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# DOUBLE POINTS OF COMPOSITIONS OF PROJECTIONS

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## 0. Introduction.

Let $X \subseteqq \boldsymbol{P}^M$ be a variety and let $\pi_L : X \to \boldsymbol{P}^N$ be the linear projection with center $L$.

Double point and ramification schemes for $\pi_L$ are defined by the following construction (see [5], [7]).

Let $G$ denote the Grassmannian of lines in $\boldsymbol{P}^M$ and let $\Sigma_L \subseteqq G$ denote the Schubert variety of lines in $\boldsymbol{P}^M$ intersecting $L$. Let $\Delta_X$ denote the diagonal of $X \times X$. There is a morphism $X \times X \setminus \Delta_X \to G$ which maps a pair of points to the line they span. Let $(X \times X)^\sim$ be the closure in $X \times X \times G$ of the graph of this morphism. Let $g : (X \times X)^\sim \to G$ be the projection on the Grassmannian. Let $P(X)$ be the inverse image of $\Delta_X \subset X \times X$ by the projection $\pi : (X \times X)^\sim \to X \times X$. The *double point scheme* $\tilde{D}(\pi_L) \subseteqq (X \times X)^\sim$ is defined to be $g^{-1}\Sigma_L$. One has that points in $\tilde{D}(\pi_L)$ are those pairs $(x_1, x_2)$ with $x_1 \neq x_2$ and $\pi_L(x_1) = \pi_L(x_2)$, together with those tangent directions in $P(X)$ on which the induced tangent map vanishes. *The ramification scheme* $\tilde{R}(\pi_L)$ is defined to be $P(X) \cap \tilde{D}(\pi_L)$.

$$\begin{array}{ccccc}
(X \times X)^\sim & & \xrightarrow{\quad g \quad} & & G \\
\cup\!\!| & \mathbin{\rotatebox[origin=c]{-45}{$\supseteqq$}} & & & \mathbin{\rotatebox[origin=c]{-45}{$\supseteqq$}} \\
P(X) & & g^{-1}\Sigma_L = \tilde{D}(\pi_L) & \longrightarrow & \Sigma_L \\
& \mathbin{\rotatebox[origin=c]{-45}{$\supseteqq$}} & \cup\!\!| & & \\
& & P(X) \cap g^{-1}\Sigma_L = \tilde{R}(\pi_L) & &
\end{array}$$

Define the double point class $\boldsymbol{D}(\pi_L)$ and the ramification class $\boldsymbol{R}(\pi_L)$ in $A.(X)$ to be

$$\boldsymbol{D}(\pi_L) = (\mathrm{pr}_1 \circ \pi)_* \left([\tilde{D}(\pi_L)]\right)$$
$$\boldsymbol{R}(\pi_L) = (\mathrm{pr}_1 \circ \pi)_* \left(\tilde{R}(\pi_L)]\right),$$

where $\mathrm{pr}_1 : X \times X \to X$ is projection on the first factor and $[\tilde{D}(\pi_L)]$ (respectively $[\tilde{R}(\pi_L)]$) is the rational equivalence class of $\tilde{D}(\pi_L)$ (respectively $\tilde{R}(\pi_L)$).

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Let $L' \supsetneqq L$ be linear subspaces such that $L'$ does not meet $X$. A point in $\tilde{D}(\pi_{L'})$ can be thought of as a line $l$ with two points $x_1, x_2 \in X$ on it such that $l$ meets $L'$.

The morphism

$$(\pi_L \times \pi_L) \circ \pi : (X \times X)^{\sim} \to X \times X \to \mathbf{P}^N \times \mathbf{P}^N$$

maps such a point $(x_1, x_2, l) \in \tilde{D}(\pi_{L'})$ to $(\pi_L(x_1), \pi_L(x_2)) \in \mathbf{P}^N \times \mathbf{P}^N$. The condition that $\pi_L(x_1) = \pi_L(x_2)$ is equivalent to either

(1) $\qquad x_1 = x_2, \quad \text{i.e. } (x_1, x_2, l) \in \tilde{R}(\pi_{L'})$

or

(2) $\qquad l$ meets $L$, $\quad$ i.e. $(x_1, x_2, l) \in \tilde{D}(\pi_L)$.

This gives the set-theoretic identity:

(*) $$\tilde{D}(\pi_{L'}) \cap ((\pi_L \times \pi_L) \circ \pi)^{-1} (\Delta_{\mathbf{P}^N}) = \tilde{D}(\pi_L) \cup \tilde{R}(\pi_{L'}).$$

In the next section we will give scheme-theoretic and cycle-theoretic versions of (*) and finally an identity in rational equivalence among the double point and ramification classes of $\pi_L$ and $\pi_{L'}$.

In particular, if $\pi_L : X \to \mathbf{P}^N$ and $\pi_{L'} = \pi_P \circ \pi_L : X \to \mathbf{P}^{N-1}$ are generic linear projections where $P$ is a point, we obtain the formula

$$\mathbf{D}(\pi_{L'}) \cdot h = \mathbf{D}(\pi_L) + \mathbf{R}(\pi_{L'})$$

in $A.(X)$, where $h$ is the class of a hyperplane.

This formula was proved by K. Johnson in [5], where he also gives applications.

All schemes are projective defined over an algebraically closed field. A variety is a reduced and irreducible scheme.

## 1. Results.

*A. Scheme-theoretic version.* With notation as above, consider the following diagram:

$$\begin{array}{ccc}
 & & \tilde{D}(\pi_{L'}) \\
 & & \cap \\
((\pi_L \times \pi_L) \circ \pi)^{-1}(\Delta_{\mathbf{P}^N}) & \subset & (X \times X)^{\sim} \\
\Big\downarrow & & \Big\downarrow \pi \\
 & & X \times X \\
 & & \Big\downarrow \pi_L \times \pi_L \\
\Delta_{\mathbf{P}^N} & \subset & \mathbf{P}^N \times \mathbf{P}^N
\end{array}$$

THEOREM 1. *Let* $X \subseteqq \boldsymbol{P}^M$ *be a variety,* $L \subset L'$ *linear subspaces of* $\boldsymbol{P}^M$, $L' \cap X = \varnothing$, *and let* $\pi_L, \pi_{L'}$ *denote the corresponding linear projections of* $X$. *Then*

i) $\big(\tilde{D}(\pi_{L'}) \cap ((\pi_L \times \pi_L) \circ \pi)^{-1}(\Delta_{\boldsymbol{P}^N})\big) \cap P(X) = \tilde{R}(\pi_{L'})$

ii) $\big(\tilde{D}(\pi_{L'}) \cap ((\pi_L \times \pi_L) \circ \pi)^{-1}(\Delta_{\boldsymbol{P}^N})\big) \setminus P(X) = \tilde{D}(\pi_L) \setminus P(X)$

*as schemes.*

PROOF. As $P(X)$ is a subscheme of $((\pi_L \times \pi_L) \circ \pi)^{-1}(\Delta_{\boldsymbol{P}^N})$ and

$$\tilde{D}(\pi_{L'}) \cap P(X) = \tilde{R}(\pi_{L'})$$

as schemes, assertion i) is obvious.

One has

$$((\pi_L \times \pi_L) \circ \pi)^{-1}(\Delta_{\boldsymbol{P}^N}) \setminus P(X) = \tilde{D}(\pi_L) \setminus P(X).$$

It suffices to prove this assertion in case $X = \boldsymbol{P}^M \setminus L$. This case can be verified directly by equations. The identity reflects the fact that two different points have the same image under projection from $L$ if and only if their secant line meets $L$. Now ii) follows by intersecting with $\tilde{D}(\pi_{L'})$ which contains $\tilde{D}(\pi_L)$.

*B. Cycle-theoretic version.*

DEFINITION. We shall call the linear projection $\pi_L : X \to \boldsymbol{P}^N$ generic if the following conditions are satisfied:

i) $\tilde{D}(\pi_L)$ is equidimensional with $\operatorname{codim}(\tilde{D}(\pi_L), (X \times X)^{\sim})$
$= \operatorname{codim}(\Sigma_L, G)$

ii) $\tilde{R}(\pi_L)$ is equidimensional with $\operatorname{codim}(\tilde{R}(\pi_L), P(X)) = \operatorname{codim}(\Sigma_L, G)$.

It is well-known ([9, Transversality lemma (1.3)]) that these conditions are satisfied for all $L$ in a non-empty Zariski open subset in the appropriate Grassmannian.

Let $P$ be the image of $L'$ under $\pi_L$. Consider the product morphism

$$\pi_P \times \pi_P : (\boldsymbol{P}^N \setminus P) \times (\boldsymbol{P}^N \setminus P) \to \boldsymbol{P}^{N-1} \times \boldsymbol{P}^{N-1}$$

where $\pi_P$ is linear projection from $P$. Define $C_P$ to be $(\pi_P \times \pi_P)^{-1}(\Delta_{\boldsymbol{P}^{N-1}})$, i.e. pairs of points collinear with $P$. $C_P$ is irreducible and smooth and $\Delta = \Delta_{\boldsymbol{P}^N} \setminus (P,P) \subseteqq C_P$ has codimension 1.

The morphism $(\pi_L \times \pi_L) \circ \pi : (X \times X)^{\sim} \to \boldsymbol{P}^N \times \boldsymbol{P}^N$ maps

$$((\pi_{L'} \times \pi_{L'}) \circ \pi)^{-1}(\Delta_{\boldsymbol{P}^{N-1}}) = ((\pi_L \times \pi_L) \circ \pi)^{-1}(\pi_P \times \pi_P)^{-1}(\Delta_{\boldsymbol{P}^{N-1}})$$

to $(\pi_P \times \pi_P)^{-1}(\Delta_{\boldsymbol{P}^{N-1}}) = C_P$.

The restriction of $(\pi_L \times \pi_L) \circ \pi : (X \times X)^{\sim} \to \boldsymbol{P}^N \times \boldsymbol{P}^N$ to $\tilde{D}(\pi_{L'})$ is therefore a morphism

$$\alpha : \tilde{D}(\pi_{L'}) \to C_P$$

and $\alpha^{-1}(\Delta) = \tilde{D}(\pi_{L'}) \cap \big((\pi_L \times \pi_L) \circ \pi\big)^{-1}(\Delta_{P^N})$.

THEOREM 2. *Let $X \subsetneqq \boldsymbol{P}^M$ be a variety, and let $\pi_L$ and $\pi_{L'}$ denote generic linear projections from linear subspaces $L \subset L'$ with* $\dim L' = \dim L + 1$. *Then*

$$\alpha^*[\Delta] = [\tilde{D}(\pi_L)] + [\tilde{R}(\pi_{L'})]$$

*as cycles on $\tilde{D}(\pi_{L'})$.*

PROOF. From Theorem 1 and the above formula for $\alpha^{-1}(\Delta)$ we have

$$\begin{aligned}[\alpha^{-1}(\Delta)] &= [\tilde{D}(\pi_{L'}) \cap \big((\pi_L \times \pi_L) \circ \pi\big)^{-1}(\Delta_{P^N})] \\ &= [\tilde{D}(\pi_L)] + [\tilde{R}(\pi_{L'})]\end{aligned}$$

as cycles because no component of $\tilde{D}(\pi_L)$ is contained in $P(X)$ since $\pi_L$ is generic.

As $\Delta \subset C_P$ is of codimension 1,

$$[\alpha^{-1}(\Delta)] = \alpha^*[\Delta].$$

*C. In rational equivalence.* Let $A_.(X)$ denote the Chow homology group, i.e., the group of algebraic cycles on $X$ modulo rational equivalence on $X$, see [2].

The space $C_P$ is the total space of the bundle

$$p : \mathcal{O}_{P^{N-1}}(1) \oplus \mathcal{O}_{P^{N-1}}(1) \to \boldsymbol{P}^{N-1}.$$

LEMMA. *The class of $\Delta$ in $A^1(C_P)$ is the first Chern class of $p^*(\mathcal{O}_{P^{N-1}}(1))$.*

PROOF. The projections

$$\mathcal{O}_{P^{N-1}}(1) \oplus \mathcal{O}_{P^{N-1}}(1) \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \mathcal{O}_{P^{N-1}}(1)$$

on the factors give rise to two global sections $s_1, s_2$ of $p^*(\mathcal{O}_{P^{N-1}}(1))$. Then $\Delta = s^{-1}(0)$, where $s$ is the global section $s_1 - s_2$, see [3].

We have in the notation of the introduction

THEOREM 3. *Let $X \subset \boldsymbol{P}^M$ be a variety, $L \subset L'$ linear subspaces with* $\dim L' = \dim L + 1$ *and let $\pi_L$ and $\pi_{L'}$ be the corresponding linear projections. Then*

$$\boldsymbol{D}(\pi_{L'}) \cap c_1(\mathcal{O}_X(1)) = \boldsymbol{D}(\pi_L) + \boldsymbol{R}(\pi_{L'})$$

*in $A_.(X)$.*

PROOF. From Theorem 2 and the Lemma we conclude that

$$[\tilde{D}(\pi_{L'})] \cap \alpha^* c_1(p^* \mathcal{O}_{P^{N-1}}(1)) = [\tilde{D}(\pi_L)] + [\tilde{R}(\pi_{L'})]$$

in $A.(\tilde{D}(\pi_{L'}))$.

Let $\mathrm{pr}_1 : X \times X \to X$ be the projection on the first factor. Then

$$\alpha^* p^* \mathcal{O}_{P^{N-1}}(1) = \pi^* \mathrm{pr}_1^* \mathcal{O}_X(1)$$

on $\tilde{D}(\pi_{L'})$.

Therefore

$$[\tilde{D}(\pi_{L'})] \cap \pi^* \mathrm{pr}_1^* c_1(\mathcal{O}_X(1)) = [\tilde{D}(\pi_L)] + [\tilde{R}(\pi_{L'})]$$

and

$$(\mathrm{pr}_1 \circ \pi)_* [\tilde{D}(\pi_{L'})] \cap c_1(\mathcal{O}_X(1)) = (\mathrm{pr}_1 \circ \pi)_* [\tilde{D}(\pi_L)] + (\mathrm{pr}_1 \circ \pi)_* [\tilde{R}(\pi_{L'})].$$

**2. Remarks.**

a. A relation of the same type as in Theorem 3 goes back to Severi [10]. Specifically, let $X \subset \boldsymbol{P}^{2n}$ be an $n$-dimensional variety with a finite number, $d$, of transversal double points. Let $P \in \boldsymbol{P}^{2n}$ and $H$ be a hyperplane containing $P$. Then he proved that for generic $P$ and $H$,

$$2\tilde{d} = 2d + \omega_n,$$

where $\tilde{d}$ is the number of transversal double points for the image $\pi_P(X \cap H)$ of the hyperplane section $X \cap H$, and $\omega_n$ is the number of $n$-planes, tangent to $X$ at smooth points, which contains $P$. Catanese [1] has given a modern account of Severi's work.

b. The first author has for morphisms $f: X \to Y$ and $g: Y \to V$ given a scheme theoretic relation among the double points of $f$ and the ramification and double points of $g \circ f: X \to V$, see [4].

Let $X, Y, V$ be quasi-projective schemes defined over an algebraically closed field and let $f: X \to Y$ and $g: Y \to V$ be morphisms. Following Laksov [8], we have double-point $Z(f)$, $Z(g \circ f)$ and ramification schemes $R(g \circ f)$ of $f$ and $g \circ f$. They are subschemes of $(X \times X)^\sim$, the blow-up of $X \times X$ along the diagonal, and $Z(f)$, $R(g \circ f)$ are subschemes of $Z(g \circ f)$.

Consider the composition

$$(f \times f) \circ \pi : (X \times X)^\sim \to X \times X \to Y \times Y,$$

where $\pi : (X \times X)^\sim \to X \times X$ is the blow-up morphism. The composition $(f \times f) \circ \pi$ maps the double-point scheme $Z(g \circ f) \subseteqq (X \times X)^\sim$ into $Y \times_V Y$. Let

$$h: Z(g \circ f) \to Y \times_V Y$$

denote the restriction of $(f \times f) \circ \pi$.

The main result of [4] is the following

THEOREM. *Let* $\operatorname{Im}(Z(f) \amalg R(g \circ f))$ *denote the subscheme of* $Z(g \circ f)$ *defined by the product of the defining ideals of* $Z(f)$ *and* $R(g \circ f)$ *in* $Z(g \circ f)$. *The diagram*

$$\begin{array}{ccc} Z(g \circ f) & \xrightarrow{h} & Y \times_V Y \\ \cup & & \cup \\ \operatorname{Im}(Z(f) \amalg R(g \circ f)) & \to & \Delta_Y \end{array}$$

*is cartesian.*

The last author has treated double points of compositions of linear projections in his Master's Thesis [11].

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