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CLASSIFICATION OF FIBRED GROUP EXTENSIONS AND H^1

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Abstract.

Let G be a group, A an abelian group. We study the equivalence classes of "fibred group extensions" defined by a certain compatible system of actions of G and A ((G,A)-bundle (see below)), which had been introduced in [3] and further studied in [4], [5]. Given a (G,A)-bundle Σ and a G-action ε on A, we define a G-module $A'(\Sigma,\varepsilon)$. We show that the set of equivalence classes of fibred extensions corresponding to the given bundle, with its natural group structure (which had already been introduced in [3]) is naturally isomorphic to $H^1(G, A'(\Sigma,\varepsilon))$.

Résumé.

Soit G un group et A un groupe abélien. On étudie les classes d'équivalence d'extensions de groupes fibrées d'un groupe G par un groupe A qui se définissent par rapport à un système d'actions de groupes (les (G,A)-fibrés, voir ci-dessous) introduit dans [3] et éduié aussi dans [4], [5]. Soit Σ un (G,A)-fibré et ε un G-action sur A. Alors nous introduisons un G-module $A'(\Sigma,\varepsilon)$. On démontre que l'ensemble des classes d'équivalence d'extensions fibrées correspondant au (G,A)-fibré et une action de G sur A donné, avec sa structure de groupe (introduite antérieurement dans [3]) est naturellement isomorphe au groupe de cohomologie $H^1(G, A'(\Sigma,\varepsilon))$.

0. Introduction.

It is well known from the classical theory of group extensions, founded by Schreier [1], [2], that, with respect to a very natural notion of equivalence, the set of equivalence classes of extensions of a group G by a group A, with prescribed homomorphism from G to the group of outer automorphisms of A, is classified by the set of equivalence classes of solutions of certain "cohomological" functional equations. When the kernel A is abelian, these equivalence classes form an abelian group, the second cohomology group of G with values in the G-module A defined by

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the prescribed action of G. In practice it is often extremely difficult to compute these cohomology groups and, even when this is possible, to find natural realizations of the extensions corresponding to the elements of these cohomology groups. However, often the problem of finding and describing group extensions is posed in the more concrete setting of transformation groups, i.e. the groups G and A are realized as transformation groups, and the extensions to be constructed are to act in a certain way compatible with these actions. In [3], we had formalized this situation by introducing the notions of a "(G,A)-bundle" and a "fibred group extension" defined with respect to such a bundle (see below). We showed there that there is a natural notion of equivalence in this setting. Furthermore, like in the classical case, one can define "relative cohomology groups" defined not only by G, A, and the action of G on A, but by those data plus the (G,A)-bundle; the elements of this cohomology group classify the equivalence classes of fibred group extensions. In fact, the classical theorem was derived as a special case of our result [3].

However, the problem of computing these "relative" cohomology groups, while often simpler than that of computing the ordinary cohomology group $H^2(G,A)$, remains difficult. It is the purpose of the present paper to facilitate the study of fibred extensions by means of the following construction. From a (G,A)-bundle Σ and an action of G on A,ε, we construct a new G-module $A'(\Sigma,\varepsilon)$, which has the property that the equivalence classes of fibred extensions corresponding to Σ and the given G-action on A are parametrized by the group $H^1(G,A'(\Sigma,\varepsilon))$. In some interesting examples (see below), the module $A'(\Sigma,\varepsilon)$ will be a trivial G-module, so that the equivalence classes of fibred extensions are parametrized by the group of characters of G with values in that trivial module.

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1.

We first repeat the definition of a "fibred group extension" first introduced in [3] (see also [4], [5]). We start with a (faithful) transformation group A acting on a set X, for which we shall use the notation (X,A), and denote the set of A-orbits in X by L. We then assume given a second transformation group (L,G). We call the quadruple $(X,A,L = X/A,G)$ satisfying these conditions a (G,A)-bundle (in [3] we used the term "compatible system of transformation groups"). We denote the natural map from X to L by P. A map s from L to X for which $P(s(l)) = l$

for all l in L will be called a section for P. We shall denote by $\mathscr{S}$ the set of all such sections. The normalizer of A in the group $S(X)$ of all bijections of X will be denoted by N. Given n in N, $\operatorname{Int}(n)$ will denote the inner automorphism defined by n restricted to A. Since N permutes the A-orbits in X, we obtain a natural homomorphism

$$p : N \to S(L).$$

We are now ready to define what we mean by a fibred group extension. We start with a (G,A)-bundle $\Sigma = (X,A,L,G)$ as above. Then a fibred extension corresponding to Σ is an element of the following set:

DEFINITION 1. $\operatorname{Ext}(\Sigma)$ = the set of pairs (E,q), where E is a subgroup of N for which the following conditions hold:

a) i) $E \cap \operatorname{Ker}(p) = A$.
   ii) $p(E) = G$,

b) q is the extension of G by A arising from the restriction of p to E:

$$1 \to A \to E \xrightarrow{q} G \to 1.$$

Obviously a group E satisfying a) determines the fibred extension. However, we find it convenient to include the extension q defined by E in our definition.

In [3] we have introduced the following equivalence relation in $\operatorname{Ext}(\Sigma)$.

DEFINITION 2. Let $\Sigma = (X,A,L,G)$ and $\Sigma' = (X',A,L',G)$ be two (G,A)-bundles, E and E' fibred extensions with respect to Σ, respectively Σ'.

a) An equivalence between E and E' is given by a pair of bijections

$R\colon\ X \to X'$
$r\colon\ E \to E'$

such that the following conditions are satisfied:

i) for all x in X, e in E, $R(x.e) = R(x).r(e)$.
ii) r is an equivalence of abstract group extensions.

b) $\operatorname{Ext}'(\Sigma)$ is $\operatorname{Ext}(\Sigma)/\sim$, where $\sim$ is the equivalence relation defined above. $[E]$ will denote the equivalence class of E.

We shall assume throughout this paper that A is abelian (one can obtain similar results without this assumption (see also [3], where A was allowed to be non-abelian). Given an action ε of G on A, we shall denote by $\operatorname{Ext}(\Sigma;\varepsilon)$ the set of fibred extensions which induce the action ε of G on A by conjugation. It is possible that $\operatorname{Ext}(\Sigma,\varepsilon)$ is empty. If it is not empty, then it is known [3] that for each (G,A)-bundle Σ, and each action ε of G on A, the set $\mathrm{Ext}'(\Sigma;\varepsilon)$ of equivalence classes of fibred extensions defined by elements in $\mathrm{Ext}(\Sigma,\varepsilon)$ is an abelian group.

We shall now study this group from a new viewpoint, as explained in the introduction. First we characterize those pairs (Σ,ε) for which $\mathrm{Ext}(\Sigma,\varepsilon)$ is not empty.

We let $A(l)$ to be the (common) stabilizer of any point x in $P^{-1}(l)$ and $A'(l) = A/A(l)$.

Definition 3. An action ε of G on A is called Σ-admissible, iff for each g in G and for each l in L, $\varepsilon(g)$ maps $A(l)$ to $A(l\,.\,g)$.

Lemma 1. *ε is Σ-admissible iff $\mathrm{Ext}(\Sigma,\varepsilon)$ is non-empty.*

Proof. It is a simple exercise to verify that for any extension E in $\mathrm{Ext}(\Sigma,\varepsilon)$, ε is Σ-admissible. To prove the converse, we show that, for ε Σ-admissible, the semi-direct product $A \rtimes_\varepsilon G$ has a realisation as a fibred extension for Σ. We consider the given action of A on X. We choose a section s for P. We then define an action of G on X by the formulae

(1) $s(l)\,.\,g = s(l\,.\,g)$

(2) $s(l)\,.\,a\,.\,g = s(l\,.\,g)\,.\,\varepsilon(g)(a)$.

One verifies that these actions of G and A uniquely define an action of the semi-direct product $A \rtimes_\varepsilon G$ on X which is in $\mathrm{Ext}(\Sigma,\varepsilon)$, which proves the lemma.

Above we had defined the projection $p: N \to S(L)$. We consider the following subgroup of $p^{-1}(G)$ defined by ε:

$$N(\varepsilon) = \{n \mid \varepsilon(p(n)) = \mathrm{Int}(n)\},$$

where $\mathrm{Int}(n)$ is conjugation by n restricted to A. We now let s be a section for P as above, and $p(\varepsilon)$ the restriction of p to $N(\varepsilon)$. We let $A(\Sigma)$ to be

$$A(\Sigma) = \prod_{l \text{ in } L} A'(l).$$

We let $A(\Sigma)$ act on X by the formula

$$x\,.\,(a(l))_{l\in L} = x\,.\,a(l), \quad \text{for } x \text{ in } P^{-1}(l).$$

We can now describe the structure of $N(\varepsilon)$:

LEMMA 2. i) $\operatorname{Ker}(p(\varepsilon)) = A(\Sigma)$.

ii) *Let G be embedded in* $N(\varepsilon)$ *by means of formulae* 1) *and* 2) *in the proof of Lemma* 1. *Then there exists a natural isomorphism* ψ_s *(which depends on s!)*

$$\psi_s : N(\varepsilon) \xrightarrow{\sim} A(\Sigma) \rtimes_\varepsilon G.$$

PROOF. i) It is clear that $A(\Sigma)$ is contained in $\operatorname{Ker}(p(\varepsilon))$. On the other hand, by definition of $N(\varepsilon)$, $\operatorname{Ker}(p(\varepsilon))$ is contained in the centralizer of A in $N(\Sigma)$. Furthermore, an element of $\operatorname{Ker}(p(\varepsilon))$ is determined by its action on each $P^{-1}(l)$. Since A acts transitively on each of these, i) follows from the known fact (which is a special case of the theorem (4) of [3]) that any abelian group B is its own centralizer in $S(B)$.

ii) is immediately verified.

From the lemma we see that, for ε as above, $A(\Sigma)$ is a G-module which contains A as a G-submodule. Henceforth we shall fix an ε-admissible action.

We denote $A(\Sigma)$, viewed as a G-module, with action defined by ε, by $A(\Sigma,\varepsilon)$.

The G-module $A'(\Sigma,\varepsilon) = A(\Sigma,\varepsilon)/A$ will play the key role in what is to follow.

It is immediate from the definition that any E in $\operatorname{Ext}(E,\varepsilon)$ is contained in $N(\varepsilon)$.

We now remind the reader of the definition of the group $H^1(G,B)$, for any G-module B (writing the action of G on the right).

DEFINITION 3.

i) $C^1(G,B) = \{f: G \to B \mid f(g \cdot g') = f(g) \cdot (f(g') \cdot g) \text{ for all } g,g' \text{ in } G\}$.

ii) $B^1(G,B) = \{f(g) = b - b \cdot g \mid b \text{ in } B\}$.

iii) $H^1(G,B) = C^1(G,B)/B^1(G,B)$.

iv) We let $[f]$ be the element in $H^1(G,B)$ defined by f in B.

When G, B are understood, we shall write H^1, B^1, C^1 for $H^1(G,B)$, etc.

We are now ready to state the principal result of this paper:

THEOREM. *Let Σ be any (G,A)-bundle, ε any Σ-admissible action of G on A. Then* $\operatorname{Ext}'(\Sigma,\varepsilon)$ *carries the structure of an abelian group. Moreover, this group is naturally isomorphic with* $H^1(G, A'(\Sigma,\varepsilon))$.

PROOF. The proof will involve the following steps:

i) Construction of a map

$$\text{th}: \operatorname{Ext}(\Sigma,\varepsilon) \to H^1(G, A'(\Sigma,\varepsilon)).$$

by means of auxiliary maps

$$\mathrm{th}^\circ\colon \mathrm{Ext}(\Sigma,\varepsilon)\times\mathscr{S} \to B^1(G, A'(\Sigma,\varepsilon)).$$

($\mathscr{S}$ is the set of sections for P, as above).

ii) Prove that the map th defines a map from $\mathrm{Ext}'(\Sigma,\varepsilon)$ to $H^1(G, A'(\Sigma,\varepsilon))$ by passing to equivalence classes. That map will be denoted by $\overline{\mathrm{th}}$.

iii) Prove that $\overline{\mathrm{th}}$ is a bijection from $\mathrm{Ext}'(\Sigma,\varepsilon)$ onto $H^1(G, A'(\Sigma,\varepsilon))$.

iv) Define the group structure on $\mathrm{Ext}'(\Sigma,\varepsilon)$, and prove that $\overline{\mathrm{th}}$ as above is a homomorphism of groups.

STEP i). We first define an auxiliary map

$$\mathrm{th}^\circ\colon \mathrm{Ext}(\Sigma,\varepsilon)\times\mathscr{S} \to B^1.$$

We shall prove that the class of th° is in fact independent of s and hence yields the map

$$\mathrm{th}\colon \mathrm{Ext}(\Sigma,\varepsilon) \to H^1(G, A'(\Sigma,\varepsilon))$$

which we wish to define. For a given section s we had the identification

$$\psi_s : N(\varepsilon) \xrightarrow{\sim} A(\Sigma) \rtimes_\varepsilon G.$$

We have already remarked that any extension in $\mathrm{Ext}(\Sigma,\varepsilon)$ is contained in $N(\varepsilon)$. Hence we can write any such extension as follows:

$$E = \{(a,g) \mid a = f(g) \bmod A\},$$

where f is some function $G \to A'(\Sigma,\varepsilon)$. In view of the formula $(a,g)\,.\,(a',g') = (a\,.\,(a'\,.\,g), g\,.\,g')$ in $A(\Sigma)\rtimes_\varepsilon G$, we obtain immediately

LEMMA 3. *f is an element of $B^1(G, A'(\Sigma,\varepsilon))$.*
We now define the maps

$$\mathrm{th}^\circ\colon \mathrm{Ext}(\Sigma,\varepsilon)\times\mathscr{S} \to B^1(G, A'(\Sigma,\varepsilon))$$

by the formula $\mathrm{th}^\circ(E,s) = f$, f as in Lemma 3, and the map

$$\mathrm{th}\colon \mathrm{Ext}(\Sigma,\varepsilon) \to H^1(G, A'(\Sigma,\varepsilon)),$$

by $\mathrm{th}(E) = [f]$. The only thing that needs to be checked is that the class of f in $H(G, A'(\Sigma,\varepsilon))$ is independent of the section s.

We note that $A(\Sigma)$ acts freely and transitively on $\mathscr{S}$. We suppose given two sections s and s' and write $s' = s\,.\,a$ for some (unique) a in $A(\Sigma)$. It is clear that the automorphism $\psi_s \circ \psi_{s'}^{-1}$ of the group $A(\Sigma)\rtimes_\varepsilon G$ is given by conjugation by the element a. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} & N(\varepsilon) & \\ {\scriptstyle \psi_{s'}} \swarrow & & \searrow {\scriptstyle \psi_s} \\ A(\Sigma) \rtimes_\varepsilon G & \xrightarrow[\psi_s \circ \psi_{s'}^{-1}]{} & A(\varepsilon) \rtimes_\varepsilon G \end{array}$$

and $\psi_s \circ \psi_{s'}^{-1}((a,g)) = (-a + a\,.\,g,g)$. Hence the cocycles f and f' differ by the coboundary $a' - a'\,.\,g$, where a' is the image of a in $A'(\Sigma,\varepsilon)$. This completes the proof that the map

$$\mathrm{th}\colon \mathrm{Ext}(\Sigma,\varepsilon) \to H^1(G, A'(\Sigma,\varepsilon))$$

as above is well defined.

ii) We need to verify that, if E' is equivalent to E, then $\mathrm{th}(E) = \mathrm{th}(E')$. Suppose (R,r) is an equivalence (Definition 2), s a section for P. By means of R, s defines a unique section s' in Σ', and it is easy to see that $\mathrm{th}^\circ(E,s) = \mathrm{th}^\circ(E',s')$, and hence that $\mathrm{th}(E) = \mathrm{th}(E')$.

iii) To show that th is indeed a bijection as claimed, we construct its inverse. As above, we fix a section s. We define a map φ_s from $B^1(G,A(\Sigma,\varepsilon))$ to $\mathrm{Ext}(\Sigma,\varepsilon)$ as follows: let f be an element of B^1, and let s be in $\mathscr{S}$. As above, we have the isomorphism

$$\psi_s : N(\varepsilon) \xrightarrow{\sim} A(\Sigma) \rtimes_\varepsilon G.$$

We let $E = \varphi_s(f)$ be the following subgroup of $N(\varepsilon) = A(\Sigma) \rtimes_\varepsilon G$:

$$E = \{(a,g) \mid a \equiv f(g) \bmod A\}.$$

Since in $A(\Sigma) \rtimes_\varepsilon G$, $(a,g)\,.\,(a',g') = (a\,.\,(a'\,.\,g), g\,.\,g')$, it follows immediately from the functional equation defining B^1 that E is a subgroup of $N(\varepsilon)$. Indeed we have the following

LEMMA 4. *E is in* $\mathrm{Ext}(\Sigma,\varepsilon)$.

PROOF. Condition ii) of Definition 1 is immediate. To verify i), we must show that $E \cap A(\Sigma) = A$. This is equivalent to showing $f(1_G) = 1_{A'(\Sigma,\varepsilon)}$, which is clearly true since $f(1) = f(1\,.\,1) = f(1)\,.\,f(1)\,.\,1$, which proves the lemma. Writing $\mathrm{th}(s)(E) = \mathrm{th}^\circ(E,s)$, it is also clear that φ_s is $(\mathrm{th}(s))^{-1}$ as a map from B^1 onto $\mathrm{Ext}(\Sigma,\varepsilon)$. Since we have just seen that $\overline{\mathrm{th}}$ defines a map from equivalence classes in $\mathrm{Ext}(\Sigma,\varepsilon)$ to elements in H^1, φ_s defines a bijection from $H^1(G,A'(\Sigma,\varepsilon)$ onto $\mathrm{Ext}'(\Sigma,\varepsilon)$. This proves iii).

iv) We shall fix a section s, as above, and use the expression for E, via ψ_s:

$$E = \{(a,g) \mid a \equiv f(g) \bmod A\}.$$

We shall write $E = E(f)$. Now we define multiplication in $\text{Ext}'(\Sigma,\varepsilon)$ as follows:

$$[E(f)] + [E(f')] = [E(f+f')]. \tag{1}$$

To see that this multiplication is well defined, one needs to verify that $[E(f)] = [E(h)]$ and $[E(f')] = [E(h')]$ implies $[E(f+f')] = [E(h+h')]$. But this is so since, as we have seen in the proof of iii), $[E(f)] = [E(f')]$ iff f and f' define the same element in H^1. By construction (formula (1)), the map $\overline{\text{th}}$ is a homomorphism of groups. This proves the theorem.

An interesting special case arises when G acts trivially on $A'(\Sigma,\varepsilon)$. In this case, there are no non-trivial co-boundaries, and $H^1(G, A'(\Sigma,\varepsilon))$ is just the group of characters of G with values in A'.

COROLLARY 1. *If the action of G on* $A'(\Sigma,\varepsilon)$ *is trivial, then* $\text{Ext}'(\Sigma,\varepsilon) = \text{Hom}(G, A')$.

An important example of this situation is the following. We let W be the symmetric group on n letters, and $L = \{1,\ldots,n\}$ with the standard action of W. We let $\{X(i)\}_{i\in L}$ be a system of linearly independent lines with 0 deleted in an n-dimensional vector space over K, X be the union of the $X(i)$, A the diagonal subgroup of $\text{SL}(n,K)$ defined by the system $X(i)$, acting on X by restriction, and ε the permutation action of W on A. We have a natural projection from X onto L, and $\Sigma = (X, A, L, W)$ is a (W, A)-bundle. Furthermore we have the exact sequence of W-modules as above

$$(1) \to A \to \prod_{l\in L} K_l^* \to K^* \to (1),$$

and W acts trivially on $A'(\Sigma,\varepsilon) = K^*$. Hence, by Corollary 1, we have

COROLLARY 2. $\text{Ext}'(\Sigma,\varepsilon) = \text{Hom}(W, K^*) \cong \mathbf{Z}/2\mathbf{Z}$.

We note that N, the normalizer of A in $\text{SL}(n,K)$ defines the (unique) non-trivial extension in $\text{Ext}'(\Sigma,\varepsilon)$. It is easy to see that its subgroup

$$\left\{(\bar{t},\sigma) \mid \bar{t} = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix},\ \sigma \in W,\ t_i = \pm 1,\ \prod_{i=1}^{N} t_i = \text{sign}(\sigma)\right\}$$

is the smallest subgroup of N projecting onto W. It is Tits' "extended Weyl group" for $\text{SL}(n,K)$ (see [6] for the definition and role of the "extended Weyl group" for the theory of semi-simple Lie algebras).

Analogous situations arising for more general groups will be studied in a later paper.

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