{\ })$ are complete invariants of $G\hat{\ }$. A fortiori, $B(G)$ which is isometrically isomorphic as a Banach algebra with $M^1(G\hat{\ })$ and $A(G)$ which is isometrically isomorphic as a Banach algebra with $L^1(G\hat{\ })$ are each a complete invariant for G. [Note that $A(G)$ and $B(G)$ are complete invariants for nonabelian G as well, cf. [21].] Thus, at least in the case, when G is abelian one might be led to make the statement that, for example, $A(G)$ and $L^1(G)$ together are a Banach algebra version of Pontriagin duality, in the sense that $L^1(G)$ takes the place of G and $A(G)$ takes the place of $G^\wedge$, and in some sense $L^1(G)$ and $A(G)$ are "dual" to each other. A principal goal of this paper is to make precise the manner in which $L^1(G)$ and $A(G)$ are dual to each other.

Let us summarize some of the above information in Diagram 1 that makes sense at least, when G is abelian.

$$\begin{tikzcd}
C_0(G)' \cong M^1(G) \arrow[drr, "\mathscr{F}"] & & M^1(G^\wedge) \cong C_0(G^\wedge)' \arrow[dll, "\mathscr{F}^{-1}"'] \\
B(G) & & B(G^\wedge)
\end{tikzcd}$$

$$\begin{tikzcd}
L^1(G) \arrow[drr, "\mathscr{F}"] & & L^1(G^\wedge) \arrow[dll, "\mathscr{F}^{-1}"'] \\
C_0(G) \supset A(G) & & A(G^\wedge) \subset C_0(G^\wedge)
\end{tikzcd}$$

Diagram 1.

So much for motivation. We now dispense totally with the abelian case. In the above diagram the left hand column still makes sense when G is a general locally compact group, in addition, $C_0(G^\wedge)$ from the right hand column still has an analogue called $C^*(G)$, the universal enveloping C^*-algebra of $L^1(G)$, cf. [3, p. 40]. Also, as Banach spaces $B(G) \cong C^*(G)'$, where $C^*(G)'$ is the collection of continuous linear functionals on $C^*(G)$, cf. [6]. Note that $B(G)$, the finite linear combinations of continuous positive definite functions, is still a commutative Banach algebra, even if G is not abelian. Also the Fourier transform $\mathscr{F}$ has an analogue, namely the universal representation $\omega: L^1(G) \to C^*(G)$, cf. [3, §2.7, §13.9], [20].

The fundamental theory of $B(G)$ and $A(G)$ for general locally compact G has been worked out in [6], [20], [21]. We must assume the reader will familiarize himself/herself with those parts of these works which are necessary to this paper. Actually a thorough reading of §13 of [3] and a mastery of the definition of $B(G)$ from [6] will be sufficient nontraditional analysis background for much of the present paper. We will need to use $W^*(G)$, the universal enveloping von Neumann algebra of $C^*(G)$, cf., [20,

§12], [3]. Recall that $C^*(G)' \cong B(G)$ and $B(G)' \cong W^*(G)$ as Banach spaces.

Finally, if A is a C*-algebra, and M_n is the C*-algebra of $n \times n$ matrices with complex entries, then $A \otimes M_n$ is in a unique way a C*-algebra, cf. [1], [18, p. 192]. We may think of $A \otimes M_n$ as the C*-algebra of $n \times n$ matrices with entries from A and we have

$$\max_{1 \leqq i,j \leqq n} \|a_{ij}\|_A \leqq \|(a_{ij})\|_{A \otimes M_n} \leqq \sum_{i,j=1}^{n} \|a_{ij}\|_A \tag{1}$$

where (a_{ij}) is an $n \times n$ matrix and $a_{ij} \in A$ for $i,j = 1,2,\ldots,n$, cf. [18, p. 192].

A linear map $\varphi: A \to A$ is *completely positive* if

$$\varphi \otimes I_n : A \otimes M_n \to A \otimes M_n$$

is positive for $n = 1,2,\ldots,$ that is

$$\varphi \otimes I_n((a_{ij})) = (\varphi(a_{ij}))$$

is positive for $n = 1,2,\ldots$ whenever (a_{ij}) is a positive element in $A \otimes M_n$.

Definition 1. We will denote by $\mathscr{P}(A)$ the collection of all completely positive maps of a C*-algebra A into itself.

If V is a topological vector space let $\mathscr{L}(V)$ be the collection of all continuous linear maps of V into itself. We say that a linear map $\varphi: A \to A$ is a *completely bounded* map of C*-algebra A into itself if $\sup_n \|\varphi \otimes I_n\|_{\mathscr{L}(A \otimes M_n)}$ is finite.

Definition 2. We denote by $\mathscr{D}(A)$ the collection of all completely bounded maps of a C*-algebra A into itself. We set $\|\varphi\|_{\mathscr{D}} = \sup_n \|\varphi \otimes I_n\|$. We will call $\mathscr{D}(A)$ the *dual algebra* of C*-algebra A.

2. Duality for groups.

We start this section by proving that the dual algebra of a C*-algebra is indeed a Banach algebra with an involution of sorts which we call here a conjugation. This conjugation is described in the following definition.

Definition 3. If A is an algebra (possibly not commutative) over $\mathbf{C}$, a conjugation $^-$, is a map $a \in A \mapsto \bar{a} \in A$ with the properties

(1) $(\bar{a})^- = a$

(2) $(a+b)^- = \bar{a} + \bar{b}$

(3) $(\lambda a)^- = \bar{\lambda}\bar{a}$

(4) $(ab)^- = \bar{a}\bar{b}$

for $a,b \in A$, $\lambda \in \mathbf{C}$, and $\bar{\lambda}$ denotes the usual complex conjugate of λ.

PROPOSITION 1. *Let A be a* C*-*algebra. The dual algebra*, $\mathscr{D}(A)$, *with the* $\|\cdot\|_{\mathscr{D}}$-*norm is a Banach algebra with an isometric conjugation.*

PROOF. The product in $\mathscr{D}(A)$ is composition, denoted $\varphi_1 \circ \varphi_2$ if $\varphi_1, \varphi_2 \in \mathscr{D}(A)$. We have

$$\begin{aligned}\|\varphi_1 \circ \varphi_2\|_{\mathscr{D}} &= \sup_n \|\varphi_1 \circ \varphi_2 \otimes I_n\| = \sup_n \|(\varphi_1 \otimes I_n) \circ (\varphi_2 \otimes I_n)\| \\ &\leqq \|\varphi_1\|_{\mathscr{D}} \|\varphi_2\|_{\mathscr{D}}.\end{aligned}$$

Also

$$\begin{aligned}\|\varphi_1 + \varphi_2\|_{\mathscr{D}} &= \sup_n \|(\varphi_1 + \varphi_2) \otimes I_n\| = \sup_n \|\varphi_1 \otimes I_n + \varphi_2 \otimes I_n\| \\ &\leqq \|\varphi_1\|_{\mathscr{D}} + \|\varphi_2\|_{\mathscr{D}}.\end{aligned}$$

Thus $\mathscr{D}(A)$ is closed under addition and multiplication, viz. composition, and the triangle inequalities hold. Moreover $\mathscr{D}(A) \subset \mathscr{L}(A)$, thus except possibly for completeness $\mathscr{D}(A)$ is a Banach algebra.

We now show that $\mathscr{D}(A)$ is complete in the $\|\cdot\|_{\mathscr{D}}$-norm. Let $\{\varphi_k\}$ be a Cauchy sequence in $\mathscr{D}(A)$, i.e., given $\varepsilon > 0$,

$$\sup_n \|\varphi_k \otimes I_n - \varphi_l \otimes I_n\| < \varepsilon$$

if k and l are sufficiently large. In the case $n = 1$, we get that $\{\varphi_k\}$ is Cauchy in $\mathscr{L}(A)$. By the completeness of $\mathscr{L}(A)$ there exists $\alpha_1 \in \mathscr{L}(A)$ such that $\lim_k \varphi_k = \alpha_1$ in $\mathscr{L}(A)$. Similarly $\{\varphi_k \otimes I_n\}$ is Cauchy in $\mathscr{L}(A \otimes M_n)$ for each $n = 1,2,3,\ldots$. Thus there exists an $\alpha_n \in \mathscr{L}(A \otimes M_n)$ for each n such that $\alpha_n = \lim_k \varphi_k \otimes I_n$ in $\mathscr{L}(A \otimes M_n)$. In particular, if $(a_{ij}) \in A \otimes M_n$

$$\|(\varphi_k(a_{ij})) - (\alpha_n((a_{ij})))\|_{A \otimes M_n} \leqq \varepsilon \|(a_{ij})\|_{A \otimes M_n}$$

for sufficiently large k. By inequality (1) above we get that

$$\max_{1 \leqq i_0, j_0 \leqq n} \|\varphi_k(a_{i_0 j_0}) - (\alpha_n((a_{ij})))_{i_0 j_0}\|_A \leqq \varepsilon \|(a_{ij})\|_{A \otimes M_n}.$$

Thus $\alpha_1(a_{i_0 j_0}) = (\alpha_n((a_{ij})))_{i_0 j_0}$. Thus $\alpha_n = \alpha_1 \otimes I_n$ for $n = 1,2,\ldots$ and

$$\lim_k \varphi_k \otimes I_n = \alpha_1 \otimes I_n$$

uniformly in n. Thus $\lim_k \varphi_k = \alpha_1$ in the $\|\cdot\|_{\mathscr{D}}$ norm.

Finally if $a \in A \mapsto a^* \in A$ is the involution in A, we define the conjugation $\varphi \in \mathscr{D}(A) \mapsto \bar{\varphi} \in \mathscr{D}(A)$ as follows:

$$\bar{\varphi}(a) = (\varphi(a^*))^*.$$

We see that $\|\bar{\varphi}\|_{\mathscr{D}} = \|\varphi\|_{\mathscr{D}}$, and we are done with the proof of Proposition 1.

We note that $\mathscr{P}(A)$ is a semigroup, since the composition of two completely positive maps is completely positive. Thus $\langle\mathscr{P}(A)\rangle$, the finite linear combinations of $\mathscr{P}(A)$ is a subalgebra of $\mathscr{D}(A)$, since $\mathscr{P}(A) \subset \mathscr{D}(A)$. Define $\mathscr{B}(A)$ to be the closure of $\langle\mathscr{P}(A)\rangle$ in $\mathscr{D}(A)$. We observe that $\mathscr{B}(A)$ is the analogue of a Fourier–Stieltjes algebra of A. We have

$$\mathscr{B}(A) \subset \mathscr{D}(A) \subset \mathscr{L}(A).$$

Our first main result is that there is a copy of $B(G)$ contained in $\langle\mathscr{P}(\mathrm{C}^*(G))\rangle \subset \mathscr{B}(\mathrm{C}^*(G))$. Before proving this we need to establish some preliminary notation and facts.

First, if $\varphi: \mathrm{C}^*(G) \to \mathrm{C}^*(G)$ and $\varphi \in \mathscr{L}(\mathrm{C}^*(G))$, we can "lift" φ to $B(G) \cong \mathrm{C}^*(G)'$, and then to $\mathrm{W}^*(G) \cong \mathrm{C}^*(G)''$ by taking transposes, i.e.

$$^{t}\varphi: B(G) \to B(G) \quad \text{and}$$

$$^{tt}\varphi: \mathrm{W}^*(G) \to \mathrm{W}^*(G)$$

where

$$\langle {}^{t}\varphi(b), a\rangle = \langle b, \varphi(a)\rangle \quad \text{for } b \in B(G) \text{ and } a \in \mathrm{C}^*(G).$$

Similarly

$$\langle {}^{tt}\varphi(x), b\rangle = \langle x, {}^{t}\varphi(b)\rangle \quad \text{for } x \in \mathrm{W}^*(G),\ b \in B(G).$$

We wish to have a similar apparatus available for $\mathrm{C}^*(G) \otimes M_n$, $B(G) \otimes M_n$, $\mathrm{W}^*(G) \otimes M_n$. To this end think of $\mathrm{C}^*(G)$ as concretely represented on H_ω, its universal representation Hilbert space. Then in a natural way, cf. [18, p. 192],

$$\mathrm{C}^*(G) \otimes M_n \subset \mathscr{L}(H_\omega \otimes H_n),$$

where H_n is an n-dimensional complex Hilbert space. Since $(\mathrm{C}^*(G) \otimes M_n)'$ algebraically identifies with $B(G) \otimes M_n$ via

$$\langle a, f\rangle = \sum_{i,j=1}^{n} \langle a_{ij}, f_{ij}\rangle$$

where $a = (a_{ij}) \in \mathrm{C}^*(G) \otimes M_n$, $f \in (\mathrm{C}^*(G) \otimes M_n)'$, $(f_{ij}) \in B(G) \otimes M_n$ (and hence we identify $B(G) \otimes M_n$ and $(\mathrm{C}^*(G) \otimes M_n)'$ as Banach spaces), by [3, 12.1.1], we can identify $(\mathrm{C}^*(G) \otimes M_n)''$ with the weak closure of $\mathrm{C}^*(G) \otimes M_n$, which is none other than $\mathrm{W}^*(G) \otimes M_n$, viewed in this case concretely as a subalgebra of $\mathscr{L}(H_\omega \otimes H_n)$.

Thus for each $\varphi \otimes I_n \in \mathscr{L}(C^*(G) \otimes M_n)$ we have

$$\begin{aligned} \varphi \otimes I_n \quad & : C^*(G) \otimes M_n \to C^*(G) \otimes M_n \\ {}^t\varphi \otimes I_n = {}^t(\varphi \otimes I_n) & : B(G) \otimes M_n \to B(G) \otimes M_n \\ {}^{tt}\varphi \otimes I_n = {}^{tt}(\varphi \otimes I_n) & : W^*(G) \otimes M_n \to W^*(G) \otimes M_n \end{aligned}$$

cf. [18, p. 200], and one of the above maps is completely bounded, respectively, completely positive, if and only if all three maps are. Note these notions make sense for duals of C*-algebras, cf. [18, p. 200].

We are concerned now primarily with the "generalized translation" operators $T_b \otimes I_n$ (introduced and studied in [24]) which are defined as follows.

DEFINITION 4. If $b \in B(G)$, $T_b \in \mathscr{L}(C^*(G))$ is determined by $\langle T_b a,d\rangle = \langle a,bd\rangle$ for all $a \in C^*(G)$, $d \in B(G)$.

REMARK. The fact that $T_b \in \mathscr{L}(C^*(G))$, indeed, $T_b \in \mathscr{B}(C^*(G)) \subset \mathscr{D}(C^*(G))$ follows from the fact that

$$T_b = \sum_{i=1}^{4} \lambda_i T_p$$

where

$$b = \sum_{i=1}^{4} \lambda_i p_i,$$

p_i positive definite for $i = 1,2,3,4$, and $T_{p_i} \in \mathscr{P}(C^*(G))$, $i = 1,2,3,4$, cf. [24, p. 502].

We are now ready to state our first main result:

THEOREM 1. *If $B(G)$ is the Fourier–Stieltjes algebra of locally compact group G, then the map $b \in B(G) \mapsto T_b \in \mathscr{B}(C^*(G)) \subset \mathscr{D}(C^*(G))$ is an isometric Banach algebra isomorphism of $B(G)$ onto a maximal abelian subalgebra B of $\mathscr{D}(C^*(G))$. The algebra B is in fact maximal abelian in $\mathscr{L}(C^*(G))$, and*

$$\| b \|_{B(G)} = \| T_b \|_{\mathscr{D}(C^*(G))} = \| T_b \|_{\mathscr{L}(C^*(G))} \text{ for } b \in B(G).$$

PROOF. The facts that

$$T_{b_1 b_2} = T_{b_1} T_{b_2} \quad \text{and} \quad T_{\lambda_1 b_1 + \lambda_2 b_2} = \lambda_1 T_{b_1} + \lambda_2 T_{b_2}$$

for $b_1, b_2 \in B(G)$, and complex numbers λ_1, λ_2 are immediate. Their verification is left to the reader.

We will show first that $b \in B(G) \mapsto T_b \in \mathscr{L}(C^*(G))$ is an isometry, that is $\| b \|_{B(G)} = \| T_b \|_{\mathscr{L}(C^*(G))}$. First

$$\begin{aligned} \| T_b \|_{\mathscr{L}(C^*(G))} &= \sup \{ \| T_b x \|_{C^*(G)} : \| x \|_{C^*(G)} \leqq 1 \} \\ &= \sup \{ | \langle T_b x, d \rangle | : \| x \|_{C^*(G)} \leqq 1, \ \| d \|_{B(G)} \leqq 1 \} \\ &\geqq \sup \{ | \langle T_b x, 1 \rangle | : \| x \|_{C^*(G)} \leqq 1 \} \\ &= \sup \{ | \langle x, b \rangle | : \| x \|_{C^*(G)} \leqq 1 \} = \| b \|_{B(G)}. \end{aligned}$$

Second

$$\begin{aligned} \| T_b x \|_{C^*(G)} &= \sup \{ | \langle x, bd \rangle | : \| d \|_{B(G)} \leqq 1 \} \\ &\leqq \sup \{ \| x \|_{C^*(G)} \| bd \|_{B(G)} : \| d \|_{B(G)} \leqq 1 \} \\ &\leqq \| x \|_{C^*(G)} \| b \|_{B(G)}. \end{aligned}$$

Hence

$$\| T_b \|_{\mathscr{L}(C^*(G)} \leqq \| b \|_{B(G)},$$

and thus finally

$$\| b \|_{B(G)} = \| T_b \|_{\mathscr{L}(C^*(G))}.$$

This also shows that

$$\| T_b \|_{\mathscr{D}} = \sup_n \| T_b \otimes I_n \|_{\mathscr{L}(C^*(G) \otimes M_n)} \geqq \| T_b \|_{\mathscr{L}(C^*(G))} = \| b \|_{B(G)}$$

for all $b \in B(G)$. Thus we have left to demonstrate that $\| T_b \|_{\mathscr{D}} \leqq \| b \|_{B(G)}$ for $b \in B(G)$. We show that

$$\| T_b \otimes I_n \|_{\mathscr{L}(C^*(G) \otimes M_n)} \leqq \| b \|_{B(G)} \quad \text{for } b \in B(G),$$

and we accomplish this by analyzing the structure of the map T_b in much the same fashion that completely positive generalized translations were analyzed in [24].

Given nonzero $b \in B(G)$ without loss of generality assume $\| b \| = 1$. Let $b = v \, . \, p$ be the polar decomposition of b with $p \in P(G)$, $v \in W^*(G)$, $\| b \| = \| p \| = 1$. Let $p(\cdot) = (\pi_p(\cdot) \xi_p | \xi_p)$. where π_p is the G.N.S. representation of G induced by p on Hilbert space H_{π_p}. View $W^*(G)$ as concretely represented via the universal representation ω on H_ω. Then for $f \in L^1(G)$ we have, for $\xi, n \in H_\omega$,

$$\begin{aligned}(T_b\omega(f)\xi|\eta) &= (\omega(bf)\xi|n)\\ &= \int_{x\in G}(\omega(x)\xi|\eta)(\pi_p(x)\pi_p(v)\xi_p|\xi_p)f(x)d\lambda(x)\\ &= \int_{x\in G}((\omega\otimes\pi_p)(x)\xi\otimes\pi_p(v)\xi_p|\eta\otimes\xi_p)f(x)d\lambda(x)\\ &= ((\omega\otimes\pi_p)(f)\xi\otimes\pi_p(v)\xi_p|\eta\otimes\xi_p)\\ &= (Q(\omega\otimes\pi_p)(f)(I\otimes\pi_p(v))Q^*\xi|\eta).\end{aligned}$$

Thus

$$T_b\omega(f) = Q(\omega\otimes\pi_p)(f)(I\otimes\pi_p(v))Q^*,$$

where Q is the projection of

$$H_\omega\otimes H_{\pi_p}\to H_\omega\otimes \mathbf{C}\xi_p\cong H_\omega$$

followed by the identification

$$x\otimes\xi_p\in H_\omega\otimes\mathbf{C}\xi_p\mapsto x\in H_\omega.$$

Note $\mathbf{C}$ denotes the complex numbers, $\mathbf{C}\xi_p$, the 1-dimensional span of ξ_p. Also by continuity f can be replaced by any x in $C^*(G)$ in the above equation.

Thus we have the following where $(\omega(x_{ij}))\in\omega(C^*(G))\otimes M_n$ (often written $C^*(G)\otimes M_n$ for simplicity)

$$T_b\otimes I_n((\omega(x_{ij}))) = (T_b\omega(x_{ij}))$$

$$= \begin{bmatrix} Q & 0 & \cdots & \cdots & 0\\ 0 & Q & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \cdots & Q\end{bmatrix} ((\omega\otimes\pi_p)(x_{ij}))$$

$$\begin{bmatrix} I\otimes\pi_p(v) & 0 & \cdots & 0\\ 0 & I\otimes\pi_p(v) & & \vdots\\ \vdots & & \ddots & \vdots\\ 0 & \cdots & \cdots & I\otimes\pi_p(v)\end{bmatrix} \begin{bmatrix} Q^* & 0 & \cdots & 0\\ 0 & Q^* & 0 & \cdots\\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & Q^*\end{bmatrix}$$

$$= T_p \otimes I_n \left[\big((\omega \otimes \pi_p)(x_{ij})\big) \begin{bmatrix} I \otimes \pi_p(v) & 0 & \cdots & \cdots & 0 \\ 0 & I \otimes \pi_p(v) & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & I \otimes \pi_p(v) \end{bmatrix} \right]$$

Now consider what happens to the norm of the matrix

$$\big(\omega(x_{ij})\big) \in C^*(G) \otimes M_n \subset \mathscr{L}(H_\omega) \otimes \mathscr{L}(H_n),$$

when it is acted on by $T_b \otimes I_n$. Viewing the action of $T_b \otimes I_n$ as a composition (as above), we see that first

$$\begin{aligned} \omega \otimes \pi_p \otimes I_n \colon \big(\omega(x_{ij})\big) &\in \mathscr{L}(H_\omega) \otimes \mathscr{L}(H_n) \\ &\mapsto \big(\omega \otimes \pi_p(x_{ij})\big) \in \mathscr{L}(H_\omega) \otimes \mathscr{L}(H_{\pi_p}) \otimes \mathscr{L}(H_n) \end{aligned}$$

where $\omega \otimes \pi_p \otimes I_n$ is a *-representation of a C*-algebra into a C*-algebra, hence

$$\|(\omega \otimes \pi_p)(x_{ij})\| \leqq \|(x_{ij})\|.$$

Now

$$\left\| \big(\omega \otimes \pi_p(x_{ij})\big) \begin{bmatrix} I \otimes \pi_p(v) & 0 & \cdots & 0 \\ & & & \vdots \\ 0 & I \otimes \pi_p(v) & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & & I \otimes \pi_p(v) \end{bmatrix} \right\|$$

$$\begin{aligned} &\leqq \|(x_{ij})\| \; \|I \otimes \pi_p(v)\| \\ &= \|(x_{ij})\| \; \|I\|_{\mathscr{L}(H_\omega)} \|\pi_p(v)\|_{\mathscr{L}(H_{\pi_p})} \\ &= \|(x_{ij})\|, \end{aligned}$$

since $\pi_p(v)$ is a partial isometry. Finally, conjugating by Q and Q^* is the same as applying the positive operator $T_p \otimes I_n$, which takes its norm at

$$\begin{bmatrix} e & 0 & \cdots & \cdots & 0 \\ 0 & e & 0 & \cdots & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & e \end{bmatrix}$$

that is,

$$\| T_p \otimes I_n \| = \| p \|_{B(G)} = \| b \|_{B(G)} = 1.$$

Thus we have for each $b \in B(G)$, that

$$\| T_b \otimes I_n \|_{\mathscr{L}(C^*(G) \otimes M_n)} \leqq \| b \|_{B(G)}.$$

By the first part of our proof we see that

$$\| T_b \otimes I_n \|_{\mathscr{L}(C^*(G) \otimes M_n)} = \| b \|_{B(G)}$$

for all $b \in B(G)$ and all n, hence $\| T_b \|_{\mathscr{D}} = \| b \|_{B(G)}$.

We now establish that

$$B = \{T_b \in \mathscr{D}(C^*(G)) : b \in B(G)\}$$

is maximal abelian in $\mathscr{D}(C^*(G))$, in fact, maximal abelian in $\mathscr{L}(C^*(G))$. For technical reasons we wish to work not with $T_b \otimes I_n \in \mathscr{L}(C^*(G) \otimes M_n)$ but with the double transpose

$${}^{tt}(T_b \otimes I_n) = {}^{tt}T_b \otimes I_n \in \mathscr{L}_\sigma(W^*(G) \otimes M_n)$$

where the subscript σ indicates the σ-weakly continuous linear operators. Note that

$$\| {}^{tt}T_b \otimes I_n \|_{\mathscr{L}_\sigma(W^*(G) \otimes M_n)} = \| T_b \otimes I_n \|_{\mathscr{L}(C^*(G) \otimes I_n)}$$

since the unit ball of $C^*(G) \otimes M_n$ is σ-weakly dense in that of $W^*(G) \otimes M_n$. Thus the map

$$b \in B(G) \mapsto {}^{tt}T_b \in \mathscr{D}_\sigma(W^*(G))$$

is an isometric isomorphism of $B(G)$ into the σ-weakly continuous, completely bounded maps of $W^*(G)$ into itself.

Now let $\Phi \in \mathscr{L}(C^*(G))$, that is ${}^{tt}\Phi \in \mathscr{L}_\sigma(W^*(G))$. Suppose $T_b\Phi = \Phi T_b$ for all $b \in B(G)$, that is ${}^{tt}T_b{}^{tt}\Phi = {}^{tt}\Phi{}^{tt}T_b$. From now on in this proof we will drop the double pre-superscript ^{tt}, thus abusing notation and identifying an operation with its double transpose. For $g \in G \subset W^*(G)$ we have

$$T_b(\Phi(g)) = T_b\Phi(g) = \Phi(T_b g) = b(g)\Phi(g).$$

Thus $\Phi(g)$ is an eigenvector of T_b for all $b \in B(G)$, with eigenvalue $b(g)$.

At this point we insert a lemma in the proof of Theorem 1. Recall that $\sigma(B(G)) \subset W^*(G)$ is the set of non zero multiplicative linear functionals of $B(G)$, i.e., the spectrum of commutative Banach algebra $B(G)$, cf. [20].

LEMMA. *For* $x \in W^*(G)$, $x \neq 0$, *we have* $T_b x = \langle b,x\rangle x$ *for all* $b \in B(G)$ *if and only if* $x \in \sigma(B(G))$.

PROOF OF LEMMA. For $b,d \in B(G)$

$$\langle bd,x\rangle = \langle d,\langle b,x\rangle x\rangle = \langle b,x\rangle\langle d,x\rangle,$$

if $T_b x = \langle b,x\rangle x$; hence $x \in \sigma(B(G))$. Conversely, if $x \in \sigma(B(G))$, then $\langle b,x\rangle\langle d,x\rangle = \langle d,T_b x\rangle$ and $\langle b,x\rangle\langle d,x\rangle = \langle d,\langle b,x\rangle x\rangle$ for all $d \in B(G)$. Thus $T_b x = \langle b,x\rangle x$ and the lemma is proved.

We now claim that for each g, $\Phi(g)$ is a scalar multiple of an element in the spectrum of $B(G)$. First we observe that $\langle 1,\Phi(g)\rangle = 0$ or $\langle 1,\Phi(g)\rangle \neq 0$. In the latter case we have that

$$T_b\left(\frac{\Phi(g)}{\langle 1,\Phi(g)\rangle}\right) = \left\langle b,\frac{\Phi(g)}{\langle 1,\Phi(g)\rangle}\right\rangle\frac{\Phi(g)}{\langle 1,\Phi(g)\rangle},$$

hence by the lemma $\dfrac{\Phi(g)}{\langle 1,\Phi(g)\rangle} \in \sigma(B(G))$. In the former case,

$$\langle b,\Phi(g)\rangle = \langle 1,T_b\Phi(g)\rangle = \langle 1,b(g)\Phi(g)\rangle = b(g)\langle 1,\Phi(g)\rangle = 0$$

for all $b \in B(G)$. Thus $\Phi(g) = 0$.

We now claim that $\Phi(g) = \Phi_g g$, where $\Phi_g \in \mathbf{C}$ for each $g \in G$. Suppose not, then $\Phi(g) \neq 0$, hence $\langle 1,\Phi(g)\rangle \neq 0$, hence $\Phi(g)/\langle 1,\Phi(g)\rangle \in \sigma(B(G))$. Now recall that any $x \in \sigma(B(G)) \setminus G$ satisfies $\langle x,a\rangle = 0$ for all $a \in A(G)$, cf. [20]. Thus if $\Phi(g) \notin \mathbf{C}g$, there exists an $a \in A(G)$ such that $a(g) = 1$, and $\langle a,\Phi(g)\rangle = 0$. Thus $\langle 1,T_a\Phi(g)\rangle = \langle a,\Phi(g)\rangle = 0$. Hence

$$0 = \langle 1,a(g)\Phi(g)\rangle = a(g)\langle 1,\Phi(g)\rangle = \langle 1,\Phi(g)\rangle$$

a contradiction. Thus $\Phi(g) = \Phi_g g$ for all $g,\Phi_g \in \mathbf{C}$. In fact, $\Phi(g) = \langle 1,\Phi(g)\rangle g$ for all g. But recall that $\Phi \in \mathscr{L}_\sigma(W^*(G))$, that is, Φ is σ-weakly continuous, hence $\Phi_g = \langle 1,\Phi(g)\rangle$ is a continuous function on G.

We claim that $\Phi = T_{\langle 1,\Phi(\cdot)\rangle}$ and that $\langle 1,\Phi(\cdot)\rangle \in B(G)$. First

$$\Phi\left(\sum_{i=1}^{n}\lambda_i g_i\right) = \sum_{i=1}^{n}\lambda_i\langle 1,\Phi(g_i)\rangle g_i$$

for $g_1,\dots,g_n \in G$, $\lambda_1,\dots,\lambda_n \in \mathbf{C}$. Thus

$$\begin{aligned}\left|\sum_{i=1}^{n} \lambda_i \langle 1, \Phi(g_i)\rangle\right| &= \left|\left\langle 1, \sum_{i=1}^{n} \lambda_i \langle 1, \Phi(g_i)\rangle g_i\right\rangle\right| \\ &= \left|\left\langle 1, \Phi\left(\sum_{i=1}^{n} \lambda_i g_i\right)\right\rangle\right| \\ &\leqq \left\|\sum_{i=1}^{n} \lambda_i g_i\right\|_{W^*(G)} \|\Phi\|_{\mathscr{L}(C^*(G))}.\end{aligned}$$

By [6, 2.24], $\langle 1, \Phi(\cdot)\rangle \in B(G)$. Since Φ and $T_{\langle 1,\Phi(\cdot)\rangle}$ agree on $G \subset W^*(G)$, and Φ and $T_{\langle 1,\Phi(\cdot)\rangle}$ are both σ-weakly continuous

$$\Phi(x) = T_{\langle 1,\Phi(\cdot)\rangle} x \quad \text{for all } x \in W^*(G).$$

This is because linear combinations of G are σ-weakly dense in $W^*(G)$. The proof of Theorem 1 is thus done.

Remark. We were able to compute an explicit upper bound on the completely bounded norm of T_b by exhibiting the formula

$$T_b\omega(f) = Q(\omega \otimes \pi_p)(f)(I \otimes \pi_p(v))Q^*.$$

If $T: A_1 \to A_2$ is a linear map of C*-algebra A_1 into C*-algebra $A_2 \subset \mathscr{L}(H)$, H a Hilbert space, we shall define a *polar decomposition of* T to be a formula of the form $T(\cdot) = Q\pi(\cdot)R^*$, where π is a *-representation of A_1 into $\mathscr{L}(K)$, K a Hilbert space, and $Q: K \to H$, $R: K \to H$ are bounded linear operators. Given such a polar decomposition, $\|T\|_{\mathscr{D}} \leqq \|Q\| \|R\|$. We know of no general theory of polar decompositions in the literature which includes the case where A_2 is an arbitrary C*-algebra.

Now we make a simple statement about the conjugation of $\mathscr{D}(C^*(G))$ restricted to B.

Proposition 2. *For all $b \in B(G)$, $(T_b)^- = T_{b^\flat}$, thus B is invariant under the conjugation operation on $\mathscr{D}(C^*(G))$. The map $b \in B(G) \mapsto T_b \in \mathscr{D}(C^*(G))$ "preserves" the conjugation $^\flat$ on $B(G)$.*

Proof. Again working with the double transpose of T_b on $W^*(G)$, we have

$$\begin{aligned}\overline{T}_b g &= (T_b g^{-1})^* = (b(g^{-1})g^{-1})^* = \overline{b(g^{-1})g} \\ &= T_{b^\flat} g \quad \text{for all } g \in G.\end{aligned}$$

We are done.

There is a dual version of Theorem 1 which is easier to prove. First note that we replace the $C^*(G)$, $B(G)$ pair with $C_0(G)$, $M^1(G)$. We denote

$$\|f\|_\infty = \sup_{x\in G} |f(x)|, \quad \text{for } f \in C_0(G).$$

Then $M^1(G)$ acts on $C_0(G)$ as "generalized translation" operators. In this case we have

$$\mu \in M^1(G) \mapsto T_\mu f \in C_0(G), \quad \text{for } f \in C_0(G),$$

where $T_\mu f = \mu * f$, i.e. convolution by μ on the left. There is an optional action,

$$\mu \in M^1(G) \mapsto T_\mu^0 f,$$

where $T_\mu^0 f = f * \mu$, convolution by μ on the right.

THEOREM 2. *If $M^1(G)$ is the Banach convolution algebra of (bounded) regular complex Borel measures on G, then the map*

$$\mu \in M^1(G) \mapsto T_\mu \in \mathscr{B}(C_0(G)) \subset \mathscr{D}(C_0(G))$$

is an isometric Banach algebra isomorphism of $M^1(G)$ onto a subalgebra M^1 of $\mathscr{D}(C_0(G))$.

PROOF. We first observe that $C_0(G)$ is a commutative C^* algebra in which the notion of positivity is simply pointwise positivity of functions. Also

$$T_\mu = \sum_{i=1}^{4} \lambda_i T_\mu$$

where

$$\mu = \sum_{i=1}^{4} \lambda_i \mu_i, \quad \mu_i \geqq 0, \quad i = 1,2,3,4.$$

Now if $\mu_i \geqq 0$ and $f \geqq 0$, $f \in C_0(G)$, we see that

$$T_{\mu_i} f(x) = \int_{yG} f(y^{-1}x) d_{\mu_i}(y) \geqq 0.$$

Note that $T_{\mu_i} f \in C_0(G)$ as can be verified easily by the reader. Thus T_{μ_i} is a positive, hence, completely positive operator on the commutative C^*-algebra $C_0(G)$, cf. [18, p. 199]. It also follows immediately from the convolution formula above that

$$\|T_\mu\|_{\mathscr{L}(C_0(G))} \leqq \|\mu\|_{M^1(G)}.$$

The equations

$$T_{\mu_1 * \mu_2} = T_{\mu_1} T_{\mu_2} \quad \text{and} \quad T_{\lambda_1\mu_1 + \lambda_2\mu_2} = \lambda_1 T_\mu + \lambda_2 T_\mu$$

for $\mu_1, \mu_2 \in M^1(G)$ and $\lambda_1, \lambda_2 \in C$ are clear.

We now show that

$$\| T_\mu \|_{\mathscr{L}(C_0(G))} = \| \mu \|_{M^1(G)}.$$

From the polar decomposition for a measure, [15, p. 126], there is a Borel measurable function h on G such that $hd|\mu| = d\mu$, and $|h(x)| = 1$ for all $x \in G$. Thus

$$\int_G \bar{h}(x) d\mu(x) = \| \mu \|_{M^1(G)} \quad \text{and} \quad \| h \|_\infty = 1.$$

By Lusin's Theorem [15, p. 53], there exists a net $\{k_\alpha\} \subseteqq C_0(G)$, $\| k_\alpha \|_\infty \leqq \| h \|_\infty = 1$, such that

$$\lim_\alpha \int k_\alpha(x) d\mu(x) = \| \mu \|_{M^1(G)}.$$

Hence

$$T_\mu k_\alpha^\vee(e) = \int_G k_\alpha^\vee(y^{-1}e) d\mu(y) = \int_G k_\alpha(y) d\mu(y).$$

Thus $\| T_\mu \|_{\mathscr{L}(C_0(G))} = \| \mu \|_{M^1(G)}$. Note that we used $k^\vee(y) = k(y^{-1})$.

We have left to show that $\| T_\mu \|_{\mathscr{D}} = \| \mu \|_{M^1(G)}$. We can show this by using the technique employed in Theorem 1, i.e., by computing an appropriate polar decomposition formula. However, as Jun Tomiyama pointed out to us, it follows from [11, Lemma 1], that if A is a *commutative* C*-algebra, and $T \in \mathscr{L}(A)$, then

$$\| T \|_{\mathscr{L}(C_0(G))} = \| T \otimes I_n \|_{\mathscr{L}(C_0(G) \otimes M_n)}$$

for $n = 1,2,\ldots$. Thus Theorem 2 is proved.

We now verify that M^1 is invariant under the conjugation in $\mathscr{D}(C_0(G))$. Note that $\langle f, \bar{\mu} \rangle = \langle \bar{f}, \mu \rangle^-$ for $f \in C_0(G)$.

PROPOSITION 3. *For all $\mu \in M^1(G)$, $(T_\mu)^- = T_{\bar{\mu}}$, thus M^1 is invariant under the conjugation operation in $\mathscr{D}(C_0(G))$. The map $\mu \in M^1(G) \mapsto T_\mu \in \mathscr{D}(C_0(G))$ preserves the conjugation $^-$ on $M^1(G)$.*

PROOF. We have

$$\overline{T_\mu}(f) = \overline{T_\mu(\overline{f})} = \overline{\mu * \overline{f}} = \bar{\mu} * f = T_{\bar{\mu}} f$$

for $f \in C_0(G)$. We are done.

REMARK. A similar investigation of T_μ^0, convolution on the right, may be done.

3. Duality for M_n

Our chief reference for this section is [13], although we will give an almost selfcontained discussion. Roughly a groupoid is much like a group except that group multiplication is not necessarily defined for every pair of elements and the unit (or identity) is not necessarily unique. See [13, p. 5] for a more precise definition. In this paper we will concern ourselves only with the principal transitive groupoid G on n-elements, where n is finite in most of this section. As a set we label

$$G = \{e_{ij}: 1 \leqq i,\ j \leqq n\}.$$

The units of G, labelled G^0, are $\{e_{ii}: 1 \leqq i \leqq n\}$. Multiplication in G is defined by $e_{ij} e_{kl} = e_{il}$ if and only if $j = k$ for $1 \leqq i,j,k,l \leqq n$, otherwise we say the product is undefined. Also inverses are given by $e_{ij}^{-1} = e_{ji}$, $1 \leqq i$, $j \leqq n$. For G there are two maps, the range map r and domain map d, of G into G^0. We define these maps as follows:

$$d(e_{ij}) = e_{ji} e_{ij} = e_{jj}, \quad r(e_{ij}) = e_{ij} e_{ji} = e_{ii}.$$

For G there is a left-Haar system, [13, p. 16], which in this case is simply a discrete measure, i.e., $\lambda\{e_{ij}\} = 1$, $1 \leqq i,j \leqq n$. The continuous functions with compact support on G, which we denote simply as $C(G)$ in this case, form an algebra with product taken to be convolution with respect to the discrete left-Haar system λ, [13, p. 48]. Note first that if $f \in C(G)$, then f_{ij} is the value of f at e_{ij}, that is $f(e_{ij})$, $1 \leqq i,j \leqq n$. In this case it is easy to see that $f \in C(G) \longleftrightarrow (f_{ij}) \in M_n$ is a one-one correspondence between $C(G)$ and the $n \times n$ matrices with complex entries, M_n. With this correspondence in mind, the formula for convolution of $f,g \in C(G)$ is

$$\begin{aligned} f * g(e_{ik}) &= \int_j f(e_{ik} e_{kj}) g(e_{jk}) d\lambda^{e_{kk}}(e_{kj}) \\ &= \sum_j f(e_{ij}) g(e_{jk}) \\ &= \sum_j f_{ij} g_{jk}. \end{aligned}$$

Note that in the first equality $d\lambda^{e_{kk}}$ is the measure in the Haar system with support $r^{-1}(e_{kk})$. Also we note that convolution of $f,g \in C(G)$ corresponds exactly to the matrix product of f and g, when f and g are viewed as matrices. Thus $C(G)$ with convolution will be written $(M_n, *)$; that is $n \times n$ complex matrices with the usual matrix product. We will write the matrix product of $x,y \in M^n$ as xy, rarely as $x * y$.

There is another product in $C(G)$, namely, the pointwise product

$$(fg)(e_{ij}) = f(e_{ij})g(e_{ij}).$$

In the matrix context we have $(f_{ij}) \circ (g_{ij}) = (f_{ij}g_{ij})$, where $\circ$ has been called the Schur product in the literature. We will write this algebra as $(M_n, \circ)$.

There are several norms of interest on M_n. Let us define some of these norms now.

DEFINITION 5. If $(x_{ij}) \in M_n$, then

(i) $$\|(x_{ij})\|_{I,r} = \sup_i \sum_j |x_{ij}|$$

(ii) $$\|(x_{ij})\|_{I,d} = \sup_j \sum_i |x_{ij}|$$

(iii) $$\|(x_{ij})\|_I = \max\{\|(x_{ij})\|_{I,r}, \|(x_{ij})\|_{I,d}\}$$

(iv) $$\|(x_{ij})\|_{L^1} = \sum_{i,j} |x_{ij}|$$

(v) $$\|(x_{ij})\|_{C^*(G)} = \sup\{|((x_{ij})\xi|\eta)| : \xi,\eta \in \mathbf{C}^n, \|\xi\|_{\mathbf{C}^n} \leqq 1, \|\eta\|_{\mathbf{C}^n} \leqq 1\}.$$

(vi) $$\|(x_{ij})\|_{C_0(G)} = \|(x_{ij})\|_\infty = \sup_{i,j} |x_{ij}|$$

(vii) $$\|(x_{ij})\|_{\mathrm{Tr}} = \mathrm{Tr}(|(x_{ij})|) = \mathrm{Tr}[(x_{ij})^*(x_{ij})]^{1/2}.$$

REMARK. In (v), $\mathbf{C}^n =$ the n-dimensional Hilbert space of n-tuples of complex numbers, and $\|\xi\|_{\mathbf{C}^n}$ is the Euclidean norm in $\mathbf{C}^n$. Thus $\|(x_{ij})\|_{C^*(G)}$ is the well known operator norm on M_n. Also in (vii), Tr is the trace and $|(x_{ij})|$ is the square root with respect to the matrix product of the matrix product $(x_{ij})^*(x_{ij})$, where $(x_{ij})^* = (\bar{x}_{ji})$.

We will denote by $(M_n, *, \|\cdot\|_{I,r})$ the $n \times n$ matrices with the matrix product, and $\|\cdot\|_{I,r}$ norm; $(M_n, \circ, \|\cdot\|_{\mathrm{Tr}})$ denotes the $n \times n$ matrices with Schur product and $\|\cdot\|_{\mathrm{Tr}}$ norm. Other similar notations will appear.

PROPOSITION 4. *The following are Banach algebras* $(M_n, *, \|\cdot\|_{I,d})$, $(M_n, *, \|\cdot\|_{I,r})$, $(M_n, *, \|\cdot\|_I)$, $(M_n, *, \|\cdot\|_{L^1})$, $(M_n, *, \|\cdot\|_{C^*(G)})$, $(M_n, \circ, \|\cdot\|_\infty)$.

PROOF. For example

$$\begin{aligned}\|(x_{ij})(y_{ij})\|_{I,r} &= \left\|\left(\sum_k x_{ik}y_{kj}\right)\right\|_{I,r}\\ &= \sup_i\left|\sum_j\sum_k x_{ik}y_{kj}\right| \leqq \sup_i \sum_k |x_{ik}|\sum_j |y_{kj}|\\ &\leqq \left(\sup_i \sum_k |x_{ik}|\right)\left(\sup_k \sum_j |y_{kj}|\right)\\ &= \|(x_{ij})\|_{I,r}\,\|(y_{ij})\|_{I,r}.\end{aligned}$$

We leave the rest of the proofs to the reader.

REMARK. $(M_n, *, \|\cdot\|_{C^*(G)})$ is the C*-algebra $C^*(G)$; $(M_n, \circ, \|\cdot\|_\infty)$ is the C*-algebra $C_0(G)$.

PROPOSITION 5. *We have that* $(M_n, \circ, \|\cdot\|_{\mathrm{Tr}})$ *is a Banach algebra.*

In the proof of Proposition 5, we use the following lemma which is probably folklore, but we learned it first from H. Araki.

Let $\mathbf{C}^n$ be as above. Let $e_i = (0,\dots,0,1,0,\dots,0)$ with 1 in the ith position, and let $K = \operatorname{span}\{e_i \otimes e_i : i = 1,\dots,n\}$ in $\mathbf{C}^n \otimes \mathbf{C}^n$.

LEMMA. *Let* $k\colon \mathbf{C}^n \otimes \mathbf{C}^n \to K$, *be the orthogonal projection; the following commutative diagram of maps gives an alternative description of the Schur product*

$$\begin{array}{ccc}(a,b) \in M^n \times M^n & \mapsto & a \circ b \in M^n\\ \downarrow & & \uparrow\\ a \otimes b \in M^n \otimes M^n & & \text{(equality)}\\ \downarrow & & \\ k(a \otimes b)k \in (M^n \otimes M^n)_k & \mapsto & \Phi(k(a \otimes b)k) \in M^n\end{array}$$

where $\Phi(\sum a_{ij}b_{ij}(e_{ij} \otimes e_{ij})) = \sum a_{ij}b_{ij}e_{ij}$, $a_{ij}, b_{ij} \in \mathbf{C}$ *and* e_{ij} *is the element of* M^n *with a* 1 *in the* ith *row,* jth *column, zeros elsewhere. In the above diagram we have* $\Phi(k(a \otimes b)k) = a \circ b$.

PROOF OF LEMMA. Let $a = \sum a_{ij}e_{ij}$, $b = \sum b_{ij}e_{ij}$, then

$$a \otimes b = \sum_{i,j,l,m} a_{ij}b_{lm}e_{ij} \otimes e_{lm}.$$

Reducing by k we get

$$k(a \otimes b)k = \sum_{i,j} a_{ij}b_{ij}e_{ij} \otimes e_{ij}.$$

Furthermore

$$\Phi(k(a \otimes b)k) = \sum_{i,j} a_{ij} b_{ij} e_{ij} = a \circ b$$

PROOF OF PROPOSITION 5. The map: $\sum_{i,j} \lambda_{ij} \mathrm{e}_{ij} \otimes \mathrm{e}_{ij} \to \sum_{i,j} \lambda_{ij} \mathrm{e}_{ij}$ is an isomorphism of $k(M^n \otimes M^n)k$ with M^n. Thus

$$\mathrm{Tr}(|k(a \otimes b)k|) = \mathrm{Tr}(|a \circ b|).$$

Hence

$$\begin{aligned} \|a \circ b\|_{\mathrm{Tr}} &= \mathrm{Tr}(|k(a \otimes b)k|) = \mathrm{Tr}(k(a \otimes b)kv) \\ &= \mathrm{Tr}((\mathrm{a} \otimes \mathrm{b})\mathrm{kvk}) \leqq \|a \otimes b\|_{\mathrm{Tr}} \|kvk\|_{C^*(G)} = \ = \|a\|_{\mathrm{Tr}} \|b\|_{\mathrm{Tr}}. \end{aligned}$$

Note v is a suitable partial isometry arising from the polar decomposition. This ends the proof of Proposition 5.

The above discussion can easily be generalized to the infinite case, i.e., the case where G is the countably infinite, discrete principal transitive groupoid. In fact we have for a separable Hilbert space H:

PROPOSITION 5′. *The collection of trace class operators on H, denoted $A(G)$, forms a commutative Banach algebra with Schur product and norm $\|\cdot\|_{\mathrm{Tr}}$.*

REMARK. Note that $A(G)$ is the $\|\cdot\|_{\mathrm{Tr}}$-norm closure of the finite rank operators. Note also that $A(G)$ does not have an identity.

REMARK. From the Hopf–von Neumann algebra point of view the Schur product is the multiplication determined by the co-multiplication $c: M_n \to M_n \otimes M_n$ which is determined by $c(e_{ij}) = e_{ij} \otimes e_{ij}$, $1 \leqq i,j \leqq n$. This point of view also leads to an alternative proof of Proposition 5, wherein the Schur product is defined using the transpose of c in the obvious manner.

Returning to the finite dimensional case, define the map

$$a \in (M_n, *, \|\cdot\|_{I,r}) \mapsto T_a \in \mathscr{L}(M_n, \circ, \|\cdot\|_{C_0(G)})$$

by $T_a: x \in C_0(G) \mapsto ax \in C_0(G)$. Note ax is the matrix product of $a,x \in M_n$. We thus have

PROPOSITION 6. *The map T_a defined above is completely bounded and $\|T_a\|_{\mathscr{D}} = \|a\|_{I,r}$. Thus*

$$a \in (M_n, *, \|\cdot\|_{I,r}) \mapsto T_a \in \mathscr{D}(M_n, \circ, \|\cdot\|_{C_0(G)})$$

is an isometric isomorphism onto a subalgebra N of $\mathscr{D}(M_n, \circ, \|\cdot\|_{C_0(G)})$.

PROOF. We show first that $\| T_a \|_{\mathscr{L}(C_0(G))} = \| a \|_{I,r}$. Thus

$$\| (a_{ij})(x_{ij}) \|_{C_0(G)} = \sup_{i,j} \left| \sum_k a_{ik} x_{kj} \right| \leqq \left(\sup_i \sum_k |a_{ik}| \right) \left(\sup_{k,j} |x_{k,j}| \right)$$
$$= \| (a_{ij}) \|_{I,r} \, \| (x_{ij}) \|_{C_0(G)}.$$

Hence $\| T_a \|_{\mathscr{L}(C_0(G))} \leqq \| a \|_{I,r}$. For the reverse inequality let $a_{ij} = u_{ij} |a_{ij}|$, $|u_{ij}| = 1$ for $1 \leqq i,j \leqq n$. Then

$$\| (a_{ij})(u_{ij})^* \|_{C_0(G)} = \sup_{i,j} \left| \sum_k |a_{ik}| \, u_{ik} \bar{u}_{jk} \right|$$

(let $= j$)
$$\geqq \sup_i \sum_k |a_{ik}| \, u_{ik} \bar{u}_{ik}$$
$$= \sup_i \sum_k |a_{ik}| = \| a \|_{I,r}.$$

Since $C_0(G)$ is a commutative C*-algebra, as in Theorem 2 we have by lemma 1, in [11], that

$$\| T_a \otimes I_n \|_{\mathscr{L}(C_0(G) \otimes M_n)} = \| T \|_{\mathscr{L}(C_0(G))},$$

$n = 1,2,\ldots$. Thus

$$\| T_a \|_{\mathscr{D}} = \| T \|_{\mathscr{L}(C_0(G))} = \| a \|_{I,r}.$$

Again we could have obtained this result directly by explicitly computing an appropriate polar decomposition. The proof of Proposition 6 is complete.

REMARK. If we let $(M_n, *, \| \cdot \|_{I,r})$ act on $(M_n, \circ, \| \cdot \|_{C_0(G)})$ on the right, i.e.,

$$x \in (M_n, \circ, \| \cdot \|_{C_0(G)}) \mapsto xa \in (M_n, \circ, \| \cdot \|_{C_0(G)})$$

we would again get a completely bounded map but with completely bounded norm $\| a \|_{I,d}$.

PROPOSITION 7. *For all $a \in (M_n, *, \| \cdot \|_{I,r})$, $(T_a)^- = T_{\bar{a}}$, thus N is invariant under the conjugation operation in $\mathscr{D}(M_n, \circ, \| \cdot \|_{C_0(G)})$. The map*

$$a \in (M_n, *, \| \cdot \|_{I,r}) \mapsto T_a \in N$$

"preserves" the conjugation $^-$ on $(M_n, *, \| \cdot \|_{I,r})$.

PROOF. We have $(T_a)^- x = (T_a\bar{x})^- = \overline{a\bar{x}} = \bar{a}x = T_{\bar{a}}x$ for $x \in (M_m, \circ, \|\cdot\|_{C_0(G)})$. Note that the complex conjugation is the isometric involution of C*-algebra $(M_n, \circ, \|\cdot\|_{C_0(G)})$ and that

$$a \in (M_n, *, \|\cdot\|_{I,r}) \mapsto \bar{a} \in (M_n, *, \|\cdot\|_{I,r})$$

is also an isometric conjugation.

REMARK. The above discussion, Proposition 6 in particular, gives a fundamental reason for considering the norms $\|\cdot\|_{I,r}$, $\|\cdot\|_{I,d}$.

We are now ready to investigate the dual version of Proposition 6. We note first that if we are to eventually have a duality in the sense of section 4, then we must find a C*-completion of $(M_n, *, \|\cdot\|_{I,r})$. This is a Banach algebra, in which the usual Hermitian conjugate involution is not isometric, since $\|x^*\|_{I,r} = \|x\|_{I,d}$. Nevertheless, with this involution we get a C*-completion in the sense of Definition 5, namely $(M_n, *, \|\cdot\|_{C^*(G)})$, i.e. usual $n \times n$ matrices with operator norm. Now we wish to compute the norm of a "Schur action"). Namely, consider the map

$$a \in (M_n, \circ, ?) \mapsto T_a \in \mathscr{L}(M_n, *, \|\cdot\|_{C^*(G)}),$$

where $T_a x = a \circ x$ for $a, x \in M_n$. We want to compute the completely bounded norm $\|T_a\|_{\mathscr{D}}$.

Let us first look at T_p, where p is a Hermitian positive matrix. Thus $T_p x = p \circ x$; and if x is positive in $C^*(G)$, then so is $p \circ x$. The norm of such a positive map is attained at the identity, I, so

$$\|T_p\|_{\mathscr{L}(C^*(G))} = \|p \circ I\|_{C^*(G)} = \sup_i |p_{ii}| = \max_i p_{ii}$$

where the diagonal entries of p are p_{ii}, $1 \leqq i \leqq n$. Note that T_p is a completely positive map. We can see this by considering Hermitian positive matrix

$$x(m) = (x_{ij}(m)) \in C^*(G) \otimes M_m,$$

[18, p.192]. The $m \times m$ matrix

$$A = \begin{bmatrix} p & p & p & \cdots & p \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ p & p & p & \cdots & p \end{bmatrix}$$

is positive definite, since given $\xi_i \in \mathbf{C}^n$, $1 \leqq i \leqq m$ we have (let $p = qq = q^* q$, (matrix product))

$$(A\xi | \xi) = \sum_{i,j} (p\xi_i | \xi_j) = \left\| \sum_{i=1}^{m} q\xi_i \right\|^2 \geqq 0.$$

Thus

$$(T_p \otimes I_m)x(m) = \begin{bmatrix} p & p & \cdots & p \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p & p & \cdots & p \end{bmatrix} \circ \left(x_{ij}(m)\right)$$

is positive since the Schur product of Hermitian positive matrices is Hermitian positive.

Now

$$\| T_p \otimes I_m \|_{\mathscr{L}(C^*(G) \otimes M_m)} = \| T_p \otimes I_m(I) \| = \left\| \begin{bmatrix} p & 0 & \cdots & 0 \\ \cdot & p & & \cdot \\ \cdot & & p & \cdot \\ \cdot & & & \cdot \\ 0 & \cdot & \cdot & p \end{bmatrix} \right\| = \sup_i |p_{ii}|.$$

Thus for $p = (p_{ij})$ a positive hermitian matrix in M_n we have

$$\| T_p \|_{\mathscr{D}} = \sup_{i,j} |p_{ij}| = \sup_i |p_{ii}|.$$

Now for $a \in M_n$,

$$\| a \|_{\mathrm{Tr}} = \sum_{i=1}^{n} \omega_{\xi_i}(|a|)$$

where $\{\xi_1, \xi_2, \ldots, \xi_n\}$ is an orthonormal basis of $\mathbf{C}^n$, $|a| = \sqrt{a^* a}$, and $\omega_\xi(a) = (a\xi | \xi)$, the inner product of $a\xi$ and ξ. Exploiting an analogy with Proposition 7 we can interpret $\omega_\xi(\cdot)$ as the sum on the "ith row" and get $\sup_{1 \leqq i \leqq n} \omega_{\xi_i}(|a|)$ as an analogue of $\| a \|_{I,r}$.

There are two places where this analogy fails to be precise. First $|a|$ is not unambiguous, there are "left and right" absolute values.

Definition 6. If $a \in M_n$ we define the right absolute value of a to be $|a|_r = \sqrt{a^* a}$ and the left absolute value of a to be $|a|_l = \sqrt{aa^*}$.

REMARK. If no subscript is given, the right absolute value is understood, i.e., $|a| = |a|_r$. Also there are corresponding right and left polar decompositions for a, that is $a = u|a|_r = |a|_l u$ for suitable partial isometry u.

The second source of imprecision in the analogy is that it is not always clear which orthonormal basis should be chosen as the "rows" in a given setting. Thus we make the following definition.

DEFINITION 7. Given a positive Hermitian matrix $p \in M_n$, and orthonormal basis $\mathscr{B} = \{\xi_1, \xi_2, \ldots, \xi_n\}$ of $\mathbf{C}^n$, we define the Haarek norm of p with respect to basis $\mathscr{B}$ to be

$$\| p \|_{H(\mathscr{B})} = \sup_{\xi_i \in \mathscr{B}} \omega_{\xi_i}(p).$$

REMARK. If matrix p is written in terms of basis $\mathscr{B}$, or equivalently $\xi_i = e_i$, $i = 1,2,\ldots,n$ is the standard basis, then we drop the $\mathscr{B}$ in the above definition and have

$$\| p \|_H = \sup_i \omega_{e_i}(p) = \sup_i |p_{ii}| = \sup_i p_{ii}$$

The above analogy leads us to an estimate for the completely bounded norm of $T_a : x \in M_n \mapsto a \circ x \in M_n$ which is good enough for our purposes but not the best possible. See the second remark following Proposition 8.

PROPOSITION 8. *The map* $T_a \in \mathscr{L}(M_n, *, \| \cdot \|_{C^*(G)})$ *defined above is completely bounded with*

$$\| a \|_\infty = \sup_{i,j} |a_{ij}| \leqq \| T_a \|_{\mathscr{D}_s} \leqq \sqrt{\| |a|_l \|_H} \sqrt{\| |a|_r \|_H} = \sqrt{\sup_i p_{ii}} \sqrt{\sup_j q_{jj}},$$

where $q = |a|_r = \sqrt{a^* a}$ *and* $p = |a|_l = \sqrt{aa^*}$. *The subscript* $\mathscr{D}_s$ *indicates the completely bounded norm with regard to the Schur product. Thus the map*

$$a \in (M_n, \circ, \| \cdot \|_{\mathscr{D}_s}) \mapsto T_a \in \mathscr{D}(M_n, *, \| \cdot \|_{C^*(G)})$$

is an isometric isomorphism onto a subalgebra Q *of* $\mathscr{D}(M_n, *, \| \cdot \|_{C^*(G)})$.

REMARK. For the proof of this result see [25].

REMARK. In the proof of Proposition 8 if one replaces p by p^α, $0 \leqq \alpha \leqq 1$, one can obtain the following better estimate:

$$\| T_a \|_{\mathscr{D}} \leqq \inf_{0 \leqq \alpha \leqq 2} \{ \sqrt{\max_i (p^\alpha)_{ii}} \sqrt{\max_j (q^{2-\alpha})_{jj}} \}.$$

PROPOSITION 9. *For all* $a \in (M_n, \circ, \|\cdot\|_{\mathscr{D}_s})$, $(T_a)^- = T_{a^*}$, *thus* Q *is invariant under the conjugation in* $\mathscr{D}(M_n, *, \|\cdot\|_{C^*(G)})$. *The map* $a \in (M_n, \circ, \|\cdot\|_\infty) \mapsto T_a \in Q$ *preserves the conjugation* $*$ *on* $(M_n, \circ, \|\cdot\|_{\mathscr{D}_s})$.

PROOF. We have $(T_a)^- x = (a \circ x^*)^* = a^* \circ x = T_{a^*} x$ for $x \in (M_n, *, \|\cdot\|_{C^*(G)})$. We are done.

4. A general notion of duality for Banach algebras.

As mentioned in the introduction we have been motivated by a search for an "explanation" of the similarities between certain duality structures that exist for groups and the $n \times n$ matrices. The existing duality theories of Tannaka, Stinespring, Tatsuuma, Takesaki, Enock and Schwartz, and others, do not conveniently lend themselves to the "explanation" we are looking for. Briefly, the theory of Takesaki may be referred to as the Hopf–von Neumann algebra approach to non-commutative (group) duality, and that of Enock and Schwartz may be referred to as the Kac algebra approach, cf. [19], [5]. One thing to be learned from these and other non-commutative generalizations of Pontriagin–Van Kampen duality is that one must leave the category of groups and enter the category of algebras to formulate an extension of Pontriagin–Van Kampen duality that includes nonabelian groups. Once this principle has been realized, an inveterate mathematician is forced to ponder the existence of a duality principle for algebras in general. For example, given an algebra A, how can one create an algebra B which is "dual to A"? How can one tell if two algebras are "dual" to one another? What might it mean for two algebras to be dual to one another?

As a step on this direction, we summarize the foregoing theorems by giving a definition of duality such that duality for group algebras and duality for matrices (as studied above) become special cases.

In the following A and B will be two Banach algebras each of which has an involution as well as a conjugation. The involutions (involved in part (1) below) are not required to be isometric here; neither, a priori are the conjugations (involved in part (2) below) required to be isometric.

DEFINITION 8. Banach algebras A and B with involutions $*A$ and $*B$, respectively, and conjugations ^-A and ^-B, respectively, are said to be *dual* if the following two conditions are satisfied:

(1) There exist Banach algebra homomorphisms i_A, i_B, and C*-algebras C*(A), C*(B) where

$$i_A\colon A \to C^*(A)$$
$$i_B\colon B \to C^*(B),$$

i_A and i_B are one-one, onto dense subalgebras of $C^*(A)$, $C^*(B)$ respectively, and i_A, i_B each preserve involutions $*A$, $*B$, respectively.

(2) There exist norm-decreasing Banach algebra isomorphisms j_A, j_B where

$$j_A\colon A \to \mathscr{D}(C^*(B))$$
$$j_B\colon B \to \mathscr{D}(C^*(A))$$

j_A, j_B being into the dual algebras of $C^*(B)$, $C^*(A)$ respectively; and j_A, j_B preserve conjugations ^-A, ^-B, respectively. If the involutions $*A$, $*B$ and the conjugations ^-A, ^-B are all isometric, we say that the duality between A and B is *semi-rigid*. If the duality between A and B is semi-rigid and the maps j_A, j_B are isometric; we say that the duality between A and B is *rigid*.

REMARK. The C*-algebras occurring in part (1) above are called C*-completions. For example, $C^*(A)$ is a C*-completion of A. In our work thus far all the Banach algebras involved have universal enveloping C*-algebras, cf. [3]. If one does not insist on universality, then one does not have uniqueness of the C*-completion. For example, if G is a non-amenable group, then the universal enveloping algebra, $C^*(G)$, is not isomorphic with $C^*_\lambda(G)$, the C*-algebra generated by the left-regular representation λ of G. Yet $C^*(G)$ and $C^*_\lambda(G)$ are C*-completions of $L^1(G)$.

REMARK. We use the term norm-decreasing to mean "not norm increasing", thus isometries are norm-decreasing. Also if A is dual to B we will say that $*B$ *is dual to* ^-A and that $*A$ *is dual to* ^-B. Also, we do not a priori require that $*A$ be distinct from ^-A, nor $*B$ be distinct from ^-B. The reader will note a slight redundancy in the definition of rigid. This redundancy being that if j_A and j_B are isometries, then ^-A and ^-B must be isometric since the corresponding commutative involutions on $\mathscr{D}(C^*(G))$ and $\mathscr{D}(C^*(G))$ are isometric by Proposition 1. Finally we see that dualities for group algebras $A(G)$ and $L^1(G)$, using their universal C*-completions, are rigid, while the corresponding dualities for groupoids are not.

REMARK. Corresponding to each conjugation and/or involution there is a notion of positivity. Our notion of duality can be given an alternate formulation in terms of these notions of positivity.

EXAMPLE. It follows from Theorem 1, Theorem 2, Proposition 2, and Proposition 3 that for locally compact group G, *$A(G)$ and $L^1(G)$ are dual in the sense of Definition 8.* We note that $L^1(G)$ has a universal C*-completion, $C^*(G)$, which serves as $C^*(L^1(G))$; and $C_0(G)$, the continuous, complex-valued functions that vanish at infinity on G, is the C*-completion of $A(G)$, that is, $C^*(A(G))$. Also $L^1(G)$ has an isometric involution,

$$f \in L^1(G) \mapsto f^\# = \Delta^{-1} f^\flat \in L^1(G)$$

and an isometric conjugation

$$f \in L^1(G) \mapsto \bar{f} \in L^1(G),$$

while $A(G)$ has isometric involution $a \in A(G) \mapsto \bar{a} \in A(G)$ and isometric conjugation $a \in A(G) \mapsto a^\flat \in A(G)$.

Example. It follows from Proposition 5, Proposition 6, Proposition 7, Proposition 8, and Proposition 9 that for G the n-transitive principle groupoid,

$$A(G) = (M_n, \circ, \|\cdot\|_{\mathrm{Tr}})$$

is dual to

$$L^1(G) = (M_n, *, \|\cdot\|_{L^1})$$

in the sense of Definition 8. Also $(M_n, \circ, \|\cdot\|_{\mathscr{D}_s})$ is dual to $(M_n, *, \|\cdot\|_{I,r})$.

Example. For G the countably infinite, discrete, principal transitive groupoid, let $A(G)$ be the trace class operators and $L^1(G)$ be the matrices satisfying $\sum_{i,j=1}^{\infty} |a_{ij}| < \infty$. Then $A(G)$ and $L^1(G)$ are dual (semi-rigidly).

Remark. In the last two examples there is a Fourier–Plancherel transform which completes the analogy with the group case.

Remark. In [4] the "dual" of a certain group arises as a set of endomorphisms of a certain C*-algebra. We take this as evidence that defining dual objects via completely bounded transformations promises to be a fruitful approach.

Acknowledgements. I wish to thank Haakon Waadeland and Magnus Landstad for their hospitality at the University of Trondheim while a part of this paper was written. Also I wish to thank Erling Størmer for pointing out an error in our early version of Proposition 8.

REFERENCES

1. W. B. Arveson, *Subalgebras of* C*-*algebras*, Acta Math. 123 (1969), 141–224.
2. J. Dixmier, *Les algèbres d'operateurs dans l'espace Hilbertien*, Deuxième édition (Cahiers Sci. 25) Gauthier-Villars, Paris, 1969.
3. J. Dixmier, *Les* C*-*algèbres et leurs représentations* (Cahiers Sci. 29), Gauthier-Villars, Paris, 1964.

4. S. Doplicher and J. E. Roberts, *Compact Lie groups associated with endomorphisms of* C* *algebras*, Bull. Amer. Math. Soc. 11 (1984), 333–338.
5. M. Enock and J. M. Schwartz, *Une dualité dans les algèbres de von Neumann*, Bull. Soc. Math. France Mém. 44 (1975), 1–144.
6. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181–236.
7. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, I (Grundlehren Math. Wiss. 115), Springer-Verlag, Berlin - Heidelberg - New York, 1963.
8. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, II (Grundlehren Math. Wiss. 152), Springer-Verlag, Berlin - Heidelberg - New York, 1970.
9. B. E. Johnson, *Isometric isomorphism of measure algebras*, Proc. Amer. Math. Soc. 15 (1964), 186–188.
10. C. Lance, *On nuclear* C*-*algebras*, J. Funct. Anal. 12 (1973), 157–176.
11. R. I. Loebl, *Contractive linear maps on* C*-*algebras*, Michigan Math. J. 22 (1975), 361–366.
12. A. B. Ramsay and M. E. Walter, *Groupoid algebras*, in preparation.
13. J. Renault, *A groupoid approach to* C*-*algebras* (Lecture Notes in Math. 793), Springer-Verlag, Berlin - Heidelberg - New York, 1980.
14. W. Rudin, *Fourier analysis on groups* (Interscience Tracts in Pure Appl. Math. 12), Interscience, New York, 1962.
15. W. Rudin, *Real and complex analysis*, McGraw-Hill Book Company, 1966.
16. Z. Semadeni, *Banach spaces of continuous functions*, Vol. 1 (Monograf. Math. 55), Polish Sci. Publ., Warsaw, 1971.
17. E. Størmer, *Regular abelian Banach algebras of linear maps of operator algebras*, J. Funct. Anal. 37 (1980), 331–373.
18. M. Takesaki, *Theory of operator algebras*, I, Springer-Verlag, Berlin - Heidelberg - New York, 1979.
19. M. Takesaki, *Duality and von Neumann algebras*, in *Lectures on operator algebras*, II (Proc., Tulane Univ., 1970) (Lecture Notes in Math. 247), pp. 665–785. Springer-Verlag, Berlin - Heidelberg - New York, 1972.
20. M. E. Walter, W*-*algebras and nonabelian harmonic analysis*, J. Funct. Anal. 11 (1972), 17–38.
21. M. E. Walter, *A duality between locally compact groups and certain Banach algebras*, J. Funct. Anal. 17 (1974), 131–160.
22. M. E. Walter, *Convolution on the reduced dual of a locally compact group*, Math. Scand. 37 (1975), 145–166.
23. M. E. Walter, *Duality theory for nonabelian locally compact groups*, Symposia Mathematica, Vol. 22, pp. 47–59. Academic Press, London - New York, 1977.
24. M. E. Walter, *Differentiation on the dual of a group: An introduction*, Rocky Mountain J. Math. 12 (1982), 497–535.
25. M. E. Walter, *On the norm of a Schur product*, Linear Algebra Appl., to appear.
26. J. G. Wendel, *On isometric isomorphisms of group algebras*, Pacific J. Math. 1 (1951), 305–311.
27. J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. 2 (1952), 251–261.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COLORADO
BOULDER, COLORADO 80309
U.S.A.