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THE EFFECT OF THE ROTATION GROUP ON THE BEHAVIOUR OF A ROTATION AUTOMORPHIC FUNCTION

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1. Introduction.

If the extended complex plane $\hat{\mathbf{C}}$ and the sphere $\{x_1^2+x_2^2+(x_3-\frac{1}{2})^2 = \frac{1}{4}\}$ in $\mathbf{R}^3$ are identified via the stereographic projection with the point $(0,0,1)$ as the center of the projection, then $\hat{\mathbf{C}}$ is called the Riemann sphere. A function f, meromorphic or holomorphic in the unit disk D, is *rotation automorphic* with respect to a Fuchsian group Γ acting on D if it satisfies the equation

$$f(T(z)) = S_T(f(z)) \tag{1}$$

where $T \in \Gamma$ and S_T is a rotation of the Riemann sphere. Then $\Sigma = \{S_T \mid T \in \Gamma\}$ is a rotation group and $T \to S_T$ is a homomorphism of Γ onto Σ.

In [4] we considered rotation automorphic functions f satisfying the condition

$$\iint_F f^*(z)^2 d\sigma_z < \infty, \tag{2}$$

where F is a fundamental domain of Γ, $f^*(z) = |f'(z)|/(1+|f(z)|^2)$ is the spherical derivative of f, and $d\sigma_z$ is the euclidean area element. Then a condition on F was derived which implies the normality of f in D, i.e.,

$$\sup_{z\in D} (1-|z|^2)f^*(z) < \infty$$

(cf. [7]). Certain restrictive conditions on F are also necessary for the normality of f, since in [3] a non-normal rotation automorphic function was constructed satisfying (2).

In [1] we took another point of view: We let Γ be an arbitrary Fuchsian group but imposed some restrictions on the rotation group $\Sigma = \{S_T \mid T \in \Gamma\}$. Then the following theorem holds:

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THEOREM 1. *Let f be a rotation automorphic function satisfying* (2). *If the rotation group Σ is discrete, then f is a normal function in D.*

Hence we have obtained a slight generalization of the following theorem of Pommerenke (cf. [8, Corollary 1]):

THEOREM 2. *Let f be an automorphic function with respect to Γ. If the condition* (2) *holds, then f is a normal function in D.*

In section 2 we shall obtain more detailed knowledge on the boundary behaviour of $f^*(z)$, and hence improve Theorem 1. The proof of our result (Theorem 3) is easily seen to generalize a result of Yamashita (cf. [9, Lemma 3.2. (II)]) as pointed out briefly in Remark 1.

Section 3 contains examples of different rotation automorphic functions. We first show that there exist rotation automorphic functions with discrete rotation groups Σ (e.g. groups Σ with 1, 3 and 7 rotation axes). Then an example is constructed such that Σ has an infinite number of rotation axes. Finally we construct an example where D/Γ is compact and Σ has $2g+1$ rotation axes, where g is the genus of D/Γ. In this example the meromorphic rotation automorphic function is bounded and hence holomorphic in F.

2. Boundary behaviour of $f^*(z)$.

Let ∂D denote the boundary of the unit disk D in the complex plane. The hyperbolic distance between the points $z_1, z_2 \in D$ is denoted by $d(z_1, z_2)$ and the hyperbolic disk $\{z \mid d(z, z_0) < r\}$ by $U(z_0, r)$. We fix the fundamental domain F of Γ to be some normal polygon in D. Let $\bar{F}_D = \bar{F} \cap D$, where $\bar{F}$ is the closure of F. The spherical area of a set $A \subset \hat{\mathbf{C}}$ is denoted by $m^*(A)$.

For proving our theorem we need the following lemma [4, Lemma]:

LEMMA. *Let $(z_n) \subset F$ be a sequence of points converging to ∂D, that is $|z_n| \to 1$. If $r > 0$, $0 < R < 1$ and $D_R = \{z \mid |z| > R\}$, then $T(U(z_n, r)) \cap D_R \neq \emptyset$ for finitely many $T \in \Gamma$ and $n \in \mathbf{N}$ only.*

We are now ready to prove the main result:

THEOREM 3. *Let f be a rotation automorphic function with respect to Γ for which the condition* (2) *holds. If the rotation group Σ is discrete, then*

$$\lim_{n\to\infty} (1-|z_n|^2) f^*(z_n) = 0$$

for every sequence of points $(z_n) \subset \bar{F}_D$ converging to ∂D.

PROOF. Suppose, on the contrary, that there is a subsequence (z_k) of (z_n) for which

$$\inf_k (1-|z_k|^2) f^*(z_k) = \alpha > 0. \tag{3}$$

We choose the hyperbolic disks $U(z_k,r)$, $r>0$, $k=1,2,\ldots$. Let

$$f_k(\zeta) = f\left(\frac{\zeta+z_k}{1+\bar{z}_k\zeta}\right).$$

By [1], $\{f_k(\zeta)\}$ is a normal family in D. We may assume, without loss of generality, that $(f_k(\zeta))$ tends to $f_0(\zeta)$ locally uniformly in D. Here f_0 is a meromorphic function or ∞ in D. If f_0 is not constant, then $f_0(U(0,r)) \supset B(a,s)$ where $B(a,s)$ is a disk on the Riemann sphere $\hat{\mathbf{C}}$ with center a and radius s.

We choose an increasing sequence of positive real numbers (R_k) tending to 1. By Lemma, there is an index sequence (h_k) such that

$$T(U(z_{h_k},r)) \cap D(0,R_k) = \varnothing \quad \text{for all } T \in \Gamma. \tag{4}$$

By [5, 5.1. Theorem], the group Σ contains m_0 rotations, that is,

$$\Sigma = \{S_{T_0}, S_{T_1}, \ldots, S_{T_{m_0-1}}\}.$$

By (4),

$$\begin{aligned} f(U(z_{h_k},r)) &\subset \bigcup_{T\in\Gamma} f(T(\bar{F}_D \setminus D(0,R_k))) \\ &= \bigcup_{i=0}^{m_0-1} S_{T_i}(f(\bar{F}_D \setminus D(0,R_k))). \end{aligned} \tag{5}$$

We denote $\bigcup_{i=0}^{m_0-1} S_{T_i}(f(\bar{F}_D \setminus D(0,R_k)))$ by A_k. By (2),

$$m^*(A_k) \to 0 \tag{6}$$

as $k\to\infty$. Since (R_k) is increasing, we have

$$A_{k+1} \subset A_k \tag{7}$$

for each $k=1,2,\ldots$. By (6) and (7) we find a point $b\in B(a,s)$, $b \neq \infty$, such that $b \notin A_k$, $k \geqq k_0$. Thus, by (5), $b \notin f(U(z_{h_k},r))$ for each $k \geqq k_0$. Now there is a $z_0 \in U(0,r)$ such that $f_0(z_0) = b$. We choose a hyperbolic disk $U(z_0,r') \subset U(0,r)$ such that $|f_0(z_0)| \leqq M < \infty$ for each $z \in U(z_0,r')$. In this disk, (f_{h_k}) converges to f_0 uniformly (also in the sense of the euclidean metric) and we may assume that all f_{h_k}, $k \geqq k_0' \geqq k_0$, are analytic in $U(z_0,r')$. By the Hurwitz theorem, there exists a sequence of points $(w_{h_k}) \subset U(z_0,r') \subset U(0,r)$ such that $f_{h_k}(w_{h_k}) = b$ for each $k \geqq k_0'' \geqq k_0'$. But this is a contradiction and thus f_0 is constant. Therefore, for the sequence of spherical derivatives $(f_{h_k}^*(0))$,

$$(1-|z_{h_k}|^2)f^*(z_{h_k}) = f_{h_k}^*(0) \to 0$$

as $k \to \infty$. This contradicts (3) and thus the theorem is proved.

Yamashita [10] considers the set $K_0(f)$ of the points $\zeta \in \partial D$ for which

$$\lim_{z \to \zeta} (1-|z|^2)f^*(z) = 0$$

along each angular domain at ζ. By an angular domain at ζ we mean a triangular domain whose vertices are ζ and two points of D.

Corollary. *Let f be a rotation automorphic function with respect to Γ for which the condition* (2) *holds. If the rotation group Σ is discrete, then $\bar{F} \cap \partial D \subset K_0(f)$.*

Proof. Let $\zeta \in \bar{F} \cap \partial D$ and let Δ be an arbitrary angular domain at ζ. We choose any sequence of points $(z_n) \subset \Delta$ converging to ζ. Then there is a positive constant M such that

$$\sup_n d(z_n, 0\zeta) \leqq M \tag{8}$$

where 0ζ is the radius from 0 to ζ. Let $T_n \in \Gamma$ such that $T_n(z_n) = z_n' \in \bar{F}$ for each $n = 1,2,\ldots$. By (8) we may apply our Lemma and thus $|z_n'| \to 1$ for $n \to \infty$. By Theorem 3 we obtain

$$(1-|z_n|^2)f^*(z_n) = (1-|z_n'|^2)f^*(z_n') \to 0$$

as $n \to \infty$. Thus the corollary is proved.

Remark 1. In [9, Lemma 3.2. (II)] Yamashita proved the following: Let g be a meromorphic function in D. Then

$$\lim_{|z| \to 1} (1-|z|^2)g^*(z) = 0 \tag{9}$$

if and only if there exists $r > 0$ such that

$$\lim_{|z| \to 1} \iint_{U(z,r)} g^*(z)^2 d\sigma_z = 0. \tag{10}$$

The condition (10) means that the spherical area of the Riemannian multiple-sheeted image of $U(z,r)$ by g tends to zero. By the proof of Theorem 3 we generalize this result as follows: If the spherical area of the image of $U(z,r)$ by g tends to zero for $|z| \to 1$, then (9) holds. We outline the proof briefly. Suppose, on the contrary, that there is a sequence of points (z_k) for which

$$\inf_k (1 - |z_k|^2) g^*(z_k) = \alpha > 0.$$

We may assume, without loss of generality, that the spherical area (without multiplicities)

$$m^*\big(g(U(z_k, r))\big) \leqq \frac{\pi}{2^{k+1}}$$

for each $k = 1, 2, \ldots$. Then

$$\begin{aligned} m^*\Big(g\Big(\bigcup_{k=1}^{\infty} U(z_k, r)\Big)\Big) &\leqq \sum_{k=1}^{\infty} m^*\big(g(U(z_k, r))\big) \\ &\leqq \sum_{k=1}^{\infty} \frac{\pi}{2^{k+1}} = \frac{\pi}{2} < \pi. \end{aligned} \tag{11}$$

Let

$$g_k(\zeta) = g\left(\frac{\zeta + z_k}{1 + \bar{z}_k \zeta}\right).$$

By (11) the family $\{g_k\}$ omits three values in $U(0,r)$ and thus is a normal family there. After this we shall continue as in the proof of Theorem 3.

Remark 2. By Remark 1 we note that the spherical area of the Riemannian image of $U(z,r)$ by g and the spherical area of the image of $U(z,r)$ by g simultaneously tend to zero as $|z| \to 1$.

Remark 3. We could compensate $\bar{F}_D$ by $G_R = \{z \mid d(z,F) < R\}$ in Theorem 3.

Remark 4. In [3] we constructed a non-normal rotation automorphic function satisfying the condition (2). For this function the rotation group Σ was infinite with one rotation axis only (0∞-axis).

This example and Theorem 3 show that if we do not restrict Γ in any way, then changing on the image side from finite Σ to infinite Σ can cause a strict difference in the behaviour of the expression $(1-|z|^2)f^*(z)$.

3. Examples.

In this section we give examples of rotation automorphic functions f holomorphic or meromorphic in D. We shall always suppose that the rotation group Σ of f contains rotations with 0∞-axis. Hence, if Σ has only one rotation axis, then f is character automorphic (cf. [8]).

Denote by Γ_0 a Fuchsian group representing a Riemann surface D/Γ_0 conformally equivalent to a sphere with three punctures. Suppose that the metric fundamental polygon F of Γ_0 is a regular non-euclidean quadrilateral with all vertices on the unit circle. Denote by s_1, s_2, s_3, s_4 the positively oriented sides of F, let T_1 and T_2 be the generating parabolic transformations of Γ_0 and suppose that

$$T_1(s_1) = s_2^{-1}, \quad T_2(s_3) = s_4^{-1}.$$

Character automorphic functions. Let $S: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be the rotation $S(w) = -w$, let Σ be the cyclic group generated by S and let F' be the upper half plane with the sides $t_1 = [-1,0]$, $t_2 = [0,1]$, $t_3 = [1,\infty]$, $t_4 = [-\infty, -1]$.

Since F and F' both are conformally equivalent to a square, there exists a conformal map $f: F \to F'$ for which

$$f(s_1) = t_1, \quad f(s_2) = t_2,$$
$$f(s_3) = t_3, \quad f(s_4) = t_4.$$

Then we have on the boundary of F

$$f \circ T_1 = S \circ f, \quad f \circ T_2 = S \circ f.$$

Hence f can be extended to D and we have obtained a character automorphic function holomorphic in D.

Rotation automorphic function with a quadratic group of rotations. The simplest discrete non-cyclic rotation group, *the quadratic group*, has three different rotation axes which are orthogonal to each other. (In [6] all discrete rotation groups are thoroughly treated.)

Let

$$F' = \{w = u + iv \mid v \geqq 0, \ |w| \geqq 1\}$$

and let the sides

$$t_1 = \{w \mid |w| = 1,\ u \leqq 0\}, \quad t_2 = \{w \mid |w| = 1,\ u \geqq 0\},$$
$$t_3 = \{w \mid u \geqq 1,\ v = 0\}, \quad t_4 = \{w \mid u \leqq -1,\ v = 0\},$$

be positively oriented with respect to F'.

Let Σ be the group generated by the rotations $S_2(w) = -w$, $S_1(w) = -1/w$. Then Σ is a quadratic group containing the rotations $S_1, S_2, S_3(w) = 1/w$ and the identity. Let $f\colon F \to F'$ be a conformal map for which

$$f(s_1) = t_1,\ f(s_2) = t_2,$$
$$f(s_3) = t_3,\ f(s_4) = t_4.$$

Let $j\colon \Gamma_0 \to \Sigma$ be the homomorphism defined by $j(T_1) = S_1$, $j(T_2) = S_2$. We can extend f to a holomorphic function $f\colon D \to \mathbf{C}$ satisfying $f \circ T = j(T) \circ f$ for all $T \in \Gamma$.

An example of a rotation automorphic function with the group of the tetrahedral rotations (i.e. a group containing 12 rotations and 7 axes) is given in [4].

Rotation automorphic function with a non-discrete group of rotations. Let Σ be the group generated by the rotations $S_1(w) = e^{-i}w$ and $S_2(w) = 1/w$, let

$$F' = \{w \mid -\tfrac{1}{2} \leqq \arg w \leqq \tfrac{1}{2}, |w| \leqq 1\}$$

and let the sides

$$t_1 = \{w \mid \arg w = \tfrac{1}{2},\ |w| \leqq 1\},$$
$$t_2 = \{w \mid \arg w = -\tfrac{1}{2},\ |w| \leqq 1\},$$
$$t_3 = \{w \mid -\tfrac{1}{2} \leqq \arg w \leqq 0,\ |w| = 1\},$$
$$t_4 = \{w \mid 0 \leqq \arg w \leqq \tfrac{1}{2},\ |w| = 1\}$$

be positively oriented with respect to F'.

The fixed points of the rotation $S_1^n \circ S_2 \circ S_1^{-n}$ are $\pm e^{-ni}$. Hence Σ has infinitely many rotation axes.

Since $S_1(t_1) = t_2^{-1}$ and $S_2(t_3) = t_4^{-1}$, we can continue the conformal map $f\colon F \to F'$, $f(s_k) = t_k$, k = 1,2,3,4, to a holomorphic rotation automorphic function $f\colon D \to \mathbf{C}$ having Σ as the group of rotations.

Rotation automorphic function with compact D/Γ. Let F be the regular non-euclidean octagon in D whose vertices are

$$\alpha_j = \tfrac{1}{2}\left(\sqrt{\sqrt{2}+1} - i\sqrt{\sqrt{2}-1}\right)e^{(j-1)i\pi/4},\ j = 1,\dots,8.$$

Then all vertices of F lie on the circle $|z| = 2^{-1/4}$ and the sum of the angles of F equals 2π. Denote by s_j the side of F starting from α_j, $j = 1, \ldots, 8$ (see Fig. 1).

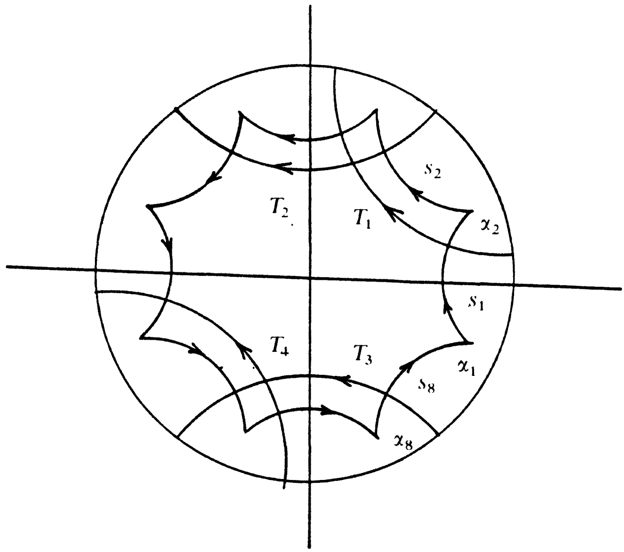

Figure 1.

In order to define a Möbius transformation T of D onto itself it suffices to give the isometric circles $I(T)$ and $I(T^{-1})$. Let

$$I(T_1) \supset s_1, \quad I(T_1^{-1}) \supset s_3,$$
$$I(T_2) \supset s_2, \quad I(T_2^{-1}) \supset s_4,$$
$$I(T_3) \supset s_8, \quad I(T_3^{-1}) \supset s_6,$$
$$I(T_4) \supset s_7, \quad I(T_4^{-1}) \supset s_5.$$

Then for instance

$$T_1(z) = i\,\frac{z\sqrt{\sqrt{2}+1}-2}{z\sqrt{2}-\sqrt{\sqrt{2}+1}},$$

and all transformations $T_j\colon D \to D$, $j = 1, \ldots, 4$, are hyperbolic. Moreover, $T_1(s_1) = s_3^{-1}$, $T_2(s_2) = s_4^{-1}$, $T_3(s_8) = s_6^{-1}$, and $T_4(s_7) = s_5^{-1}$.

Let Γ be the Fuchsian group generated by T_1, T_2, T_3, and T_4. Then Γ has F as a metric fundamental polygon. Now $T_1(\alpha_1) = \alpha_4$, $T_2^{-1}(\alpha_4) = \alpha_3$, $T_1^{-1}(\alpha_3) = \alpha_2$, and $T_2(\alpha_2) = \alpha_5$. Hence

$$T_2\big(T_1^{-1}\big(T_2^{-1}\big(T_1(\alpha_1)\big)\big)\big) = \alpha_5.$$

Similarly, $T_4^{-1}(\alpha_5) = \alpha_8$, $T_3(\alpha_8) = \alpha_7$, $T_4(\alpha_7) = \alpha_6$, and $T_3^{-1}(\alpha_6) = \alpha_1$. Hence

$$T_3^{-1}\big(T_4\big(T_3\big(T_4^{-1}(\alpha_5)\big)\big)\big) = \alpha_1.$$

Since the sum of the angles of F equals 2π, it follows that the relation

(12) $$T_3^{-1} \circ T_4 \circ T_3 \circ T_4^{-1} \circ T_2 \circ T_1^{-1} \circ T_2^{-1} \circ T_1 = \mathrm{id}$$

holds. The relation (12) is a basis for all relations in Γ.

Let F' be the octagon whose sides lie on the unit circle and whose vertices are

$$\beta_j = e^{i\pi(2j-3)/8}, \quad j = 1,\dots,8.$$

Denote by t_j the side of F' starting from β_j, $j = 1,\dots,8$ (see Fig. 2). Define rotations S_j, $j = 1,2$, of the Riemann sphere $\hat{\mathbb{C}}$ as follows:

$$S_1(z) = i/z,$$
$$S_2(z) = -1/z.$$

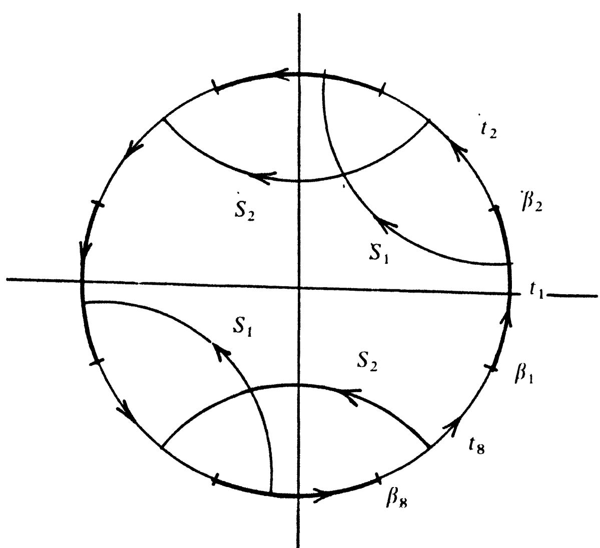

Figure 2.

Then S_1 and S_2 generate a discrete group Σ of dihedral rotations having five different axes. Moreover, $S_1(t_1) = t_3^{-1}$, $S_2(t_2) = t_4^{-1}$, $S_2(t_8) = t_6^{-1}$, and $S_1(t_7) = t_5^{-1}$.

Define $j(T_1^{\pm 1}) = j(T_4^{\pm 1}) = S_1$ and $j(T_2^{\pm 1}) = j(T_3^{\pm 1}) = S_2$. Since $S_2(S_1(z)) = iz$, we have

$$j(T_3^{-1}) \circ j(T_4) \circ j(T_3) \circ j(T_4^{-1}) \circ j(T_2) \circ j(T_1^{-1}) \circ j(T_2^{-1}) \circ j(T_1)$$
$$= (S_2 \circ S_1)^4 = \mathrm{id}.$$

Since (12) is the basis relation in Γ, j extends to a surjective homomorphism $j: \Gamma \to \Sigma$.

Let f be the conformal map of F onto F' for which $f(\alpha_j) = \beta_j$, $j = 1, \ldots, 8$. Then we have

$$\begin{aligned} f \circ T_1 &= S_1 \circ f \quad \text{on } s_1, \\ f \circ T_2 &= S_2 \circ f \quad \text{on } s_2, \\ f \circ T_3 &= S_2 \circ f \quad \text{on } s_8, \\ f \circ T_4 &= S_1 \circ f \quad \text{on } s_7. \end{aligned}$$

Define in $T(F)$, $T \in \Gamma$, f by

$$f | T(F) = j(T) \circ (f | F) \circ T^{-1}.$$

It follows that f is a well-defined meromorphic rotation automorphic function in D having Σ as the group of rotations and satisfying the following conditions:

(i) D/Γ is a compact surface of genus 2,
(ii) Σ has 5 axes,
(iii) $|f(z)| \leqq 1$ for all $z \in F$.

The above construction applies evidently to every genus $g > 1$. The number of axes of Σ is then $2g + 1$.

If f is either automorphic or character-automorphic and D/Γ is compact, then f cannot be bounded in F unless f reduces to a constant (cf. [2]).

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