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THE ROSETTES

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1.

In this paper we will consider a class of plane closed curves.

We will assume throughout this paper that they are positively oriented.

DEFINITION 1. C^2, plane closed curves of positive curvature will be called rosettes.

A rosette can cut itself.

By s, L and k we will denote the arc length, the length and the curvature of a fixed rosette respectively.

Let us consider the rosette C, $s \mapsto z(s) = x(s) + iy(s)$, $s \in [0, L]$. The tangent and normal vectors to C at the point $z(s)$ are denoted by $T(s)$ and $N(s)$, respectively. Let us fix $a \in (0, L)$ and $\alpha \in (0, 2\pi)$ such that $T(a) = e^{i\alpha} T(0)$.

Let

$$K(s) = \int_0^s k(r)dr.$$

Thus we have $K(s) = \arg T(s) - \arg T(0)$.

The following conditions are equivalent

$$K(\varphi(s)) = K(s) + K(a), \tag{1}$$

$$\begin{cases} K'(\varphi(s))\varphi'(s) = K'(s) \\ K(\varphi(0)) = K(a), \end{cases} \tag{2}$$

$$\begin{cases} k \circ \varphi \cdot \varphi' = k \\ \varphi(0) = a. \end{cases} \tag{3}$$

Let us note that the condition (1) is equivalent to

$$\arg T(\varphi(s)) - \arg T(s) = \arg T(a) - \arg T(0) = \alpha.$$

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Hence we have

$$T(\varphi(s)) = e^{i\alpha}T(s) \quad \text{for } s \in [0, L]. \tag{4}$$

With respect to this relation we will say that φ does not change the angle α.

2. Orthodiameter pairs.

Let ψ denote the solution of (3) where a is the smallest number which satisfies the condition $T(a) = -T(0)$. Thus we have

$$T \circ \psi = -T. \tag{5}$$

We introduce the functions and vectors:

$$\begin{aligned} p &= z - z \circ \psi \\ \delta &= -\langle p, N \rangle, \\ \Delta &= \langle p, T \rangle, \\ 2\lambda &= \frac{1}{k} + \frac{1}{k \circ \psi}, \end{aligned} \tag{6}$$

where $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product in the plane. The function λ will be called a mean radius of curvature. From the formulas (6) we immediately get the conditions

$$p' = 2\lambda k T, \tag{7}$$

$$\delta' = k\Delta. \tag{8}$$

DEFINITION 2. A pair of points of a rosette which lie on the same normal line will be called an orthodiameter pair.

THEOREM 1. *Each rosette has at least one orthodiameter pair.*

PROOF. We have

$$\oint k\Delta ds = \oint \delta' ds = 0.$$

Thus the function Δ has at least two zeros. They determine the orthodiameter pair.

REMARK. The above result is known for convex figures (see [4, Problem 6]).

The ellipse with unequal axes has exactly two orthodiameter pairs.

The formula (8) implies the following characterization of ovals with constant width (see [2], [3], [4]). We have

$$\text{constant width} \Leftrightarrow \langle p, N \rangle = \text{const} \Leftrightarrow \delta' = 0 \Leftrightarrow \Delta = 0.$$

Thus we obtain

THEOREM 2. *If each point of an oval C belongs to an orthodiameter pair, then C has a constant width.*

The inverse statement is well known (see [4]).

3. The α-podic points.

DEFINITION 3. A pair of points of a rosette such that
- tangent lines at these points form the oriented angle α,
- curvatures at these points are equal,

will be called the α-podic pair.

Thus an antipodal pair of an oval (see [2]) is a π-podic pair.

Let φ denote the solution of (3) which does not change the angle α and let

$$\begin{aligned} &\varphi^n = \varphi \circ \ldots \circ \varphi \qquad n = 1,2,\ldots \\ &\sigma = \frac{1}{k} - \frac{1}{k \circ \varphi}, \\ &\xi = \varphi - \mathrm{id}. \end{aligned} \tag{9}$$

We have

$$\int k(s)\sigma(s)ds = s - \varphi(s) + \text{const}, \tag{10}$$

$$\int k(s)\sigma(\varphi^n(s))ds = \varphi^n(s) - \varphi^{n+1}(s) + \text{const}, \quad n = 1,2,3,\ldots \tag{11}$$

Really, let $s = \varphi(t)$. Then with respect to (3) we have $k(s)ds = k(t)dt$. Hence we get

$$\int f(s)k(s)ds = \int f(\varphi(t))k(t)dt \tag{12}$$

for an arbitrary continuous function f.

Let us take $f(s) = 1/k(\varphi^n(s))$. Then we obtain

$$\int \frac{k(s)}{k(\varphi^n(s))} ds = \int \frac{k(t)}{k(\varphi^{n+1}(t))} dt.$$

Hence we get

$$\int \frac{k(s)}{k(\varphi^n(s))} ds = \varphi^n(s) + \text{const}.$$

It immediately implies (10) and (11).

THEOREM 3. *Each rosette has at least three α-podic pairs for an arbitrary $\alpha \in (0, \pi)$.*

PROOF. The formula (10) implies the equality $\oint k(s)\sigma(s)ds = 0$. σ has at least two zeros $a,b \in [0, L]$ with $a < b$. Let us assume that σ has exactly two zeros. Making use of (11) we obtain $\oint k(s)\sigma(\varphi(s))ds = 0$. Thus $\varphi(a) = b$ and $\varphi(b) = a + L$. It means that the tangent lines at the points a,b are parallel. In this way we obtain a contradiction.

The above theorem can be considered as a prolongation of Blaschke–Süss theorem (see [2]).

Now, we will give a characterization of points of an α-podic pair.

Let us note that $\xi(s)$ denotes the length of an arc contained between the points s and $\varphi(s)$. The formulas (3) and (9) imply the relation

$$\xi' = -k\sigma. \tag{13}$$

Thus the extremes of ξ can only be at points of an α-podic pair.

THEOREM 4. *For an oval the following conditions are equivalent:*

1° *an oval has a center of symmetry,*

2° $\xi(s) = \psi(s) - s \equiv \frac{1}{2}L$,

3° *each of its points belongs to a π-podic pair.*

PROOF. Let $2w = z \circ \psi + z$.

3° ⇒ 2°. $\sigma \equiv 0$ implies $\xi' \equiv 0$. Thus we have

$$0 = \int_a^{\psi(a)} \xi'(s)ds = L - 2\xi(a)$$

for an arbitrary $a \in [0, L]$.

1° ⇒ 2°. We have $0 = 2w' = T \circ \psi \cdot \psi' + T$. Hence we get $\psi' \equiv 1$ and $\psi(s) = s + \frac{1}{2}L$.

2° ⇒ 1°. w reduces to a point because $w' = 0$.

2° ⇒ 3°. $\xi' \equiv 0$ implies $\sigma \equiv 0$.

4. Rosettes with a constant mean radius of curvature.
We will prove some integral formula for rosettes.

THEOREM 5. *The following formula holds for a rosette:*

$$\oint k(s)\lambda(s)ds = L. \tag{14}$$

PROOF. Making use of (3) and (6) we obtain

$$\oint k(s)\lambda(s)ds = \tfrac{1}{2}\oint(1+\psi'(s))ds = L.$$

The relation (14) implies

THEOREM 6. *The perimeter of a rosette which has a constant mean radius of curvature $\lambda(s) \equiv c$ and the index j is equal to πcj.*

REMARK. Index $j = 1/2\pi(K(L))$, (see [1]).

With respect to the relation $\Delta' = 2k\lambda - k\delta$ we get the following implication

$$\text{constant width} \Leftrightarrow \Delta = 0 \Rightarrow \Delta' = 0 \Leftrightarrow \delta = 2\lambda.$$

Thus in the particular case Theorem 6 reduces to the Barbier theorem (see [2]).

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