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ON A PROBLEM OF FROBENIUS

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1.

In this paper the numbers are non-negative integers, if not expressly mentioned that they may be negative.

Let $k > 0$ and $A_k = \{a_0, a_1, \ldots, a_k\}$ a set with g.c.d. $(a_0, \ldots, a_k) = 1$. If n can be written on the form $n = x_0 a_0 + \ldots + x_k a_k$ we shall say that n is *dependent* on the *basis* A_k.

The problem of Frobenius consists in determining the largest integer $g(A_k) = g(a_0, \ldots, a_k)$ not dependent on A_k.

We also touch lightly into the problem of determining $n(A_k)$, i.e. the number of integers not depending on A_k.

It is well known that $g(a_0, a_1) = a_1(a_0 - 1) - a_0$ and $n(a_0, a_1) = \frac{1}{2}(a_1 - 1)(a_0 - 1)$, see Sylvester [6].

In section 2 we define a class of bases which we call regular. In section 3 we prove a basic lemma for regular bases and some lemmas which may help us to decide whether a basis is regular. In section 4 we give a recursion formulae to determine g for regular bases, and also two more special theorems.

We use our method to improve some results obtained by Hofmeister [3] and [2], Selmer [5], and Temkin [7] and to determine g for an almost arithmetic set, i.e. all but one of the basis elements form an arithmetic sequence (see Rødseth [4]).

2.

If one of the basis elements, say a_k, is dependent on the others, then clearly a_k can be removed from the basis without altering the values of g and n.

Throughout the paper we assume $a_0 > 1$ and $(a_0, a_1) = 1$. For $i = 1, \ldots, k$ we determine b_i by

$$a_i \equiv a_1 b_i (\operatorname{mod} a_0) \text{ with } 0 \leqq b_i < a_0, \; b_1 = 1.$$

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We can assume $b_i > 0$, $b_i \neq b_j$ for $i \neq j$, and $a_i < a_1 b_i$. Otherwise the basis is dependent. Then there exist $c_1, \ldots, c_k$ such that $a_i = a_1 b_i - a_0 c_i$, $c_1 = 0$.

We write $b_{k+1} = a_0$, $c_{k+1} = a_1$, and $a_{k+1} = 0$. If needed we may reindex $a_2, \ldots, a_k$ such that

$$(2.1) \qquad \begin{cases} a_i = a_1 b_i - a_0 c_i & \text{for } i = 1, \ldots, k+1, \\ 1 = b_1 < b_2 < \ldots < b_{k+1} = a_0 & \text{and} \\ 0 = c_1 < c_2 < \ldots < c_{k+1} = a_1. \end{cases}$$

(If $c_i \geqq c_j$ for $i < j$, then a_j is dependent on a_0, a_1, a_i.) If (2.1) holds we say that the basis is *ordered*. The basis is then fully determined by the sets

$$B_k = \{1 = b_1, \ldots, b_{k+1} = a_0\} \quad \text{and} \quad C_k = \{0 = c_1, \ldots, c_{k+1} = a_1\}.$$

Because $b_1 = 1$ every n can be written on the form

$$(2.2) \qquad n = \sum_{i=1}^{j} x_i b_i, \quad 0 < j \leqq k+1.$$

We call the number $\sum_{i=1}^{j} x_i c_i$ associated to the form (2.2). If $\sum_{i=1}^{l} x_i b_i < b_{l+1}$ for all $l < j$ then we call (2.2) the regular representation of n by $b_1, \ldots, b_j$ (Hofmeister [2]). Abbreviated we call it j-regular. This representation is unique and easy to determine. We now define:

$$(2.3) \qquad R(n,j) = \sum_{i=1}^{j} x_i c_i \quad \text{where } n = \sum_{i=1}^{j} x_i b_i \text{ is } j\text{-regular},$$

and

$$(2.4) \qquad M(n,j) = \max \left\{ \sum_{i=1}^{j} x_i c_i \;\middle|\; n = \sum_{i=1}^{j} x_i b_i \right\}.$$

Clearly $R(n,j) \leqq M(n,j)$, $R(n + m b_j, j) = R(n,j) + m c_j$, $R(n,j) = R(n, j+1)$ if $n < b_{j+1}$.

We define $R(n) = R(n,k)$. Then, for $n < b_{j+1}$ and $j \leqq k$ we have $R(n,j) = R(n)$.

If

$$(2.5) \qquad R(n, k+1) = M(n, k+1) \quad \text{for all } n$$

then

$$\begin{aligned} a_i = a_1 b_i - a_0 c_i &= R(b_{k+1} b_i, k+1) - R(b_{k+1} b_i, i) \\ &\geqq M(b_{k+1} b_i, k+1) - M(b_{k+1} b_i, i) \geqq 0 \end{aligned}$$

and for $i < k+1$ is $a_i = 0$ impossible, because $(a_0, a_1) = 1$ and $b_i < a_0$.

DEFINITION. If the basis is ordered and (2.5) holds, we shall say that the basis is *regular*.

3.

We use a lemma by Brauer and Shockley [1] in the following form:

LEMMA 1. *Let*

$$t_l = \min\{t \mid t \equiv a_1 l \,(\mathrm{mod}\, a_0),\ l < a_0 \text{ and } t \text{ dependent on } a_1, \dots, a_k\}.$$

Then

$$g(a_0, \dots, a_k) = \max\{t_l \mid l < a_0\} - a_0.$$

(We remark that t_l is defined only for $l < a_0$.)

LEMMA 2. *Let* $l < a_0$ *and* $l = \sum_{i=1}^{k} x_i b_i$ *be k-regular. A necessary and sufficient condition that*

$$t_l = a_1 l - a_0 R(l) = \sum_{i=1}^{k} x_i a_i \quad \text{for all } l < a_0 \tag{3.1}$$

is that the basis should be regular.

PROOF. 1) *The condition is sufficient.* Let

$$\sum_{i=1}^{k+1} y_i a_i \equiv a_1 l \ (\mathrm{mod}\, a_0).$$

From (2.1) and because $a_{k+1} = 0$ follows

$$\sum_{i=1}^{k} y_i a_i = a_1 \sum_{i=1}^{k+1} y_i b_i - a_0 \sum_{i=1}^{k+1} y_i c_i \equiv a_1 l \,(\mathrm{mod}\, a_0).$$

Hence $\sum_{i=1}^{k+1} y_i b_i = l + n a_0$ with $n \geqq 0$. Thus, because the basis is regular

$$\begin{aligned}\sum_{i=1}^{k} y_i a_i &\geqq a_1(l + na_0) - a_0 M(l + na_0, k+1)\\ &= a_1(l + na_0) - a_0 R(l + na_0, k+1) = a_1 l - a_0 R(l) = \sum_{i=1}^{k} x_i a_i\end{aligned}$$

and (3.1) follows.

2) If the basis is not regular there exists an $l < a_0$ and an $n \geqq 0$ such that $R(l + na_0, k+1) < M(l + na_0, k+1)$. Let

$$l + na_0 = \sum_{i=1}^{k+1} z_i b_i \quad \text{with} \quad M(l + na_0, k+1) = \sum_{i=1}^{k+1} z_i c_i.$$

We then have

$$\sum_{i=1}^{k} z_i a_i = \sum_{i=1}^{k+1} z_i a_i = a_1(l+na_0) - a_0 M(l+na_0, k+1)$$
$$< a_1(l+na_0) - a_0 R(l+na_0, k+1) = a_1 l - a_0 R(l)$$

and (3.1) does not hold.

LEMMA 3. *Suppose* $R(n,j) = M(n,j)$ *for all* $n,j \leqq k+1$. *Then* $R(pn+qm,j) \geqq pR(n,j)+qR(m,j)$ *for all* p, n, q *and* m.

PROOF. Let $n = \sum_{i=1}^{j} x_i b_i$ and $m = \sum_{i=1}^{j} y_i b_i$ both be j-regular. Then

$$R(pn+qm,j) = M(pn+qm,j) \geqq \sum_{i=1}^{j} (px_i+qy_i)c_i = pR(n,j)+qR(m,j).$$

We now write

$$(3.2) \quad b_{i+1} = q_i b_i - s_i, \quad s_i < b_i, \quad q_i = \left\langle \frac{b_{i+1}}{b_i} \right\rangle \geqq 2, \quad i = 1,\dots,k,$$

and prove

LEMMA 4. *Suppose* $j < k+1$ *and*

$$(3.3) \qquad R(n,j) = M(n,j) \quad \textit{for all } n.$$

a) *A necessary and sufficient condition that*

$$(3.4) \qquad R(n,j+1) = M(n,j+1) \quad \textit{for all } n$$

is

$$(3.5) \qquad c_{j+1} \geqq q_j c_j - R(s_j),$$

b) *If* (3.5) *holds for* $j = 2,\dots,k$, *then* $R(n,j) = M(n,j)$ *holds for all* n *and all* $j \leqq k+1$. *The basis is regular.*

PROOF. a) *The condition is necessary*. From (3.2) and (3.4) follow $R(b_{j+1}+s_j, j+1) = M(q_j b_j, j+1)$ or $c_{j+1} + R(s_j) \geqq q_j c_j$, that is (3.5).

The condition is sufficient. From (3.3) follows $R(n,j+1) = M(n,j+1)$ for $n < b_{j+1}$. For $n \geqq b_{j+1}$ we write

$$(3.6) \qquad n = \sum_{i=1}^{j} x_i b_i + t b_{j+1}, \quad \text{where} \quad p = \sum_{i=1}^{j} x_i b_i \text{ is } j\text{-regular and } t > 0.$$

The associated number is $m(n,j+1,t) = R(p,j) + tc_{j+1}$. We have then $m(n,j+1,t-1) = R(p+b_{j+1},j) + (t-1)c_{j+1}$. Because (3.5), (3.3), and Lemma 3

$$\begin{aligned} m(n,j+1,t) - m(n,j+1,t-1) &= c_{j+1} + R(p,j) - R(p+b_{j+1},j) \\ &\geqq q_j c_j - R(s_j) + R(p,j) - R(p+b_{j+1},j) \\ &\geqq q_j c_j + R(p,j) - R(p+q_j b_j,j) = 0. \end{aligned}$$

It follows that $m(n,j+1,t)$ is maximal for t maximal i.e. when (3.6) is $(j+1)$-regular, and (3.4) follows.

b) Clearly $R(n,2) = M(n,2)$ for all n, and b) follows by induction.

Temkin [7] introduced an ordered basis with $a_0 = b_{k+1} \leqq 2b_2$ and $a_1 = c_{k+1} \geqq 2c_k$. We prove

Lemma 5. *Suppose that the basis $\{a_0, \dots, a_k\}$ is ordered and $a_0 = b_{k+1} \leqq 2b_2$. Then a necessary and sufficient condition that the basis should be regular is*

(3.7) $$a_1 = c_{k+1} \geqq \max\{c_i + c_j - R(b_i + b_j - a_0) \mid 2 \leqq i \leqq j \leqq k\}.$$

Proof. 1) If the basis is regular, we have (for $2 \leqq i \leqq j \leqq k$)

$$R\big(a_0 + (b_i + b_j - a_0), k+1\big) = M(b_i + b_j, k+1)$$

or

$$a_1 + R(b_i + b_j - a_0) \geqq c_i + c_j$$

and (3.7) follows.

2) If $n < b_{k+1}$, then clearly

(3.8) $$R(n,k+1) = M(n,k+1).$$

We now suppose (3.7) to be true and have to prove that (3.8) holds for all $n \geqq b_{k+1}$. Thus, we assume $n \geqq b_{k+1}$ and write

$$n = \sum_{i=1}^{k} x_i b_i + tb_{k+1} = s + tb_{k+1}, \quad \text{where}$$

$$M(s,k) = \sum_{i=1}^{k} x_i c_i \quad \text{and} \quad t \geqq 0.$$

The number associated with this representation of n is

$$M(n,k+1,t) = M(s,k) + tc_{k+1}.$$

We assume $t < [n/b_{k+1}]$ and so $s \geqq b_{k+1}$. Then it is clearly sufficient to prove

$$M(n,k+1,t+1) \geqq M(n.k+1,t). \tag{3.9}$$

Because $s \geqq b_{k+1}$ we have $M(s,k) \geqq c_k$. If $M(s,k) = c_k$, then (3.9) obviously holds. If $M(s,k) > c_k$, then $x_2 + \ldots + x_k \geqq 2$.

There are two cases:

1) There is a $l > 1$ with $x_l \geqq 2$. In this case we put $y_l = x_l - 2$ and $y_i = x_i$ for $i \neq l$. Then

$$n = \sum_{i=1}^{k} y_i b_i + 2b_l - a_0 + (t+1)b_{k+1}.$$

We then have

$$\begin{aligned} M(n,k+1,t+1) &\geqq \sum_{i=1}^{k} y_i c_i + R(2b_l - a_0) + (t+1)c_{k+1} \\ &= M(n,k+1,t) - 2c_l + R(2b_l - a_0) + a_1. \end{aligned}$$

Because (3.7) we see that (3.9) is true.

2) There is a $l > 1$ with $x_l = 1$ and a $h > l$ with $x_h = 1$. We put $y_l = y_h = 0$ and $y_i = x_i$ for $i \neq l, h$. We obtain (3.9) in a similar way as in case 1).

From Lemmas 5 and 4 follows: Suppose there is an l $(2 < l < k+1)$ such that $2b_2 \geqq b_l$ and

$$c_l \geqq \max\{c_i + c_j - R(b_i + b_j - b_l) \mid 2 \leqq i \leqq j \leqq l-1\}$$

and that (3.5) holds for $j = l, \ldots, k$. Then the basis is regular.

LEMMA 6. a)

$$R(n+1,j) - R(n,j) \leqq \max\{c_i - R(b_i - 1) \mid 2 \leqq i \leqq j\}.$$

b) *If*

$$c_{i+1} \geqq q_i c_i - R(s_i) \quad \textit{for } i = 2, \ldots, j-1 \tag{3.10}$$

then $R(n+1,j) - R(n,j) \leqq c_j - R(b_j - 1)$.

PROOF. a) Let $n = \sum_{i=1}^{j} x_i b_i$ be j-regular and $s < j$ be the largest suffix with $1 + \sum_{i=1}^{s} x_i b_i = b_{s+1}$ (this is true for $s = 0$). Then

$$n + 1 = (1 + x_{s+1}) b_{s+1} + \sum_{i=s+2}^{j} x_i b_i$$

is j-regular. We have

$$R(n+1,j)-R(n,j)=c_{s+1}-\sum_{i=1}^{s}x_i c_i=c_{s+1}-R(b_{s+1}-1),$$

and a) follows.

b) Let $2\leqq i\leqq j-1$. From (3.10), Lemma 4, b), and Lemma 3 follows:

$$\begin{aligned} c_{i+1}-R(b_{i+1}-1) &\geqq q_i c_i - R(s_i,i)-R(b_{i+1}-1,i)\\ &\geqq q_i c_i - R(q_i b_i-1,i)\\ &= q_i c_i - R\big((q_i-1)b_i+b_i-1,i\big)=c_i-R(b_i-1), \end{aligned}$$

and b) follows.

4.

We assume that the basis A_k is regular with

$$B_k=\{1=b_1,\ldots,b_{k+1}=a_0\}\quad\text{and}\quad C_k=\{0=c_1,\ldots,c_{k+1}=a_1\},$$

and shall derive a recursion formula to determine g. First, some definitions:

1)

$$L_i=\left\{\sum_{i=1}^{k}x_i b_i\;\middle|\; l=\sum_{i=1}^{k}x_i b_i \text{ is } k\text{-regular, } 0\leqq l<b_i\right\},\quad i=1,\ldots,k+1.$$

i.e. L_i is the ordered set of the k-regular representations of the numbers $0,\ldots,b_i-1$.

2) Replacing b_i by a_i for all i, L_i (by Lemma 2) becomes

$$T_i=\{t_l\mid 0\leqq l<b_i\}.$$

3) Let S be an ordered set and p a number. We write

$$S+p=\{x\mid x=s+p,\; s\in S\}.$$

4)

$$\dot{\bigcup_{x=0}^{r}}(S+xp)=\varnothing\quad\text{for } r<0.$$

5) $y^{+}=\operatorname{Min}\{1,y\}$.

Let now

$$b_i-1=\sum_{s=1}^{i-1}r_{i,s}b_s\quad\text{be } k\text{-regular},\quad i=1,\ldots,k+1.$$

Then $L_1 = \{0\}$, $L_2 = \bigcup_{x=0}^{r_{2.1}} (L_1 + xb_1)$ and generally for $1 < i \leqq k+1$:

$$(4.1)\quad \begin{cases} L_i = \bigcup_{j=i-1}^{1} L_{i,j} \text{ where} \\ L_{i,j} = \bigcup_{x=0}^{r_{i,j}-1} (L_j + xb_j + \sum_{s=j+1}^{i-1} r_{i,s} b_s) \text{ for } j > 1, \text{ and} \\ L_{i,1} = \bigcup_{x=0}^{r_{i,1}} (L_1 + xb_1 + \sum_{s=2}^{i-1} r_{i,s} b_s). \end{cases}$$

We need only a little consideration to see this. E.g. let $1 \leqq p < q \leqq i-1$ and $L_{i,q} \neq \emptyset$, $L_{i,m} = \emptyset$ for $p < m < q$, $L_{i,p} \neq \emptyset$. Thus $r_{i,q} > 0$, $r_{i,m} = 0$, and $r_{i,p} > 0$ for $p > 1$. Then the last element in $L_{i,q}$ is

$$\sum_{s=1}^{q-1} r_{q,s} b_s + (r_{i,q} - 1) b_q + \sum_{s=q+1}^{i-1} r_{i,s} b_s = -1 + \sum_{s=q}^{i-1} r_{i,s} b_s.$$

The next element in L_i is the first element in $L_{i,p}$, that is

$$\sum_{s=p+1}^{i-1} r_{i,s} b_s = \sum_{s=q}^{i-1} r_{i,s} b_s.$$

Because of Lemma 1 we are only interested in $\max T_{k+1}$. Replacing b_i by a_i for all i, $L_{i,j}$ becomes $T_{i,j}$ and from (4.1) we obtain:

THEOREM 1.

$$\max T_{i,j} = \left(\max T_j + (r_{i,j} - 1) a_j + \sum_{s=j+1}^{i-1} r_{i,s} a_s \right) r_{i,j}^{+} \quad \textit{for } j > 1,$$

$$(4.2)\quad \max T_{i,1} = \sum_{s=1}^{i-1} r_{i,s} a_s = a_1 (b_i - 1) - a_0 R(b_i - 1) \quad \textit{and}$$

$$\max T_i = \max \{ \max T_{i,j} \mid 1 \leqq j \leqq i-1 \}.$$

By Lemma 1 we have

$$(4.3)\qquad g(A_k) = \max T_{k+1} - a_0.$$

By Lemma 2 we have

$$t_{l+1} - t_l = a_1 - a_0 \big(R(l+1) - R(l) \big), \quad l < a_0 - 1.$$

If there is a $j \leqq k$ with

$$a_1 > a_0 \max\{c_i - R(b_i - 1) \mid 2 \leqq i \leqq j\}, \tag{4.4}$$

then by Lemma 6, a) it follows $t_{l+1} - t_l > 0$ for all $l < b_{j+1} - 1$, and so

$$\max T_i = \max T_{i,1} = a_1(b_i - 1) - a_0 R(b_i - 1) \quad \text{for all } i \leqq j+1.$$

This may be useful by determining $\max T_{k+1}$. We also obtain

THEOREM 2. *If the basis is regular and* (4.4) *holds for* $j = k$, *then* $g(a_0, \ldots, a_k) = g(a_0, a_1) - a_0 R(a_0 - 1)$.

REMARK. Selmer [5] proved:

$$n(a_0, \ldots, a_k) = \frac{1}{a_0} \sum_{l=0}^{a_0 - 1} t_l - \tfrac{1}{2}(a_0 - 1).$$

From this theorem and Lemma 2 follows:

$$n(a_0, \ldots, a_k) = n(a_0, a_1) - \sum_{l=0}^{a_0 - 1} R(l)$$

if and only if the basis is regular.

In the next theorem we do not suppose that the basis is regular.

THEOREM 3. *Let* $a_i = a_1 b_i - a_0 c_i$, $i = 1, \ldots, k$, *where* $1 = b_1 < \ldots < b_k$ *and* $0 = c_1 < \ldots < c_k$. *We write* $b_{i+1} = q_i b_i - s_i$, $s_i < b_i$, $i = 2, \ldots, k-1$ *and suppose*

$$c_{i+1} \geqq q_i c_i - R(s_i), \quad i = 2, \ldots, k-1 \tag{4.5}$$

and

$$a_1 > a_0(c_k - R(b_k - 1)). \tag{4.6}$$

Let m be the largest suffix with $b_m < a_0$. *Then*

$$g(a_0, \ldots, a_k) = g(a_0, \ldots, a_m) = g(a_0, a_1) - a_0 R(a_0 - 1).$$

PROOF. We have $a_1(x+1) - a_0 R(x+1) - (a_1 x - a_0 R(x)) = a_1 - a_0(R(x+1) - R(x))$. From (4.6) and Lemma 6 follow that $a_1 x - a_0 R(x)$ is an increasing function of x.

If $m < k$ and $m < i \leqq k$ we determine p by $b_i \equiv p \pmod{a_0}$, $p < a_0$. Then

$$a_i = a_1 b_i - a_0 R(b_i) \equiv a_1 p - a_0 R(p) \pmod{a_0}$$

and because $p < b_i$ there exists a z_0 such that

$$a_i = a_1 p - a_0 R(p) + z_0 a_0 = \sum_{j=0}^{m} z_j a_j$$

where $p = \sum_{j=1}^{m} z_j b_j$ is m-regular. It follows that $g(a_0,\dots,a_k) = g(a_0,\dots,a_m)$.

We write $a_0 = qb_m - s$, $s < b_m$. From (4.5), (4.6), and Lemma 6 follows

$$a_1 > a_0\big(c_m - R(b_m - 1)\big). \tag{4.7}$$

Further, from (4.5), Lemma 4. b), and Lemma 3 follows

$$\begin{aligned} a_1 > a_0 c_m - a_0 R(b_m - 1, m) &\geqq a_0 c_m - R(a_0 b_m - a_0, m) \\ &= a_0 c_m - R\big((a_0 - q)b_m + s, m\big) = qc_m - R(s). \end{aligned}$$

Thus the basis $a_0,\dots,a_m$ is regular and by (4.7) and Theorem 2 we have $g(a_0,\dots,a_m) = g(a_0,a_1) - a_0 R(a_0 - 1)$.

REMARK. Theorem 3 is a generalization of a result by Hofmeister ([3, p. 79]). The proof of this assertion will not be included.

5.

EXAMPLE 1. $a_{i+1} = v_i a_i + d$, $v_i > 0$, $i = 0,\dots,k-1$ and $(a_0,d) = 1$.. d may be negative.

By induction we find $a_i = a_1 b_i - a_0 c_i$, where

$$b_0 = 0,\ b_1 = 1,\ b_{i+1} = v_i b_i + 1 = (v_i + 1)b_i - v_{i-1} b_{i-1}$$

and

$$c_0 = -1,\ c_1 = 0,\ c_{i+1} = v_i c_i + v_0 = (v_i + 1)c_i - v_{i-1} c_{i-1} \quad \text{for } i = 1,\dots,k-1.$$

We have $c_k - R(b_k - 1) = v_0$. From Theorem 3 follows:

For $d > 0$ is $g(A_k) = g(a_0,a_1) - a_0 R(a_0 - 1)$. This is a result by Hofmeister ([3, p. 83–84]).

We assume now $a_0 > b_k$, $d < 0$ and that (3.5) holds for $j = k$. Then the basis $a_0,\dots,a_k$ is regular. The representation $b_{i+1} - 1 = v_i b_i$ is k-regular. By use of (4.2) we obtain

$$\max T_i = a_i - (i-1)d \quad \text{for } i = 2,\dots,k.$$

Further by (4.3)

$$g(A_k) = \max\left\{\sum_{j=i}^{k} x_j a_j - (i-1)d \mid 1 \leqq i \leqq k\right\} - a_0,$$

where

$$a_0 - 1 = b_{k+1} - 1 = \sum_{j=1}^{k} x_j b_j \text{ is } k\text{-regular.}$$

EXAMPLE 2. $a_i = a_0 + p^{i-1}d,\ i = 1,\dots,k,\ (a_0, d) = 1,\ p \geqq 2.$ d may be negative.

This basis was introduced by Hofmeister [2] for $d > 0$. Selmer [5] treated it for $p = 2,\ d = 1$ and $a_0 > (k-4)2^{k-1} + 1$.

First, we use Theorem 3. We have $a_i = a_1 b_i - a_0 c_i$, where $b_i = p^{i-1}$ and $c_i = b_i - 1$. It is easy to see that the condition (4.5) holds.

(Let $n = \sum_{i=1}^{k} x_i b_i$ be k-regular. We write $S(n) = \sum_{i=1}^{k} x_i$. Then $R(n) = n - S(n)$.).

From Theorem 3 we obtain: If $a_1 > a_0(c_k - R(b_k - 1)) = a_0 S(b_k - 1) = a_0(p-1)(k-1)$, that is $d > a_0(pk - k - p)$, then

$$g(A_k) = a_0(S(a_0 - 1) - 1) + d(a_0 - 1).$$

See Hofmeister [2, p. 31]. We now suppose $a_0 > b_k$. Let then

$$(5.1)\quad a_0 - 1 = \sum_{i=1}^{k} x_i b_i \text{ be } k\text{-regular } (x_k > 0 \text{ and } x_i < p \text{ for } i < k).$$

Then

$$a_0 = (x_k + 1)b_k - \sum_{i=1}^{k-1} (p - 1 - x_i) b_i.$$

We suppose

$$a_1 \geqq (x_k + 1)c_k - \sum_{i=1}^{k-1} (p - 1 - x_i) c_i = a_0 + pk - k - p - S(a_0 - 1)$$

that is $d \geqq pk - k - p - S(a_0 - 1)$. Then by Lemma 4 the basis is regular, and we can use Theorem 1.

We have

$$b_i - 1 = \sum_{j=1}^{i-1} (p-1) b_j \text{ is } k\text{-regular, } i = 2,\dots,k.$$

By induction it is easy to prove that

$$\max T_i = (p-1)\sum_{j=1}^{i-1} a_j, \quad i = 2,\dots,k.$$

From (5.1), (4.2), and (4.3) we obtain

$$g(A_k) = -a_0 + \max\left\{(p-1)\sum_{j=1}^{i-1} a_j - a_i + \sum_{j=i}^{k} x_j a_j,\ \sum_{j=1}^{k} x_j a_j \,\middle|\, 2 \leqq i \leqq k\right\}.$$

In discussing this formula, we distinguish three cases. We will not go into details but only state the results.

1) $S(a_0-1) = (p-1)(k-1) + x_k$.

Then $a_0 = (1+x_k)p^{k-1}$. For $d > -1-x_k$ is

$$g(A_k) = x_k a_k + (pk-k-p)a_0 + d(p^{k-1}-1).$$

2) $S(a_0-1) = (p-1)(k-1) + x_k - 1$.

Then there exists an $r < k$ such that $a_0 = (1+x_k)p^{k-1} - p^{r-1}$.

We obtain: For $d > 0$, $g = x_k a_k - a_r + (pk-k-p)a_0 + d(p^{k-1}-1)$. For $0 > d \geqq -x_k$

(5.2) $$g = (x_k-1)a_k + (pk-k-p)a_0 + d(p^{k-1}-1).$$

3) $S(a_0-1) < (p-1)(k-1) + x_k - 1$.

Let r be the largest suffix less than k with $x_r < p-1$. For $x_r = p-2$ let s be the largest suffix with $x_s < p-1$ and $s < r$. For $x_r < p-2$, let $s = r$.

We obtain:

For $d \geqq \max\{pk-k-p-S(a_0-1), 1\}$ and $a_0 > d(p^r - p^{r-1} - p^{s-1})$

$$g = x_k a_k - a_{r+1} + (pk-k-p)a_0 + d(p^{k-1}-1).$$

For $0 > d \geqq pk-k-p-S(a_0-1)$, the result is (5.2).

In all three cases we have: If $0 > d \geqq pk-k-p-S(a_0-1)$, then

$$g = ([a_0/p^{k-1}]-1)a_k + (pk-k-p)a_0 + d(p^{k-1}-1).$$

Example 3. Suppose $a_i = a_1 b_i - a_0 c_i$, $i = 1,\dots,k$, $1 = b_1 < b_2 < \dots < b_{k+1} = a_0$ and $0 = c_1 < c_2 < \dots < c_{k+1} = a_1$. Suppose further that $a_0 \leqq 2b_2$ and

$$a_1 \geqq \max\{c_i + c_j - R(b_i + b_j - a_0) \mid 2 \leqq i \leqq j \leqq k\}.$$

By Lemma 5 the basis is regular. We use Theorem 1 and easily obtain

$$g = \max\{(b_i - 1)a_1 - (c_{i-1} + 1)a_0 \mid i = 2, \ldots, k+1\}.$$

For $a_1 \geqq 2c_k$ this is a result by Temkin [7].

EXAMPLE 4. The basis $A_{k+1} = \{a_0, \ldots, a_k, p\}$, $a_i = a_0 - id$, $i = 1, \ldots, k$, $(a_0, d) = 1$, $a_0 > kd$ and $p > 0$.

Rødseth [4] has solved the problem of determining $g(A_{k+1})$.

Although in general the basis is not regular, our method may be used. We determine s_1 and r_1 by

$$p = a_1 s_1 - a_0 r_1, \quad r_1 < s_1 < a_0 \quad \text{and} \quad r_1 < a_1.$$

Further, we write $s_0 = a_0$ and $r_0 = a_1$ and

$$\begin{array}{lll} s_0 = q_1 s_1 - s_2, & r_0 = q_1 r_1 - r_2, & 0 \leqq s_2 < s_1, \\ s_1 = q_2 s_2 - s_3, & r_1 = q_2 r_2 - r_3, & 0 \leqq s_3 < s_2, \\ \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots, \\ s_{m-1} = q_m s_m, & r_{m-1} = q_m r_m - r_{m+1}, & 0 = s_{m+1} < s_m \end{array}$$

($r_1, \ldots, r_{m+1}$ may be negative).

Then $a_1 s_i - a_0 r_i = P_i p$, where $P_0 = 0$, $P_1 = 1$ and $P_{i+1} = q_i P_i - P_{i-1}$.

We consider now the basis

$$A_{k+i} = \{a_0, \ldots, a_0 - kd, P_i p, \ldots, P_1 p\},$$

where

$$B_{k+i} = \{1, \ldots, k, s_i, s_{i-1}, \ldots, s_1, s_0\} \quad \text{and}$$
$$C_{k+i} = \{0, \ldots, k-1, r_i, r_{i-1}, \ldots, r_1, r_0\},$$

where i is defined by the condition $r_{i+1} \leqq R(s_{i+1}, k)$ and $r_j > R(s_j, k)$ for $j \leqq i$. (If $i = 0$, then p is dependent on $a_0, \ldots, a_k$ and $A_{k+i} = \{a_0, \ldots, a_k\}$.) It is easy to prove that $i \geqq 0$, $s_i > k$, $r_i > k-1$ and that A_{k+i} is regular. Obviously $g(A_{k+1}) = g(A_{k+i})$. Let $s_i - 1 = m + nk$ and $s_i - s_{i+1} - 1 = u + vk$ be k-regular representations. By using Theorem 1, it is not difficult to prove that

$$g(A_{k+1}) = -d + (n + m^+ - 1)(a_0 - kd) + \\ + (P_{i+1} - 1)p - \min\{P_i p, (n - v + m^+ - u^+)(a_0 - kd)\}.$$

Added in proof.

Finally we will prove the following generalization of Theorem 2:

THEOREM 4. *Let the basis* $A_k = \{a_0, \ldots, a_k\}$ *be ordered,* $a_i = a_i b_i - a_0 c_i$ *and*

(5.3) $$a_1 > a_0 \max\{c_i - M(b_i - 1, k) | 2 \leqq i \leqq k\}.$$

Then

$$g(A_k) = g(a_0, a_1) - a_0 M(a_0 - 1, k).$$

PROOF. Let

$$n = \sum_{i=1}^{k} x_i b_i \quad \text{with } M(n,k) = \sum_{i=1}^{k} x_i c_i, \quad n > 0.$$

Suppose $x_j > 0$. Then clearly $M(n - b_j, k) = M(n,k) - c_j$. Hence

$$\begin{aligned} M(n,k) - M(n-1,k) &= M(n,k) - M(n - b_j + b_j - 1, k) \\ &\leqq M(n,k) - M(n - b_j, k) - M(b_j - 1, k) \\ &= c_j - M(b_j - 1, k). \end{aligned}$$

From (5.3) follows

$$a_1 n - a_0 M(n,k) - \big(a_1(n-1) - a_0 M(n-1,k)\big) > 0 \quad \text{for all } n > 0.$$

Therefore (see Lemma 1) $t_l = a_1 l - a_0 M(l,k)$ and

$$g(A_k) = a_1(a_0 - 1) - a_0 M(a_0 - 1, k) - a_0 = g(a_0, a_1) - a_0 M(a_0 - 1, k).$$

EXAMPLE. Let the basis A be defined by

$$B = \{1, 7, 23, 40, a_0 = 47\} \quad \text{and} \quad C = \{0, 3, 11, 19, a_1\}.$$

From Theorem 4 follows: for $a_1 > 141$ is $g(A) = 46a_1 - 1081$. Let further A' be the basis defined by

$$\begin{aligned} B' &= \{1, 7, 23, 28, 30, 35, 37, 40, 44, 46, a_0 = 47\} \quad \text{and} \\ C' &= \{0, 3, 11, 12, 14, 15, 17, 19, 20, 22, a_1\}. \end{aligned}$$

Then A' is regular for $a_1 \geqq 25$ (follows from a generalization of Lemma 5). In addition is $g(A) = g(A')$. Using Theorem 1 we find

$$\begin{aligned} g(A) &= 20a_1 - 329 && \text{for } a_1 = 25 \textit{ or } 26, \\ g(A) &= 45a_1 - 987 && \text{for } 26 < a_1 < 94, \\ g(A) &= 46a_1 - 1081 && \text{for } a_1 > 94. \end{aligned}$$

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