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ON THE GROWTH OF ALGEBROID SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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1. Introduction.

It has been shown by Bank [1] that the growth of meromorphic solutions of linear differential equations, hence of algebraic differential equations, with meromorphic coefficients cannot be estimated uniformly in terms of the growth of the coefficients alone. Later on, such uniform estimates for the growth of meromorphic solutions were developed by Bank [3] and Bank–Laine [4]. In the general situation of algebraic differential equations $\Omega(z,y) = 0$, where

$$\Omega(z,y) = \sum_{j=1}^{n} \Omega_j(z,y) = \sum_{j=1}^{n} \sum_{i_0+\ldots+i_p=j} a_i(z) y^{i_0}(y')^{i_1}\ldots(y^{(p)})^{i_p}$$

is a differential polynomial in y with meromorphic coefficients, quantities needed to obtain such an estimate for $y(z)$ are, essentially, the growth of the coefficients and the counting functions for the poles and zeros of the solution, see [3, Lemma 4] for the nonhomogeneous case $(\Omega_j(z,y(z)) \not\equiv 0$ for some $j)$ and [4, Theorem 4], [4, p. 125] for the homogeneous case $(\Omega_j(z,y(z)) \equiv 0$ for all $j)$.

An immediate question arises whether similar estimates may be found for algebroid solutions of linear and algebraic differential equations with meromorphic coefficients. The first step into this direction was taken by Xiao and He [10, Theorem 3] by generalizing [3, Lemma 4]. This paper contains the corresponding generalizations of [4, Theorem 4] and [4, Theorem 3] as well as a similar uniform estimate for the growth of algebroid solutions of the equation

$$\Omega(z,y) = R(z,y)$$

with meromorphic coefficients, $R(z,y)$ being irreducible and rational in y.

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2. Notation and main results.

All meromorphic functions to be considered here are assumed to be meromorphic in the complex plane. Respectively when considering a ν-valued algebroid function w defined by

$$\Psi(z,w) := B_\nu(z)w^\nu + B_{\nu-1}(z)w^{\nu-1} + \dots + B_0(z) = 0,$$

we always assume that the coefficients $B_j(z)$, $j = 0, \dots, \nu$, are meromorphic functions in the complex plane (and therefore we may assume them to be entire). We shall apply the usual notations and basic results of Nevanlinna theory of value distribution, see e.g. [7] for the meromorphic case and [8], [9] for the algebroid case.

We mostly consider algebraic differential equations

$$\Omega(z,y) = \sum_{i \in I} a_i(z) y^{i_0} \dots (y^{(k)})^{i_k} = 0 \tag{1}$$

with meromorphic coefficients $a_i(z)$ and with a finite set I of multi-indices $i = (i_0, \dots, i_k)$. The (total) degree of a single term of multi-index $i \in I$ in Ω is denoted by

$$|i| := i_0 + \dots + i_k$$

and its weight by

$$\|i\| := i_1 + 2i_2 + \dots + ki_k.$$

We usually write

$$\Omega(z,y) = \sum_{j=0}^{n} \Omega_j(z,y) = \sum_{j=0}^{n} \sum_{|i|=j} a_i(z) u^{i_0} \dots (y^{(k)})^{i_k}, \tag{2}$$

thus presenting the homogeneous part $\Omega_j(z,y)$ (of total degree j) of $\Omega(z,y)$ separately. For a homogeneous part $\Omega_j(z,y)$ of $\Omega(z,y)$, we denote by $A_j(z)$ the sum of all coefficients $a_i(z)$ in $\Omega_j(z,y)$ having multi-indices of maximal weight, i.e. for

$$k := \max_{|i|=j} \|i\|$$

we have

$$A_j(z) = \sum_{|i|=j, \|i\|=k} a_i(z).$$

Finally, we denote

$$\Phi(r) := \max_{i \in I} \big(\log r, T(r, a_i(z))\big).$$

We recall [10, Theorem 3] due to Xiao and He:

THEOREM A. *Let $y(z)$ be an algebroid solution of (1) such that $y(z)$ does not satisfy all homogeneous equations $\Omega_j(z,y)=0$, $j=0,\dots,n$, see (2). Then there exists a constant $K>0$ such that*

$$T(r,y) \leqq KH(r),$$

outside a possible set of finite linear measure, where

$$H(r) := \bar{N}(r,y)+\bar{N}(r,1/y)+N_3(r,y)+\Phi(r).$$

An obvious application of [2, § 2] proves the following modification of Theorem A:

THEOREM 1. *Let $y(z)$ be an algebroid solution of (1) such that $y(z)$ does not satisfy all homogeneous equations $\Omega_j(z,y)=0$, $j=1,\dots,n$, see (2). For any $\sigma>1$, there exist positive constants A and r_0 such that for all $r \geqq r_0$,*

$$T(r,y) \leqq AH(\sigma r),$$

where

$$H(r) := \bar{N}(r,y)+\bar{N}(r,1/y)+N_3(r,y)+\Phi(r).$$

Therefore, the growth of $y(z)$ can be estimated in this case uniformly in terms of the growth of the coefficients and the counting functions for the branch points, distinct poles and distinct zeros of $y(z)$.

To prove the corresponding uniform estimate in the homogeneous case, we need the following

LEMMA 2. *Let $y(z)$ be an algebroid function and denote $w = y'/y$. For any $a>1$, there exist positive constants C, C_1 and r_0 such that for all $r \geqq r_0$,*

$$T(r,y) \leqq C\big(rN(ar,y)+r^2\exp\big(C_1\Psi(ar)\big)\big),$$

where

$$\Psi(r) := T(r,w)+N(r,w)\log r+N(r,w)\log^+ N(r,w).$$

The uniform estimate corresponding to Theorem 1 in the homogeneous case now reads as follows:

THEOREM 3. *Let $y(z)$ be an algebroid solution of (1) also satisfying all homogeneous equations $\Omega_j(z,y)=0$, $j=1,\dots,n$, see (2). If for some j such that $\Omega_j \not\equiv 0$ we have $A_j(z)\not\equiv 0$, then for any $\sigma>1$ there exist positive constants C, C_1 and r_0 such that for all $r \geqq r_0$,*

$$T(r,y) \leqq C\Big(rN(\sigma r,y)+r^2\exp\big(C_1H(\sigma r)\log(rH(\sigma r))\big)\Big),$$

where

$$H(r) := \bar{N}(r, 1/y) + \bar{N}(r,y) + N_3(r,y) + \Phi(r).$$

Therefore the growth of $y(z)$ can be estimated in this case uniformly in terms of the growth of the coefficients and the counting functions for the branch points, poles and distinct zeros of $y(z)$.

An immediate corollary to Theorem 1 and Theorem 3 concerns with the special case of linear differential equations.

THEOREM 4. *Let $y(z)$ be an algebroid solution of linear differential equation*

$$\sum_{j=0}^{n} f_j(z) y^{(j)} = f(z)$$

with meromorphic coefficients. Denote

$$\Theta(r) := \max\bigl(\log r, T(r,f), T(r,f_0), \ldots, T(r,f_n)\bigr).$$

(A) *If $f(z) \not\equiv 0$, then for any $\sigma > 1$, there exist positive constants A and r_0 such that for all $r \geqq r_0$,*

$$T(r,y) \leqq AH_1(\sigma r),$$

where

$$H_1(r) := \bar{N}(r, 1/y) + N_3(r,y) + \Theta(r).$$

(B) *If $f(z) \equiv 0$, then for any $\sigma > 1$, there exist positive constants C, C_1 and r_0 such that for all $r \geqq r_0$,*

$$T(r,y) \leqq C\Bigl(rN(\sigma r, 1/y) + r^2 \exp\bigl(C_1 H_1(\sigma r) \log(rH_1(\sigma r))\bigr)\Bigr),$$

where

$$H_1(r) := \bar{N}(r, 1/y) + N_3(r,y) + \Theta(r).$$

Our final result concerns with the differential equation

$$\Omega(z,y) = R(z,y), \tag{3}$$

where $\Omega(z,y)$ is defined by (2),

$$R(z,y) = \sum_{k=0}^{p} a_k(z) y^k \Big/ \sum_{j=0}^{q} b_j(z) y^j$$

is an irreducible rational function in y with meromorphic coefficients and $p > q + \lambda$ where $\lambda = \max_{i \in I} |i|$. We now get

THEOREM 5. *Let $y(z)$ be an algebroid solution of (3) which satisfies the above conditions. For any $\sigma > 1$, there exist positive constants K and r_0 such that for all $r \geqq r_0$,*

$$T(r,y) \leqq KF(\sigma r),$$

where

$$F(r) := \bar{N}(r,y) + \Sigma(r),$$

$$\Sigma(r) := \Phi(r) + \sum_{k=0}^{p} T(r,a_k) + \sum_{j=0}^{q} T(r,b_j).$$

3. Proof of Lemma 2.

This lemma generalizes [2, Lemma 7] into the algebroid case. Our proof applies the same basic idea as proof of [2, Lemma 7]. Therefore some details may be taken from [2] and will be omitted here.

Suppose $y(z)$ is a ν-valued algebroid function. Its logaritmic derivative $w(z) = y'(z)/y(z)$ may also be considered as a ν-valued algebroid function. Therefore, let $B_0(z), \ldots, B_\nu(z)$ be entire functions such that

$$B_\nu(z)w^\nu + B_{\nu-1}(z)w^{\nu-1} + \ldots + B_0(z) = 0$$

and denote

$$f_j(z) := B_j(z)/B_\nu(z), \ \ j = 0, \ldots, \nu - 1.$$

Let $\{a_{n,j}\}$ and $\{b_{m,j}\}$ be the zeros and poles, respectively, of $f_j(z)$, each arranged in order of increasing moduli. Moreover, let $\{\alpha_i\}$, $\{\beta_k\}$ and $\{\gamma_l\}$ denote, respectively, the sequence of zeros, poles and branch points of w, each arranged again in order of increasing moduli. Clearly, $\{\beta_k\} \subseteqq \bigcup_j \{b_{m,j}\}$ and $\{\alpha_i\} \subseteqq \{a_{n,0}\}$. By the Poisson–Jensen formula we get

$$\log|f_j(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left| f_j(\mathrm{Re}^{i\theta}) \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \varphi)}\right| d\theta - \\ - \sum_{|a_{n,j}| < R} \log\left|\frac{R^2 - \bar{a}_{n,j}z}{R(z - a_{n,j})}\right| + \sum_{|b_{m,j}| < R} \log\left|\frac{R^2 - \bar{b}_{m,j}z}{R(z - b_{m,j})}\right|,$$

where $z = re^{i\varphi}$, $z \notin \{a_{n,j}\}$, $z \notin \{b_{m,j}\}$ and $R = \sigma r$ with $\sigma^3 = a$. Similarly as in [2], we further get

$$(4) \qquad \log|f_j(z)| \leqq \frac{\sigma+1}{\sigma-1} m(\sigma r, f_j) + \sum \log(2\sigma r) + \sum \log \frac{1}{|r - |b_{m,j}||},$$

where sums extend over all poles $b_{m,j}$ such that $|b_{m,j}| < \sigma r$. Following [2, p. 59–60] we may assume that the sequence $\{\beta_k\}$ is non-empty and we may find $k_0 \in \mathbf{N}$ such that $\sum_{k>k_0} \tau_k$, where

$$\tau_k := (N(\sigma|\beta_k|, w))^{-\sigma},$$

converges. Therefore the set

$$E' := [0, |\beta_{k_0}| + 1] \cup \bigcup_{k=k_0}^{\infty} [|\beta_k| - \tau_k, |\beta_k| + \tau_k]$$

is of finite linear measure. Suppose $r \notin E'$. Then $|r - |\beta_k|| > \tau_k$ for $k > k_0$ and $|r - |\beta_k|| \geqq 1$ for $k \leqq k_0$. Therefore, for all $b_{m,j}$ satisfying $|b_{m,j}| < \sigma r$ there exists $\beta_{k_{m,j}} \in \{\beta_k\}$ such that $b_{m,j} = \beta_{k_{m,j}}$ and so

$$\begin{aligned} \log|r - |b_{m,j}||^{-1} &= \log|r - |\beta_{m,j}||^{-1} \leqq \log \tau_{m,j}^{-1} \\ &= \sigma \log N(\sigma|\beta_{m,j}|, w) \leqq \log^+ N(\sigma^2 r, w) \end{aligned}$$

holds. Clearly, there are at most $n(\sigma r, f_j)$ terms in the two sums of (4). Therefore, if $r \notin E'$, then

$$\log|f_j(z)| \leqq \frac{\sigma+1}{\sigma-1} m(\sigma r, f_j) + n(\sigma r, f_j) \log 2\sigma r + \sigma n(\sigma r, f_j) \log^+ N(\sigma^2 r, w).$$

By [8, p. 716, (16)] we know that $n(\sigma r, f_j) \leqq n(\sigma r, w)$. Hence

$$\log|f_j(z)| \leqq \frac{\sigma+1}{\sigma-1} T(\sigma r, f_j) + n(\sigma r, w) \log 2\sigma r + \sigma n(\sigma r, w) \log^+ N(\sigma^2 r, w)$$

for $|z| = r \notin E'$. Clearly, this inequality holds for all $j = 0, \ldots, n$, if we assume that

$$|z| = r \notin E := \{\alpha_i\} \cup \{\gamma_l\} \cup E'.$$

By [8, p. 716, (17)] we find a constant $\gamma \in \mathbf{R}$ such that

$$T(r, f_j) \leqq T(r, w) + \gamma$$

holds for all $j = 1, \ldots, v$. Therefore

$$\begin{aligned} \log|f_j(z)| \leqq \frac{\sigma+1}{\sigma-1}(vT(\sigma r, w) + \gamma) + n(\sigma r, w) \log 2\sigma r + \\ + \sigma n(\sigma r, w) \log^+ N(\sigma^2 r, w), \end{aligned}$$

where the right hand side is independent of j. Let w_j be any determination of $w, j = 1, \ldots, \nu$, and denote

$$B(z) := \max(|B_0(z)|, \ldots, |B_\nu(z)|).$$

From

$$|w_j(z)| = (|B_\nu(z)|\,|w_j(z)|^{\nu-1})^{-1}\,|B_{\nu-1}(z)(w_j(z))^{\nu-1} + \ldots + B_0(z)|$$

we immediately get

$$|w_j(z)| \leqq \nu B(z)|B_\nu(z)|^{-1}.$$

Therefore

$$\log|w_j(z)| \leqq \log\nu + \log\frac{B(z)}{|B_\nu(z)|} = \log\nu + \max_{0\leqq j\leqq\nu}\log\frac{|B_j(z)|}{|B_\nu(z)|}$$

$$= \log\nu + \max_{0\leqq j\leqq\nu}\log|f_j(z)|.$$

Hence, for some $\delta \in \mathsf{R}$,

$$\log|w_j(z)| \leqq V(r)$$

(5)

$$:= \frac{\sigma+1}{\sigma-1}\nu T(\sigma r, w) + n(\sigma r, w)\log 2\sigma r + \sigma n(\sigma r, w)\log^+ N(\sigma^2 r, w) + \delta$$

holds for all $j = 1, \ldots, \nu$.

Let now $\varepsilon > 0$ be such that $y(z)$ has no zeros or poles on $0 < |z| \leqq \varepsilon$. By the Jensen formula [9, p. 203, (20)] there exists a constant $\lambda_0 > 0$ such that for all $r > 0$,

$$T(r, 1/y) = T(r, y) + h(r),$$

where $|h(r)| \leqq \lambda_0$. Denote then $b := \nu^{-1}n(0, y) + \nu^{-1}n(0, 1/y)$.

We next prove that on $|z| = r$ we have

(6) $$\log^+|y_j(z)| \leqq B(r)$$

for all determinations $y_j(z)$ of $y(z)$ and all $r \notin E$, where

(7) $$B(r) := \frac{r}{\nu\varepsilon}(2n(r, y) + re^{V(r)}) + \lambda_0 + b\log r + 2\pi r e^{V(r)}.$$

Suppose that (7) does not hold. Then there exists at least one $y_j(z)$, $r \notin E$ and $z_0 = re^{i\theta_0}$ such that $\log^+|y_j(z_0)| > B(r) > 0$. Therefore

$$\log|y_j(z_0)| > B(r) > B(r) - 2\pi r e^{V(r)} > 0.$$

Let now $z_1 = re^{i\theta_1}$ be an arbitrary point on $|z| = r$ distinct from z_0 and let Γ_j be the arc on $|z| = r$ joining z_0 and z_1 counterclockwise. Since $r \notin E$, $w(z)$ has no poles, no zeros and no branch points on $|z| = r$. Therefore, $y_j(z)$ is analytic and nowhere zero on some simply-connected neighbourhood of Γ_j. Hence we can take an analytic branch of $\log y_j(z)$ on this neighbourhood and

$$\log y_j(z_0) - \log y_j(z_1) = \int_{\Gamma_j} w_j(\zeta)d\zeta$$

holds for this branch. Exponentiation now gives, together with (5),

$$\begin{aligned} |y_j(z_0)| &\leqq |y_j(z_1)| \exp\left|\int_{\Gamma_j} w_j(\zeta)d\zeta\right| \\ &\leqq |y_j(z_1)| \exp\left(\int_{\Gamma_j} e^{\log|w_j(\zeta)|}d|\zeta|\right) \\ &\leqq |y_j(z_1)| \exp(2\pi r e^{V(r)}) \end{aligned}$$

and this holds for all z_1 on $|z| = r$. Therefore

$$\begin{aligned} \log|y_j(z_1)| &\geqq \log|y_j(z_0)| - 2\pi r e^{V(r)} > B(r) - 2\pi r e^{V(r)} \\ &= \frac{r}{\nu\varepsilon}(2n(r,y) + re^{V(r)}) + \lambda_0 + b\log r > 0 \end{aligned} \tag{8}$$

for all z_1 on $|z| = r$. Hence

$$m(r,y_j) > B(r) - 2\pi r e^{V(r)}$$

and

$$m(r,y) = \frac{1}{\nu}\sum_{j=1}^{\nu} m(r,y_j) > B(r) - 2\pi r e^{V(r)}. \tag{9}$$

Moreover, (8) implies that $|y_j(z)| \geqq 1$ on $|z| = r$ and therefore

$$m(r,1/y) = \frac{1}{\nu}\sum_{j=1}^{\nu} m(r,1/y_j) = 0.$$

By the definitions of $N(r,y)$ and ε, we obtain

$$N(r,y) \leqq \frac{r}{\nu\varepsilon} n(r,y) + \frac{1}{\nu} n(0,y)\log r$$

and

$$N(r,1/y) \leqq \frac{r}{\nu\varepsilon} n(r,1/y) + \frac{1}{\nu} n(0,1/y)\log r.$$

Let now C_r denote the closed curve over $|z| = r$ on the Riemann surface of $y(z)$. By the argument principle we get

$$n(r, 1/y) - n(r,y) = \frac{1}{2\pi vi} \int_{C_r} w(\zeta)d\zeta.$$

In view of (5), we further obtain

$$n(r, 1/y) \leqq n(r,y) + \frac{1}{2\pi v} \sum_{j=1}^{v} \int_0^{2\pi} e^{\log|w_j(\zeta)|} |d\zeta| \leqq n(r,y) + re^{V(r)}.$$

Then

$$m(r,y) \leqq N(r,y) + N(r,1/y) + m(r,1/y) + \lambda_0 = N(r,y) + N(r,1/y) + \lambda_0$$
$$\leqq \frac{r}{v\varepsilon} n(r,y) + \frac{1}{v} n(0,y)\log r + \frac{r}{v\varepsilon} n(r,1/y) + \frac{1}{v} n(0,1/y)\log r + \lambda_0$$
$$\leqq \frac{r}{v\varepsilon} 2n(r,y) + b\log r + \lambda_0 + \frac{r^2}{v\varepsilon} e^{V(r)}$$

and this clearly contradicts (9). Therefore, (6) holds.

By (6), if $r \notin E$, then

$$m(r,y) = \frac{1}{v} \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |y_j(z)|\, d\theta \leqq B(r). \tag{10}$$

By the definition of $B(r)$, we find a $r_1 > 0$ such that

$$B(r) \leqq \frac{2r}{v\varepsilon} n(r,y) + \frac{4r^2}{\varepsilon} e^{V(r)}. \tag{11}$$

Since

$$\frac{1}{v} n(r,w) \leqq \frac{2\sigma - 1}{\sigma - 1} N(\sigma r, w), \tag{12}$$

the definition of $V(r)$ gives positive constants C_1 and r_2 such that

$$V(r) \leqq \frac{\sigma+1}{\sigma-1} vT(\sigma r, w) + v\frac{2\sigma-1}{\sigma-1} N(\sigma^2 r, w)\log(2\sigma r) +$$
$$+ v\sigma\frac{2\sigma-1}{\sigma-1} N(\sigma^2 r, w)\log^+ N(\sigma^2 r, w) + \delta \leqq C_1 \Psi(\sigma^2 r)$$

holds for all $r \geqq r_2$, $r \notin E$. Adding $N(r,y)$ to both sides of (10) and making use of (11) and (12) we obtain positive constants C_2 and r_3 such that

$$
\begin{aligned}
T(r,y) &\leqq \frac{2r}{\varepsilon}\,\frac{2\sigma-1}{\sigma-1}N(\sigma r,y)+N(r,y)+\frac{4r^2}{\varepsilon}e^{C_1\Psi(\sigma^2 r)} \\
&\leqq C_2\bigl(rN(\sigma r,y)+r^2\exp\bigl(C_1\Psi(\sigma^2 r)\bigr)\bigr)
\end{aligned} \tag{13}
$$

holds for all $r \geqq r_3$, $r \notin E$. Since both sides of (13) are nondecreasing functions, $\sigma > 1$, E is of finite linear measure and $\sigma^3 = a$, a standard reasoning (see again [2, § 2]) results in positive constants C and r_0 such that

$$T(r,y) \leqq C\bigl(rN(ar,y)+r^2\exp\bigl(C_1\Psi(ar)\bigr)\bigr)$$

holds for all values of $r \geqq r_0$.

4. Proof of Theorem 3.

An immediate corollary to Lemma 2 is the following

LEMMA 6. *Let $y(z)$ be an algebroid function and denote $w = y'/y$. Then for any $\alpha > 1$, there exist positive constants A, B and r'_0 such that for all $r \geqq r'_0$,*

$$T(r,y) \leqq A\bigl(rN(\alpha r,y)+r^2\exp\bigl(BT(\alpha r,w)\log\bigl(rT(\alpha r,w)\bigr)\bigr)\bigr).$$

We may now proceed to prove Theorem 3 by observing first that w satisfies

$$y^{(n)} = \bigl(w^n + P_{n-1}(w)\bigr)y, \tag{14}$$

where $P_{n-1}(w)$ is a polynomial in w and its derivatives of total degree at most $n-1$ with constant coefficients. Substituting (14) into $\Omega_j(z,y,\ldots,y^{(n)}) = 0$ we get

$$\Omega_j(z,\ldots,y^{(n)}) = \bigl(A_j(z)w^k + Q_{k-1}(w)\bigr)y^j = 0$$

where

$$k := \max_{|i|=j} \|i\|$$

and $Q_{k-1}(w)$ is a polynomial in w and its derivatives, of total degree at most $k-1$, with coefficients which are linear combinations of the original coefficients $a_i(z)$, $|i| = j$. Clearly we may assume

$$A_j(z)w^k + Q_{k-1}(w) = 0,$$

hence

$$m(r,A_j w) \leqq K_1 \Phi(r) + o(T(r,w)) \tag{15}$$

for some $K_1 > 0$ outside of a possible exceptional set of finite linear measure. The estimate (15) follows as an application of the Clunie lemma, see [4, Lemma 1], the remark on [6, p. 278] and the original proof in [5] for the meromorphic case. Obviously

$$N(r,A_j w) \leqq N(r,A_j) + N(r,w) \leqq K_2 \Phi(r) + \bar{N}(r,1/y) + \bar{N}(r,y) + N_3(r,y)$$

for some $K_2 > 0$. Hence

$$T(r,w) \leqq K_3 \Phi(r) + \bar{N}(r,1/y) + \bar{N}(r,y) + N_3(r,y) + o(T(r,w))$$

for some $K_3 > 0$ outside of a possible exceptional set of finite linear measure. Therefore there exist $K > 0$ and $r_2 \geqq r_0'$ (from Lemma 6) such that, given $\beta > 1$,

$$T(r,w) \leqq K\Phi(\beta r) + \bar{N}(\beta r, 1/y) + \bar{N}(\beta r, y) + N_3(\beta r, y)$$

holds for all $r \geqq r_2$. The conclusion of Theorem 3 now follows from Lemma 6 by choosing $C = A$, C_1 conveniently $\geqq 2BK$, $r_0 \geqq r_2$, and $\alpha\beta \leqq \sigma$.

5. Proof of Theorem 4.

Before proceeding to prove Theorem 4 we should perhaps present some examples to show that algebroid functions may satisfy homogeneous linear differential equations with meromorphic coefficients. We list here four such examples:

(1) The 2-valued algebraic function defined by

$$zy^2 - 1 = 0$$

satisfies the linear differential equation

$$2zy' + y = 0.$$

(2) The ν-valued algebroid function defined by

$$(\sin z) y^\nu - 1 = 0$$

satisfies the linear differential equation

$$y' + \frac{1}{\nu}(\cot z) y = 0.$$

(3) The ν-valued algebroid function defined by

$$y^\nu - \sin z = 0$$

satisfies the linear differential equation

$$y'' + \frac{1}{\nu}(1 + (1 - 1/\nu)\cot^2 z)y = 0.$$

(4) The Bessel function $y = J_m(z)$ with rational order $m = j/\nu$, where j, ν are mutually prime, is a ν-valued algebroid function satisfying

$$z^2 y'' + zy' + (z^2 - j^2/\nu^2)y = 0,$$

see [6, p. 277].

To prove now Theorem 4, we observe at once that $y(z)$ cannot have a pole at point z_0 where all coefficients $f_j(z)$ take finite, non-zero values. Therefore there exist positive constants K' and r' such that

$$\bar{N}(r,y) \leqq K'\Phi(r)$$

for all $r \geqq r'$. The assertion (A) (respectively (B)) follows from Theorem 1 (respectively Theorem 3) by adjusting if needed, the positive constants A and r_0 in Theorem 1 (respectively C, C_1 and r_0 in Theorem 3).

6. Proof of Theorem 5.

Writing

$$\Omega(z,y) = R(z,y) = \frac{P(z,y)}{Q(z,y)}$$

in the form

$$Q(z,y)\Omega(z,y) = P(z,y) \tag{16}$$

we see at once by the Clunie lemma and the assumption $p > q + \lambda$ that

$$m(r,y) = O(\Sigma(r)) + S(r,y).$$

On the other hand, poles of $P(z,y(z))$ may rise from the poles of $y(z)$ and of the coefficients of $P(z,y(z))$ only. By (16), the same conclusion is true for the poles of $Q(z,y(z))\,\Omega(z,y(z))$, i.e. they may rise from the poles of $y(z)$ and the poles of the coefficients of $Q(z,y(z))$ and of $\Omega(z,y(z))$ only. Clearly the poles of $\Omega(z,y(z))$ which rise from the poles of $a_i(z)$ contribute $\leqq \sum_{i\in I} N(r,a_i)$ to $N(r,\Omega(z,y(z)))$. Moreover, the poles of $\Omega(z,y(z))$ which rise from the poles of $y(z)$ contribute $\leqq \lambda N(r,y) + \bar{\mu}\nu\bar{N}(r,y)$ to $\bar{N}(r,\Omega(z,y(z)))$, where

$$\bar{\mu} := \max_{i \in I} \left(\sum_{\alpha=1}^{n} \alpha i_\alpha \right).$$

To prove this fact, let z_0 be a pole of $y(z)$, where γ determinations of $y(z)$ have a pole at z_0 of multiplicity $n(z_0, y)$, i.e.

$$\begin{cases} y(z) = (z - z_0)^{-\frac{n(z_0,y)}{\gamma}} g(z), & g(z_0) \neq 0, \infty \\ y^{(\alpha)}(z) = (z - z_0)^{-\frac{n(z_0,y)+\alpha\gamma}{\gamma}} g_\alpha(z), & g_\alpha(z_0) \neq 0, \infty \end{cases}$$

holds. Therefore, the multiplicity of the pole of $\Omega(z, y(z))$ at z_0 is

$$\begin{aligned} & n(z_0, \Omega(z, y(z))) = \\ & \max_{i \in I} \big(i_0 n(z_0, y) + i_1(n(z_0, y) + \gamma) + \ldots + i_n(n(z_0, y) + n\gamma)\big) \\ & \leqq \lambda n(z_0, y) + \bar{\mu} v \;. \end{aligned}$$

and therefore

$$N(r, \Omega(z, y(z))) \leqq \lambda N(r, y) + \bar{\mu} v \bar{N}(r, y) + O(\Sigma(r)). \tag{17}$$

From [6, p. 278] and (17) we now obtain

$$\begin{aligned} pN(r, y) + O(\Sigma(r)) = N(r, P(z, y)) &= N(r, Q(z, y)\, \Omega(z, y)) \\ &\leqq N(r, Q(z, y)) + N(r, \Omega(z, y)) \\ &\leqq (q + \lambda) N(r, y) + \bar{\mu} v \bar{N}(r, y) + O(\Sigma(r)) \end{aligned}$$

and since $p - q - \lambda \geqq 1$,

$$N(r, y) \leqq \frac{\bar{\mu} v}{p - q - \lambda} \bar{N}(r, y) + O(\Sigma(r)). \tag{18}$$

From (16) and (17) we further get

$$T(r, y) = m(r, y) + N(r, y) \leqq \frac{\bar{\mu} v}{p - q - \lambda} \bar{N}(r, y) + K' \Sigma(r) + S(r, y)$$

for some $K' > 0$. The assertion of Theorem 5 now follows by standard reasoning [2, § 2].

7. A final remark.

In the same way as in [4, p. 125] we may determine the quantities which are needed to get a uniform estimate for the growth of algebroid solutions of algebraic differential equations (1). By Theorem 1 and Theorem 3, this concerns that case only, where $y(z)$ satisfies all homogeneous equations $\Omega_j(z, y) = 0$ and where $A_j(z) \equiv 0$ for all $j = 0, \ldots, n$.

REFERENCES

1. S. Bank, *A note on algebraic differential equations whose coefficients are entire functions of finite order*, Ann. Scuola Norm. Sup. Pisa 26 (1972), 291–297.
2. S. Bank, *On determining the growth of meromorphic solutions of algebraic differential equations having arbitrary entire coefficients*, Nagoya Math. J. 49 (1973), 53–65.
3. S. Bank, *On the growth of meromorphic solutions of linear differential equations having arbitrary entire coefficients*, Ann. Mat. Pura Appl. 107 (1975), 279–289.
4. S. Bank and I. Laine, *On the growth of meromorphic solutions of linear and algebraic differential equations*, Math. Scand. 40 (1977), 119–126.
5. J. Clunie, *On integral and meromorphic functions*, J. London Math. Soc. 37 (1962), 17–27.
6. F. Gackstatter and I. Laine, *Zur Theorie der gewöhnlichen Differentialgleichungen im Komplexen*, Ann. Polon. Math. 38 (1980), 259–287.
7. W. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.
8. H. Selberg, *Über die Wertverteilung der algebroiden Funktionen*, Math. Z. 31 (1930), 709–728.
9. E. Ullrich, *Über den Einfluß der Verzweigtheit einer Algebroide auf ihre Wertverteilung*, J. Reine Angew. Math. 167 (1932), 198–220.
10. Xiao Xiuzhi and He Yuzan, *Meromorphic and algebroid solutions of higher-order algebraic differential equations*, Sci. Sinica A 26 (1983), 1034–1043.

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